The electron as a particle

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And I laugh to see them whirl and flee,
Like a swarm of golden bees.
Shelley *The Cloud*

1.1 Introduction

In the popular mind the electron lives as something very small that has something to do with electricity. Studying electromagnetism does not change the picture appreciably. You learn that the electron can be regarded as a negative point charge and it duly obeys the laws of mechanics and electromagnetism. It is a particle that can be accelerated or decelerated but cannot be taken to bits.

Is this picture likely to benefit an engineer? Yes, if it helps him to produce a device. Is it a *correct* picture? Well, an engineer is not concerned with the truth; that is left to philosophers and theologians: the prime concern of an engineer is the utility of the final product. If this physical picture makes possible the birth of the vacuum tube, we must deem it useful; but if it fails to account for the properties of the transistor then we must regard its appeal as less alluring. There is no doubt, however, that we can go quite far by regarding the electron as a particle even in a solid—the subject of our study.

What does a solid look like? It consists of atoms. This idea originated a few thousand years ago in Greece, and has had some ups and downs in history, but today its truth is universally accepted. Now if matter consists of atoms, they must be somehow piled upon each other. The science that is concerned with the spatial arrangement of atoms is called crystallography. It is a science greatly revered by crystallographers; engineers are respectful, but lack enthusiasm. This is because the need to visualize structures in three dimensions adds to the hard enough task of thinking about what the electron will do next. For this chapter, let us assume that all materials crystallize in the simple cubic structure of Fig. 1.1, with the lattice ions fixed (it is a solid) and some electrons are free to wander between them. This will shortly enable us to explain Ohm's law, the Hall effect and several other important events. But if you are sceptical about over simplification, look forward to Fig. 5.3 to see how the elemental semiconductors crystallize in the diamond structure, or get a greater shock with Fig. 5.4 which shows a form of carbon that was discovered in meteorites but has only recently been fabricated in laboratories.

Let us specify our model a little more closely. If we postulate the existence of a certain number of electrons capable of conducting electricity, we must also say that a corresponding amount of positive charge exists in the solid. It must look electrically neutral to the outside world. Second, in analogy with our picture of gases, we may assume that the electrons bounce around in the

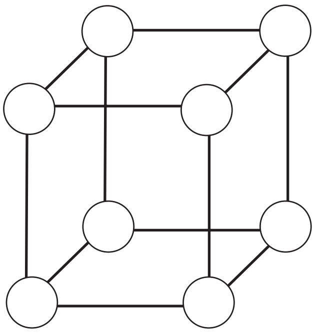

Fig. 1.1
Atoms crystallizing in a cubical lattice.

interatomic spaces, colliding occasionally with lattice atoms. We may even go further with this analogy and claim that in equilibrium the electrons follow the same statistical distribution as gas molecules (that is, the Maxwell–Boltzmann distribution) which depends strongly on the temperature of the system. The average kinetic energy of each degree of freedom is then $\frac{1}{2}kT$ where T is absolute temperature and k is Boltzmann's constant. So we may say that the mean thermal velocity of electrons is given by the formula*

$$\tfrac{1}{2}mv_{\mathrm{th}}^2 = \tfrac{3}{2}kT \tag{1.1}$$

v_{th} is the thermal velocity, and m is the mass of the electron.

because because particles moving in three dimensions have three degrees of freedom.[†]

We shall now calculate some observable quantities on the basis of this simplest model and see how the results compare with experiment. The success of this simple model is somewhat surprising, but we shall see as we proceed that viewing a solid, or at least a metal, as a fixed lattice of positive ions held together by a jelly-like mass of electrons approximates well to the modern view of the electronic structure of solids. Some books discuss mechanical properties in terms of dislocations that can move and spread; the solid is then pictured as a fixed distribution of negative charge in which the lattice ions can move. These views are almost identical; only the external stimuli are different.

1.2 The effect of an electric field—conductivity and Ohm's law

Suppose a potential difference U is applied between the two ends of a solid length L. Then an electric field

$$\mathcal{E} = \frac{U}{L} \tag{1.2}$$

is present at every point in the solid, causing an acceleration

$$a = \frac{e}{m}\mathcal{E}. \tag{1.3}$$

Thus, the electrons, in addition to their random velocities, will acquire a velocity in the direction of the electric field. We may assume that this directed velocity is completely lost after each collision, because an electron is much lighter than a lattice atom. Thus, only the part of this velocity that is picked up in between collisions counts. If we write τ for the average time between two collisions, the final velocity of the electron will be $a\tau$ and the average velocity

$$v_{\mathrm{average}} = \tfrac{1}{2}a\tau. \tag{1.4}$$

This is simple enough but not quite correct. We should not use the *average* time between collisions to calculate the average velocity but the actual times and then the average. The correct derivation is fairly lengthy, but all it gives is a factor of 2.[‡] Numerical factors like 2 or 3 or π are generally not worth worrying

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[*] We shall see later that this is not so for metals but it is nearly true for conduction electrons in semiconductors.

[†] This is often called the Drude model after the man who proposed it a good hundred years ago.

[‡] See, for example, W. Shockley, *Electrons and holes in semiconductors*, D. van Nostrand, New York, 1950, pp. 191–5.

about in simple models, but just to agree with the formulae generally quoted in the literature, we shall incorporate that factor 2, and use

$$v_{\text{average}} = a\tau. \tag{1.5}$$

The average time between collisions, τ, has many other names; for example, mean free time, relaxation time, and collision time. Similarly, the average velocity is often referred to as the mean velocity or drift velocity. We shall call them 'collision time' and 'drift velocity', denoting the latter by v_{D}.

The relationship between drift velocity and electric field may be obtained from eqns (1.3) and (1.5), yielding

$$v_{\text{D}} = \left(\frac{e}{m}\tau\right)\mathcal{E}, \tag{1.6}$$

where the proportionality constant in parentheses is called the 'mobility'. This is the only name it has, and it is quite a logical one.

The higher the mobility, the more mobile the electrons.

Assuming now that all electrons drift with their drift velocity, the total number of electrons crossing a plane of unit area per second may be obtained by multiplying the drift velocity by the density of electrons, N_{e}. Multiplying further by the charge on the electron we obtain the electric current density

$$J = N_e e v_{\text{D}}. \tag{1.7}$$

Notice that it is only the drift velocity, created by the electric field, that comes into the expression. The random velocities do not contribute to the electric current because they average out to zero.*

We can derive similarly the relationship between current density and electric field from eqns (1.6) and (1.7) in the form

$$J = \frac{N_{\text{e}} e^2 \tau}{m}\mathcal{E}. \tag{1.8}$$

This is a linear relationship which you may recognize as Ohm's law

$$J = \sigma\mathcal{E}, \tag{1.9}$$

where σ is the electrical conductivity. When first learning about electricity you looked upon σ as a bulk constant; now you can see what it comprises of. We can write it in the form

$$\begin{aligned}\sigma &= \left(\frac{e}{m}\tau\right)(N_{\text{e}}e)\\ &= \mu_{\text{e}}(N_{\text{e}}e).\end{aligned} \tag{1.10}$$

In metals, incidentally, the mobilities are quite low, about two orders of magnitude below those of semiconductors; so their high conductivity is due to the high density of electrons.

That is, we may regard conductivity as the product of two factors, charge density ($N_{\text{e}}e$) and mobility (μ_{e}). Thus, we may have high conductivities because there are lots of electrons around or because they can acquire high drift velocities, by having high mobilities.

Ohm's law further implies that σ is a constant, which means that τ must be independent of electric field.[†] From our model so far it is more reasonable

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[*] They give rise, however, to *electrical noise* in a conductor. Its value is usually much smaller than the signals we are concerned with so we shall not worry about it, although some of the most interesting engineering problems arise just when signal and noise are comparable.

[†] It seems reasonable at this stage to assume that the charge and mass of the electron and the number of electrons present will be independent of the electric field.

to assume that l, the distance between collisions (usually called the mean free path) in the regularly spaced lattice, rather than τ, is independent of electric field. But l must be related to τ by the relationship,

$$l = \tau(v_{\text{th}} + v_{\text{D}}). \tag{1.11}$$

Since v_{D} varies with electric field, τ must also vary with the field unless

$$v_{\text{th}} \gg v_{\text{D}}. \tag{1.12}$$

In a typical metal $\mu_{\text{e}} = 5 \times 10^{-3}\,\text{m}^2\,\text{V}^{-1}\,\text{s}^{-1}$, which gives a drift velocity v_{D} of $5 \times 10^{-3}\,\text{m s}^{-1}$ for an electric field of $1\,\text{V m}^{-1}$.

As Ohm's law is accurately true for most metals, this inequality should hold. The thermal velocity at room temperature according to eqn (1.1) (which actually gives too low a value for metals) is

$$v_{\text{th}} = \left(\frac{3kT}{m}\right)^{1/2} \cong 10^5 \text{ m s}^{-1}. \tag{1.13}$$

Thus, there will be a constant relationship between current and electric field accurate to about 1 part in 10^8.*

This important consideration can be emphasized in another way. Let us draw the graph (Fig. 1.2) of the distribution of particles in velocity space, that is with rectilinear axes representing velocities in three dimensions, v_x, v_y, v_z. With no electric field present, the distribution is spherically symmetric about the origin. The surface of a sphere of radius v_{th} represents all electrons moving in all possible directions with that r.m.s. speed. When a field is applied along the x-axis (say), the distribution is minutely perturbed (the electrons acquire some additional velocity in the direction of the x-axis) so that its centre shifts from (0, 0, 0) to about ($v_{\text{th}}/10^8$, 0, 0).

Taking copper, a field of $1\,\text{V m}^{-1}$ causes a current density of $10^8\,\text{A m}^{-2}$. It is quite remarkable that a current density of this magnitude can be achieved with an almost negligible perturbation of the electron velocity distribution.

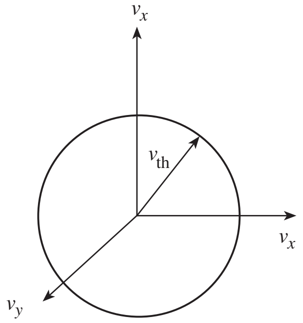

Fig. 1.2
Distributions of electrons in velocity space.

1.3 The hydrodynamic model of electron flow

By considering the flow of a charged fluid, a sophisticated model may be developed. We shall use it only in its crudest form, which does not give much of a physical picture but leads quickly to the desired result.

The equation of motion for an electron is

$$m\frac{\mathrm{d}v}{\mathrm{d}t} = e\mathscr{E}. \tag{1.14}$$

If we now assume that the electron moves in a viscous medium, then the forces trying to change the momentum will be resisted. We may account for this by adding a 'momentum-destroying' term, proportional to v. Taking the

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* This is less true for semiconductors as they violate Ohm's law at high electric fields.

proportionality constant as ζ eqn (1.14) modifies to

$$m\left(\frac{\mathrm{d}v}{\mathrm{d}t}+\zeta v\right)=e\mathscr{E}. \tag{1.15}$$

ζ may be regarded here as a measure of the viscosity of the medium.

In the limit, when viscosity dominates, the term $\mathrm{d}v/\mathrm{d}t$ becomes negligible, resulting in the equation

$$mv\zeta=e\mathscr{E}, \tag{1.16}$$

which gives for the velocity of the electron

$$v=\frac{e}{m}\frac{1}{\zeta}\mathscr{E}. \tag{1.17}$$

It may be clearly seen that by taking $\zeta = 1/\tau$ eqn (1.17) agrees with eqn (1.6); hence we may regard the two models as equivalent and, in any given case, use whichever is more convenient.

1.4 The Hall effect

Let us now investigate the current flow in a rectangular piece of material, as shown in Fig. 1.3. We apply a voltage so that the right-hand side is positive. Current, by convention, flows from the positive side to the negative side, that is in the direction of the negative z-axis. But electrons, remember, flow in a direction opposite to conventional current, that is from left to right. Having sorted this out let us now apply a magnetic field in the positive y-direction. The force on an electron due to this magnetic field is

$$e(\mathbf{v}\times\mathbf{B}). \tag{1.18}$$

To get the resultant vector, we rotate vector **v** into vector **B**. This is a clockwise rotation, giving a vector in the negative x-direction. But the charge of the electron, e, is negative; so the force will point in the positive x-direction; the electrons are deflected upwards. They cannot move farther than the top end of the slab, and they will accumulate there. But if the material was electrically neutral before, and some electrons have moved upwards, then some positive ions at the bottom will be deprived of their compensating negative charge. Hence an electric field will develop between the positive bottom layer and the

Equilibrium is established when the force due to the transverse electric field just cancels the force due to the magnetic field.

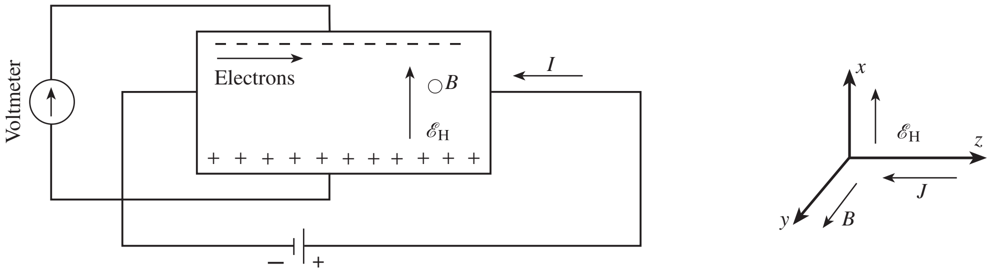

Fig. 1.3
Schematic representation of the measurement of the Hall effect.

negative top layer. Thus, after a while, the upward motion of the electrons will be prevented by this internal electric field. This happens when

$$\mathcal{E}_{\mathrm{H}} = vB. \tag{1.19}$$

Expressed in terms of current density,

R_{H} is called the *Hall coefficient.*

$$\mathcal{E}_{\mathrm{H}} = R_{\mathrm{H}} J B, \qquad R_{\mathrm{H}} = \frac{1}{N_{\mathrm{e}} e}. \tag{1.20}$$

In this experiment $\mathcal{E}_{\mathrm{H}}$, J, and B are measurable; thus R_{H}, and with it the density of electrons, may be determined.

What can we say about the direction of $\mathcal{E}_{\mathrm{H}}$? Well, we have taken meticulous care to find the correct direction. Once the polarity of the applied voltage and the direction of the magnetic field are chosen, the electric field is well and truly defined. So if we put into our measuring apparatus one conductor after the other, the measured transverse voltage should always have the same polarity. Yes . . . the logic seems unassailable. Unfortunately, the experimental facts do not conform. For some conductors and semiconductors the measured transverse voltage is in the *other* direction.

How could we account for the different sign? One possible way of explaining the phenomenon is to say that in certain conductors (and semiconductors) electricity is carried by positively charged particles. Where do they come from? We shall discuss this problem in more detail some time later; for the moment just accept that mobile positive particles may exist in a solid. They bear the unpretentious name 'holes'.

To incorporate holes in our model is not at all difficult. There are now two species of charge carriers bouncing around, which you may imagine as a mixture of two gases. Take good care that the net charge density is zero, and the new model is ready. It is actually quite a good model. Whenever you come across a new phenomenon, try this model first. It might work.

Returning to the Hall effect, you may now appreciate that the experimental determination of R_{H} is of considerable importance. If only one type of carrier is present, the measurement will give us immediately the sign and the density of the carrier. If both carriers are simultaneously present it still gives useful information but the physics is a little more complicated (see Examples 1.7 and 1.8).

In our previous example we took a typical metal where conduction takes place by electrons only, and we got a drift velocity of $5 \times 10^{-3}\ \mathrm{m\,s^{-1}}$. For a magnetic field of 1 T the transverse electric field is

$$\mathcal{E}_{\mathrm{H}} = Bv = 5 \times 10^{-3}\ \mathrm{V\,m^{-1}}. \tag{1.21}$$

The corresponding electric field in a semiconductor is considerably higher because of the higher mobilities.

1.5 Electromagnetic waves in solids

So far as the propagation of electromagnetic waves is concerned, our model works very well indeed. All we need to assume is that our holes and electrons obey the equations of motion, and when they move, they give rise to fields in accordance with Maxwell's theory of electrodynamics.

It is perfectly simple to take holes into account, but the equations, with holes included, would be considerably longer, so we shall confine our attention to electrons.

We could start immediately with the equation of motion for electrons, but let us first review what you already know about wave propagation in a medium characterized by the constants permeability, μ, dielectric constant, ϵ, and conductivity, σ (it will not be a waste of time).

First of all we shall need Maxwell's equations:

$$\frac{1}{\mu}\nabla \times \mathbf{B} = \mathbf{J} + \epsilon \frac{\partial \mathcal{E}}{\partial t}, \tag{1.22}$$

$$\nabla \times \mathcal{E} = -\frac{\partial \mathbf{B}}{\partial t}. \tag{1.23}$$

Second, we shall express the current density in terms of the electric field as

$$\mathbf{J} = \sigma \mathcal{E}. \tag{1.24}$$

It would now be a little more elegant to perform all the calculations in vector form, but then you would need to know a few vector identities, and tensors (quite simple ones, actually) would also appear. If we use coordinates instead, it will make the treatment a little lengthier, but not too clumsy if we consider only the one-dimensional case, when

$$\frac{\partial}{\partial x} = 0, \quad \frac{\partial}{\partial y} = 0. \tag{1.25}$$

Assuming that the electric field has only a component in the x-direction (see the coordinate system in Fig. 1.3), then

$$\nabla \times \mathcal{E} = \begin{vmatrix} \mathbf{e}_x & \mathbf{e}_y & \mathbf{e}_z \\ 0 & 0 & \dfrac{\partial}{\partial z} \\ \mathcal{E}_x & 0 & 0 \end{vmatrix} = \frac{\partial \mathcal{E}_x}{\partial z}\mathbf{e}_y, \tag{1.26}$$

where $\mathbf{e}_x$, $\mathbf{e}_y$, $\mathbf{e}_z$ are the unit vectors. It may be seen from this equation that the magnetic field can have only a y-component. Thus, eqn (1.23) takes the simple form

$$\frac{\partial \mathcal{E}_x}{\partial z} = -\frac{\partial B_y}{\partial t}. \tag{1.27}$$

We need further

$$\nabla \times \mathbf{B} = \begin{vmatrix} \mathbf{e}_x & \mathbf{e}_y & \mathbf{e}_z \\ 0 & 0 & \dfrac{\partial}{\partial z} \\ 0 & B_y & 0 \end{vmatrix} = \frac{\partial B_y}{\partial z}\mathbf{e}_x, \tag{1.28}$$

which, combined with eqn (1.24), brings eqn (1.22) to the scalar form

$$-\frac{\partial B_y}{\partial z} = \mu\sigma\mathcal{E}_x + \mu\epsilon\frac{\partial \mathcal{E}_x}{\partial t}. \tag{1.29}$$

Thus, we have two fairly simple differential equations to solve. We shall attempt the solution in the form*

ω represents frequency, and k is the wavenumber.

$$\mathcal{E}_x = \mathcal{E}_{x_0} \exp\{-\mathrm{i}(\omega t - kz)\} \tag{1.30}$$

and

$$B_y = B_{y_0} \exp\{-\mathrm{i}(\omega t - kz)\}. \tag{1.31}$$

Then,

$$\frac{\partial}{\partial z} \equiv \mathrm{i}k, \quad \frac{\partial}{\partial t} \equiv -\mathrm{i}\omega, \tag{1.32}$$

which reduces our differential equations to the algebraic equations

$$\mathrm{i}k\mathcal{E}_x = \mathrm{i}\omega B_y \tag{1.33}$$

and

$$-\mathrm{i}kB_y = (\mu\sigma - \mathrm{i}\omega\mu\epsilon)\mathcal{E}_x. \tag{1.34}$$

This is a homogeneous equation system. By the rules of algebra, there is a solution, apart from the trivial $\mathcal{E}_x = B_y = 0$, only if the determinant of the coefficients vanishes, that is

$$\begin{vmatrix} -\mathrm{i}k & \mathrm{i}\omega \\ \mu\sigma - \mathrm{i}\omega\mu\epsilon & \mathrm{i}k \end{vmatrix} = 0. \tag{1.35}$$

Expanding the determinant we get

$$k^2 - \mathrm{i}\omega(\mu\sigma - \mathrm{i}\omega\mu\epsilon) = 0. \tag{1.36}$$

Different people call this equation by different names. Characteristic, determinantal, and dispersion equation are among the names more frequently used. We shall call it the *dispersion equation* because that name describes best what is happening physically.

Essentially, the equation gives a relationship between the frequency, ω, and the wavenumber, k, which is related to phase velocity by $v_{\mathrm{P}} = \omega/k$. Thus, unless ω and k are linearly related, the various frequencies propagate with different velocities and at the boundary of two media are refracted at different angles. Hence the name dispersion.

A medium for which $\sigma = 0$ and μ and ϵ are independent of frequency is nondispersive. The relationship between k and ω is simply

$c_{\mathrm{m}} \leqslant c$ is the velocity of the electromagnetic wave in the medium.

$$k = \omega\sqrt{\mu\epsilon} = \frac{\omega}{c_{\mathrm{m}}}. \tag{1.37}$$

Solving eqn (1.36) formally, we get

$$k = (\omega^2\mu\epsilon + \mathrm{i}\omega\mu\sigma)^{1/2}. \tag{1.38}$$

Thus, whenever $\sigma \neq 0$, the wavenumber is complex. What is meant by a complex wavenumber? We can find this out easily by looking at the exponent of eqn (1.30). The spatially varying part is

$$\begin{aligned} \exp(\mathrm{i}kz) &= \exp\left\{\mathrm{i}(k_{\mathrm{real}} + \mathrm{i}k_{\mathrm{imag}})z\right\} \\ &= \exp(\mathrm{i}k_{\mathrm{real}}z)\exp(-k_{\mathrm{imag}}z). \end{aligned} \tag{1.39}$$

Hence, if the imaginary part of k is positive, the amplitude of the electromagnetic wave declines exponentially.†

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* We have here come face to face with a dispute that has raged between physicists and engineers for ages. For some odd reason the physicists (aided and abetted by mathematicians) use the symbol i for $\sqrt{-1}$ and the exponent $-\mathrm{i}(\omega t - kz)$ to describe a wave travelling in the z-direction. The engineers' notation is j for $\sqrt{-1}$ and $\mathrm{j}(\omega t - kz)$ for the exponent. In this course we have, rather reluctantly, accepted the physicists' notations so as not to confuse you further when reading books on quantum mechanics.

† The negative sign is also permissible though it does not give rise to an exponentially increasing wave as would follow from eqn (1.39). It would be very nice to make an amplifier by putting a piece of lossy material in the way of the electromagnetic wave. Unfortunately, it violates the principle of conservation of energy. Without some source of energy at its disposal no wave can grow. So the wave which seems to be exponentially growing is in effect a decaying wave which travels in the direction of the negative z-axis.

If the conductivity is large enough, the second term is the dominant one in eqn (1.38) and we may write

$$k \cong (\mathrm{i}\omega\mu\sigma)^{1/2} = \frac{\pm(\mathrm{i}+1)}{\sqrt{2}}(\omega\mu\sigma)^{1/2}. \tag{1.40}$$

So if we wish to know how rapidly an electromagnetic wave decays in a good conductor, we may find out from this expression. Since

$$k_{\mathrm{imag}} = \left(\frac{\omega\mu\sigma}{2}\right)^{1/2} \tag{1.41}$$

the amplitude of the electric field varies as

$$|\mathcal{E}_x| = \mathcal{E}_{x_0} \exp\left\{-\left(\frac{\omega\mu\sigma}{2}\right)^{1/2} z\right\}. \tag{1.42}$$

The distance δ at which the amplitude decays to $1/\mathrm{e}$ of its value at the surface is called the *skin depth* and may be obtained from the equation

$$1 = \left(\frac{\omega\mu\sigma}{2}\right)^{1/2} \delta, \tag{1.43}$$

yielding

$$\delta = \left(\frac{2}{\omega\mu\sigma}\right)^{1/2}. \tag{1.44}$$

You have seen this formula before. You need it often to work out the resistance of wires at high frequencies. I derived it solely to emphasize the major steps that are common to all these calculations.

We can now go further, and instead of taking the constant σ, we shall look a little more critically at the mechanism of conduction. We express the current density in terms of velocity by the equation

$$\mathbf{J} = N_{\mathrm{e}}e\mathbf{v}. \tag{1.45}$$

The symbol $\mathbf{v}$ still means the average velocity of electrons, but now it may be a function of space and time, whereas the notation v_{D} is generally restricted to d.c. phenomena.

This is really the same thing as eqn (1.7). The velocity of the electron is related to the electric and magnetic fields by the equation of motion

$$m\left(\frac{\mathrm{d}\mathbf{v}}{\mathrm{d}t} + \frac{\mathbf{v}}{\tau}\right) = e(\mathcal{E} + \mathbf{v} \times \mathbf{B}). \tag{1.46}$$

$1/\tau$ is introduced again as a 'viscous' or 'damping' term

We are looking for linearized solutions leading to waves. In that approximation the quadratic term $\mathbf{v} \times \mathbf{B}$ can be clearly neglected and the total derivative can be replaced by the partial derivative to yield

$$m\left(\frac{\partial\mathbf{v}}{\partial t} + \frac{\mathbf{v}}{\tau}\right) = e\mathcal{E}. \tag{1.47}$$

Assuming again that the electric field is in the x-direction, eqn (1.47) tells us that the electron velocity must be in the same direction. Using the rules set out in eqn (1.32) we get the following algebraic equation

$$m v_x \left(-\mathrm{i}\omega + \frac{1}{\tau}\right) = e\mathcal{E}_x. \tag{1.48}$$

The current density is then also in the x-direction:

$$\begin{aligned} J_x &= N_\mathrm{e} e v_x \\ &= \frac{N_\mathrm{e} e^2 \tau}{m} \frac{1}{1 - \mathrm{i}\omega\tau} \mathcal{E}_x \\ &= \frac{\sigma}{1 - \mathrm{i}\omega\tau} \mathcal{E}_x, \end{aligned} \tag{1.49}$$

where σ is defined as before. You may notice now that the only difference from our previous $(J - \mathcal{E})$ relationship is a factor $(1 - \mathrm{i}\omega\tau)$ in the denominator. Accordingly, the whole derivation leading to the expression of k in eqn (1.38) remains valid if σ is replaced by $\sigma/(1 - \mathrm{i}\omega\tau)$. We get

$$\begin{aligned} k &= \left(\omega^2 \mu\epsilon + \mathrm{i}\omega\mu \frac{\sigma}{1 - \mathrm{i}\omega\tau}\right)^{1/2} \\ &= \omega(\mu\epsilon)^{1/2} \left(1 + \frac{\mathrm{i}\sigma}{\omega\epsilon(1 - \mathrm{i}\omega\tau)}\right)^{1/2}. \end{aligned} \tag{1.50}$$

If $\omega\tau \ll 1$, we are back where we started from, but what happens when $\omega\tau \gg 1$? Could that happen at all? Yes, it can happen if the signal frequency is high enough or the collision time is long enough. Then, unity is negligible in comparison with $\mathrm{i}\omega\tau$ in eqn (1.50), leading to

$$k = \omega(\mu\epsilon)^{1/2} \left(1 - \frac{\sigma}{\omega^2 \epsilon\tau}\right)^{1/2}. \tag{1.51}$$

Introducing the new notation

$$\omega_\mathrm{p}^2 \equiv \frac{N_\mathrm{e} e^2}{m\epsilon} = \frac{(N_\mathrm{e} e^2/m)\tau}{\epsilon\tau} = \frac{\sigma}{\epsilon\tau} \tag{1.52}$$

we get

$$k = \omega(\mu\epsilon)^{1/2} \left(1 - \frac{\omega_\mathrm{p}^2}{\omega^2}\right)^{1/2}. \tag{1.53}$$

Hence, as long as $\omega > \omega_\mathrm{p}$, the wavenumber is real. If it is real, it has (by the rules of the game) no imaginary component; so the wave is not attenuated. This is quite interesting. By introducing a slight modification into our model, we may come to radically different conclusions. Assuming previously $J = \sigma\mathcal{E}$, we worked out that if any electrons are present at all, the wave is bound to decay. Now we are saying that for sufficiently large $\omega\tau$ an electromagnetic wave may travel across our conductor without attenuation. Is this possible? It seems to contradict the empirical fact that radio waves cannot penetrate metals. True; but that is because radio waves have not got high enough frequencies; let us try light waves. Can they penetrate a metal? No, they can not. It is another empirical fact that metals are not transparent. So we should try even higher frequencies. How high? Well, there is no need to go on guessing, we can work out the threshold frequency from eqn (1.52). Taking the electron density in a typical metal as 6×10^{28} per m^3, we then get

$$\begin{aligned} f_\text{p} &= \frac{1}{\pi}\left(\frac{N_\text{e}e^2}{m\epsilon_0}\right)^{1/2} \\ &= \frac{1}{2\pi}\left\{\frac{6 \times 10^{28}(1.6 \times 10^{-19})^2}{9.11 \times 10^{-31} \times 8.85 \times 10^{-12}}\right\}^{1/2} \\ &= 2.2 \times 10^{15}\,\text{Hz}. \end{aligned} \tag{1.54}$$

At this frequency range you are probably more familiar with the wavelengths of electromagnetic waves. Converting the above calculated frequency into wavelength, we get

$$\lambda = \frac{c}{f_\text{p}} = \frac{3 \times 10^8}{2.2 \times 10^{15}} = 136\,\text{nm}. \tag{1.55}$$

Thus, the threshold wavelength is well below the edge of the visible region (400 nm). It is gratifying to note that our theory is an agreement with our everyday experience; metals are not transparent.

There is one more thing we need to check. Is the condition $\omega\tau \gg 1$ satisfied? For a typical metal at room temperature, the value of τ is usually above 10^{-14} s, making $\omega\tau$ of the order of hundreds at the threshold frequency.

By making transmission experiments through a thin sheet of metal, the critical wavelength can be determined. The measured and calculated values are compared in Table 1.1. The agreement is not too bad, considering how simple our model is.

Before going further I would like to say a little about the relationship of transmission, reflection, and absorption to each other. The concepts are simple and one can always invoke the principle of conservation of energy if in trouble.

Let us take the case when $\omega\,\tau \gg 1$; k is given by eqn (1.54), and our conductor fills half the space, as shown in Fig. 1.4. What happens when an electromagnetic wave is incident from the left?

1. $\omega > \omega_\text{p}$. The electromagnetic wave propagates in the conducxtor. There is also some reflection, depending on the amount of mismatch. Energy

Table 1.1 *Threshold wavelengths for alkali metals*

Metal	Observed wavelength (nm)	Calculated wavelength (nm)
Cs	440	360
Rb	360	320
K	315	290
Na	210	210
Li	205	150

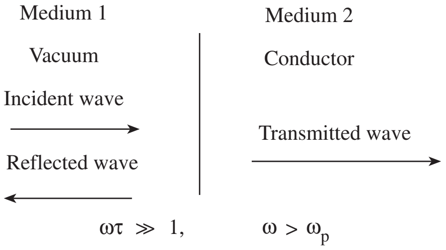

Fig. 1.4
Incident electromagnetic wave partly reflected and partly transmitted.

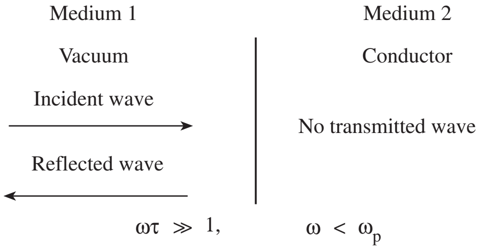

Fig. 1.5
Incident electromagnetic wave reflected by the conductor.

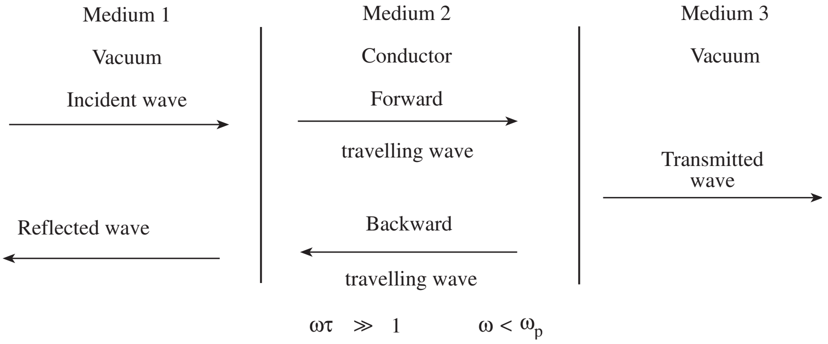

Fig. 1.6
Incident electromagnetic wave transmitted to medium 3. The amplitude of the wave decays in medium 2 but without any energy absorption taking place.

conservation says

$$\text{energy in the incident wave} = \text{energy in the transmitted wave} \\ + \text{energy in the reflected wave.}$$

Is there any absorption? No, because $\omega\tau \gg 1$.

2. $\omega < \omega_p$. In this case k is purely imaginary; the electromagnetic wave decays exponentially. Is there any absorption? No. Can the electromagnetic wave decay then? Yes, it can. Is this not in contradiction with something or other? The correct answer may be obtained by writing out the energy balance. Since the wave decays and the conductor is infinitely long, no energy goes out at the right-hand side. So everything must go back. The electromagnetic wave is reflected, as shown in Fig. 1.5. The energy balance is energy in the incident wave = energy in the reflected wave.

3. Let us take now the case shown in Fig. 1.6 when our conductor is of finite dimension in the z-direction. What happens now if $\omega < \omega_p$? The wave

now has a chance to get out at the other side, so there is a flow of energy, forwards and backwards, in the conductor. The wider the slab, the smaller is the amplitude of the wave that appears at the other side because the amplitude decays exponentially in the conductor. There is decay, but no absorption. The amplitudes of the reflected and transmitted waves rearrange themselves in such a way as to conserve energy.

If there is a smaller amplitude transmitted, there will be a larger amplitude reflected.

If we choose a frequency such that $\omega\tau \ll 1$, then, of course, dissipative processes do occur and some of the energy of the electromagnetic wave is converted into heat. The energy balance in the most general case is

$$\begin{aligned}\text{energy in the wave} &= \text{energy in the transmitted wave}\\ &\quad + \text{energy in the reflected wave}\\ &\quad + \text{energy absorbed.}\end{aligned}$$

A good example of the phenomena enumerated above is the reflection of radio waves from the ionosphere. The ionosphere is a layer which, as the name suggests, contains ions. There are free electrons and positively charged atoms, so our model should work. In a metal, atoms, and electrons are closely packed; in the ionosphere, the density is much smaller, so that the critical frequency ω_p is also smaller. Its value is a few hundred megahertz. Thus, radio waves below this frequency are reflected by the ionosphere (this is why short radio waves can be used for long-distance communication) and those above this frequency are transmitted into space (and so can be used for space or satellite communication). The width of the ionosphere also comes into consideration, but at the wavelengths used (it is the width in wavelengths that counts) it can well be regarded as infinitely wide.

1.6 Waves in the presence of an applied magnetic field: cyclotron resonance

In the presence of a constant magnetic field, the characteristics of electromagnetic waves will be modified, but the solution can be obtained by exactly the same technique as before. The electromagnetic eqns (1.22) and (1.23) are still valid for the a.c. quantities; the equation of motion should, however, contain the constant magnetic field, which we shall take in the positive z-direction. The applied magnetic field, $\mathbf{B}_0$, may be large, hence $\mathbf{v} \times \mathbf{B}_0$ is not negligible; it is a first-order quantity. Thus, the linearized equation of motion for this case is

$$m\left(\frac{\partial \mathbf{v}}{\partial t} + \frac{\mathbf{v}}{\tau}\right) = e(\mathscr{E} + \mathbf{v} \times \mathbf{B}_0). \tag{1.56}$$

In order to satisfy this vector equation, we need both the v_x and v_y components. That means that the current density, and through that the electric and magnetic fields, will also have both x and y components.

Writing down all the equations is a little lengthy, but the solution is not more difficult in principle. It may again be attempted in the exponential form, and $\partial/\partial z$ and $\partial/\partial t$ may again be replaced by $\mathrm{i}k$ and $-\mathrm{i}\omega$, respectively. All the differential equations are then converted into algebraic equations, and by making the determinant of the coefficients zero we get the dispersion equation. I shall not go through the detailed derivation here because it would take up a great deal of time, and the resulting dispersion equation is hardly more complicated than eqn (1.50). All that happens is that ω in the $\omega\tau$ term is replaced by $\omega \pm$

ω_c. Thus, the dispersion equation for transverse electromagnetic waves in the presence of a longitudinal d.c. magnetic field is

$$k = \omega(\mu\epsilon)^{1/2}\left(1 + \frac{\mathrm{i}\sigma}{\omega\epsilon\,\{1 - \mathrm{i}(\omega \pm \omega_c)\tau\}}\right)^{1/2}, \tag{1.57}$$

where

$$\omega_c = \frac{e}{m}B_0. \tag{1.58}$$

The plus and minus signs give circularly polarized electromagnetic waves rotating in opposite directions. To see more clearly what happens, let us split the expression under the square root into its real and imaginary parts. We get

$$k = \omega\sqrt{(\mu\epsilon)}\left(1 - \frac{\omega_p^2\tau^2(1 - \omega_c/\omega)}{1 + (\omega - \omega_c)^2\tau^2} + \mathrm{i}\frac{\omega_p^2\tau}{\omega}\frac{1}{1 + (\omega - \omega_c)^2\tau^2}\right)^{1/2}. \tag{1.59}$$

This looks a bit complicated. In order to get a simple analytical expression, let us confine our attention to semiconductors where ω_p is not too large and the applied magnetic field may be large enough to satisfy the conditions,

$$\omega_c \gg \omega_p \quad \text{and} \quad \omega_c\tau \gg 1. \tag{1.60}$$

We intend to investigate now what happens when ω_c is close to ω. The second and third terms in eqn (1.59) are then small in comparison with unity; so the square root may be expanded to give

$$k = \omega\sqrt{\mu\epsilon}\left(1 + \frac{\mathrm{i}}{2}\frac{\omega_p^2\tau}{\omega}\frac{1}{1 + (\omega - \omega_c)^2\tau^2}\right). \tag{1.61}$$

The attenuation of the electromagnetic wave is given by the imaginary part of k. It may be seen that it has a maximum when $\omega_c = \omega$. Since ω_c is called the cyclotron* frequency this resonant absorption of electromagnetic waves is known as *cyclotron resonance*. The sharpness of the resonance depends strongly on the value of $\omega_c\tau$, as shown in Fig. 1.7, where Im k, normalized to its value at $\omega/\omega_c = 1$, is plotted against ω/ω_c. It may be seen that the resonance is hardly noticeable at $\omega_c\tau = 1$.

The role of $\omega_c\tau$ is really analogous to that of Q in a resonant circuit. For good resonance we need a high value of $\omega_c\tau$.

The curves have been plotted using the approximate eqn (1.61); nevertheless the conclusions are roughly valid for any value of ω_p. If you want more accurate resonance curves, use eqn (1.59).

Why is there such a thing as cyclotron resonance? The calculation from the dispersion equation provides the figures, but if we want the reasons, we should look at the following physical picture.

Suppose that at a certain point in space the a.c. electric field is at right angles to the constant magnetic field, B_0. The electron that happens to be at that point will experience a force at right angles to B_0 and will move along the arc of a

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* After an accelerating device, the cyclotron, which works by accelerating particles in increasing radii in a fixed magnetic field.

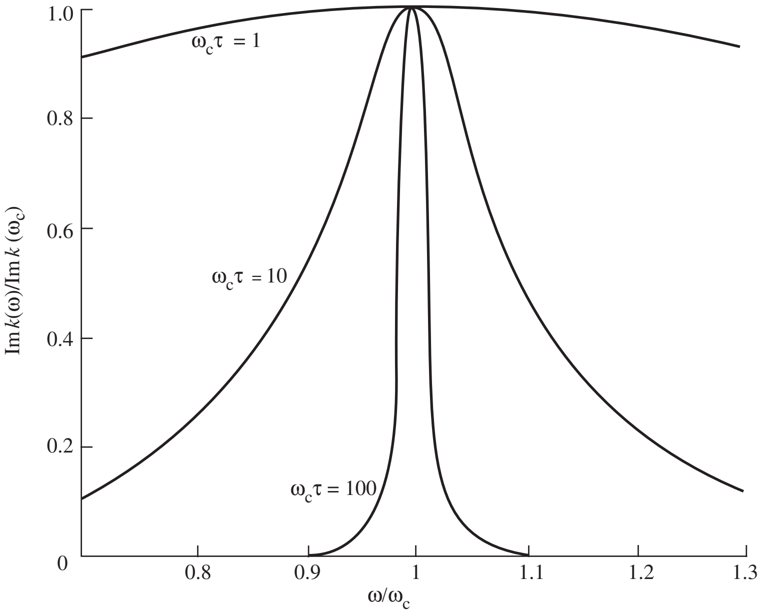

Fig. 1.7
Cyclotron resonance curves computed from eqn (1.61). There is maximum absorption when the frequency of the electromagnetic wave agrees with the cyclotron frequency.

circle. We can write a force equation. When the direction of motion is along the direction of $\mathcal{E}$ the magnetic and centrifugal forces are both at right angles to it, thus

$$B_0 e v = \frac{m v^2}{r}. \tag{1.62}$$

r is the instantaneous radius of curvature of the electron's path.

Consequently, the electron will move with an angular velocity

$$\omega_c = \frac{v}{r} = \frac{e}{m} B_0. \tag{1.63}$$

The orbits will *not* be circles, for superimposed on this motion is an acceleration varying with time in the direction of the electric field. Now if the frequency of the electric field, ω, and the cyclotron frequency, ω_c, are equal, the amplitude of the oscillation builds up. An electron that is accelerated north in one half-cycle will be ready to go south when the electric field reverses, and thus its speed will increase again. Under resonance conditions, the electron will take up energy from the electric field; and that is what causes the attenuation of the wave. Why is the $\omega_c \tau > 1$ condition necessary? Well, τ is the collision time; $\tau = 1/\omega_c$ means that the electron collides with a lattice atom after going round one radian. Clearly, if the electron is exposed to the electric field for a considerably shorter time than a cycle, not much absorption can take place. The limit might be expected when the electron covers one radian of a circle between collisions. This gives $\omega_c \tau = 1$.

Notice that any increase in *speed* must come from the electric field; the acceleration produced by a magnetic field changes direction, not speed, since the force is always at right angles to the direction of motion.

Now we may again ask the question: what is cyclotron resonance good for? There have been suggestions for making amplifiers and oscillators with the aid of cyclotron resonance, where by clever means the sign of attenuation is reversed, turning it into gain. As far as I know none of these devices reached the ultimate glory of commercial exploitation. If cyclotron resonance is no good for devices, is it good for something else? Yes, it is an excellent measurement tool.

It is used as follows: we take a sample, put it in a waveguide and launch an electromagnetic wave of frequency, ω. Then we apply a magnetic field and measure the output electromagnetic wave while the strength of magnetic field is varied. When the output is a minimum, the condition of cyclotron resonance is satisfied. We know ω so we know ω_c; we know the value of the magnetic field, B_0 so we can work out the mass of the electron from the formula

$$m = \frac{eB_0}{\omega}. \tag{1.64}$$

But, you would say, what is the point in working out the mass of the electron? That's a fundamental constant, isn't it? Well, it is, but not in the present context. When we put our electron in a crystal lattice, its mass will appear to be different. The actual* value can be measured directly with the aid of cyclotron resonance. So once more, under the pressure of experimental results we have to modify our model. The bouncing billiard balls have variable mass. Luckily, the charge of the electron does remain a fundamental constant. We must be grateful for small favours.

The charge of the electron is a fundamental constant; the mass of an electron is not.

1.7 Plasma waves

Electromagnetic waves are not the only type of waves that can propagate in a solid. There are sound waves and plasma waves as well. We know about sound waves; but what are plasma waves? In their simplest form they are density waves of charged particles in an electrically neutral medium. So they exist in a solid that has some mobile carriers. The main difference between this case and the previously considered electromagnetic case is that now we permit the accumulation of space charge. At a certain point in space, the local density of electrons may exceed the local density of positive carriers. Then an electric field arises, owing to the repulsive forces between these 'unneutralized' electrons. The electric field tries to restore the equilibrium of positive and negative charges. It drives the electrons away from the regions where they accumulated. The result is, of course, that the electrons overshoot the mark, and some time later, there will be a deficiency of electrons in the same region. An opposite electric field is then created which tries to draw back the electrons, etc. This is the usual case of harmonic oscillation. Thus, as far as an individual electron is concerned, it performs simple harmonic motion.

If we consider a one-dimensional model again, where everything is the same in the transverse plane, then the resulting electric field has a longitudinal component only. A glance at eqn (1.26), where $\nabla \times \mathcal{E}$ is worked out, will convince you that if the electric field has a z-component, only then $\nabla \times \mathcal{E} = 0$, that is $\mathbf{B} = 0$. There is no magnetic field present; the interplay is solely between the charges and the electric field. For this reason these density waves are often referred to as electrostatic waves.

If $\mathbf{B} = 0$, then eqn (1.22) takes the simple form

$$\mathbf{J} + \epsilon \frac{\partial \mathcal{E}}{\partial t} = 0. \tag{1.65}$$

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* The actual value is called, quite reasonably, the effective mass.

We need the equation of motion, which for longitudinal motion will have exactly the same form as for transverse motion, namely

$$m\frac{\partial \mathbf{v}}{\partial t} = e\mathscr{E}, \tag{1.66}$$

where we have neglected the damping term $m\mathbf{v}/\tau$.*

Current density and velocity are related again by

$$J = N_{\mathrm{e}_0}ev. \tag{1.67}$$

N_{e_0} is the equilibrium density of electrons.

We have changed over to scalar quantities.

Finally, there is the equation giving the relationship between the local density of electrons and the electric field, Poisson's equation,

$$\frac{\partial \mathscr{E}}{\partial z} = \frac{1}{\epsilon}N_{\mathrm{e}}e. \tag{1.68}$$

Substituting $\mathscr{E}$ from eqn (1.66) and J from eqn (1.67) into (1.65), we obtain

$$N_{\mathrm{e}_0}ev + \frac{\epsilon m}{e}\frac{\partial^2 v}{\partial t^2} = 0. \tag{1.69}$$

Following again our favourite method of replacing $\partial/\partial t$ by $-\mathrm{i}\omega$, eqn (1.69) reduces to

$$v\left\{N_{\mathrm{e}_0}e + \frac{\epsilon m}{e}(-\omega^2)\right\} = 0. \tag{1.70}$$

Since v must be finite, this means

$$N_{\mathrm{e}_0}e - \frac{\epsilon m}{e}\omega^2 = 0, \tag{1.71}$$

or, rearranging,

$$\omega^2 = \frac{N_{\mathrm{e}_0}e^2}{m\epsilon}. \tag{1.72}$$

This is our dispersion equation. It is a rather odd one because k does not appear in it. A relationship between k and ω gives the allowed values of k for a given ω. If k does not appear in the dispersion equation, *all* values of k are allowed. On the other hand, there is only a single value of ω allowed. Looking at it more carefully, we may recognize that it is nothing else but ω_{p}, the frequency we met previously as the critical frequency of transparency for electromagnetic waves. Historically, it was first discovered in plasma oscillations (in gas discharges by Langmuir); so it is more usual to call it the 'plasma frequency', and that is where the subscript p comes from.

Summarizing, a lossless plasma wave may have any wavelength and may propagate at any velocity, but only at one single frequency, the plasma frequency.

What are plasmas good for? This is a very difficult question to answer. We have touched only the simplest case. Plasma physics covers a vast field with immense potentialities (e.g. fusion), but the number of devices based on some sort of charge imbalance is very limited.† This is because plasma waves are generally harmful. It is more the concern of physicists and engineers to *get rid* of plasma waves than to set them up.

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[*] Ignoring losses will considerably restrict the applicability of the formulae derived, but our aim here is to show no more than the simplest possible case.

[†] Microwave tubes belong to a special category. They do employ density waves but the electrons' space charge is not compensated.

1.8 Heat

When the aim is to unravel the electrical properties of materials, should we take a detour and discuss heat? In general, no, we should not do that but when the two subjects overlap a little digression is permissible. I want to talk here first about the relationship between the electrical conductivity and heat conductivity, and then point out some discrepancies suggesting that something is seriously wrong with our model.

We have already discussed electrical conductivity. Heat conductivity is the same kind of thing but involves heat. An easy but rather unpleasant way of learning about it is to touch a piece of metal in freezing weather. The heat from your finger is immediately conducted away and you may get frostbite. Now back to that relationship. Denoting heat conductivity by K, it was claimed around the middle of the nineteenth century that for metals

$$\frac{K}{\sigma} = C_{\mathrm{WF}} T \tag{1.73}$$

where C_{WF}, the so-called Wiedemann–Franz constant, was empirically derived. It was taken as

$$C_{\mathrm{WF}} = 2.31 \times 10^{-8}\,\mathrm{W\,S^{-1}K^{-2}}. \tag{1.74}$$

How well is the Wiedemann–Franz law satisfied? Very well, as Table 1.2 shows. Can it be derived from our model in which our electrons bounce about in the solid? Yes, that is what Drude did in about 1900. Let us follow what he did.

At equilibrium, the average energy of an electron (eqn 1.1) is $E = \frac{3}{2}kT$. The specific heat C_V is defined as the change in the average energy per unit volume with temperature

$$C_V = N_{\mathrm{e}} \frac{\mathrm{d}E}{\mathrm{d}T} = N_{\mathrm{e}} \left(\frac{3}{2}\right) k. \tag{1.75}$$

Let us now consider heat flow, assuming that all the heat is carried by the electrons. We shall take a one-dimensional model in which the electrons move only in the x direction. If there is a heat flow the average energy may change slightly from point to point. Taking an interval from $x - \ell$ to $x + \ell$ (remember ℓ

Table 1.2 *Electrical and thermal conductivities measured at 293 K*

Metal	σ ($10^7\,\Omega^{-1}\,\mathrm{m}^{-1}$)	K ($\mathrm{W\,m^{-1}\,K^{-1}}$)	C_{WF} ($10^{-8}\,\mathrm{W\,\Omega\,K^{-2}}$)
Silver	6.15	423	2.45
Copper	5.82	387	2.37
Aluminium	3.55	210	2.02
Sodium	2.10	135	2.18
Cadmium	1.30	102	2.64
Iron	1.00	67	2.31

is the mean free path) the average at the two boundaries will be $E - (\mathrm{d}E/\mathrm{d}x)\ell$ and $E + (\mathrm{d}E/\mathrm{d}x)\ell$ respectively. Referring now to a result from the kinetic theory of gases that the number of particles flowing in a given direction per unit surface per unit time is $\frac{1}{6}N_\mathrm{e}v_\mathrm{th}$, the net flow across the plane at x is

$$\text{net energy flow} = (1/3)N_\mathrm{e}v_\mathrm{th}\left(\frac{\mathrm{d}E}{\mathrm{d}x}\right)\ell. \tag{1.76}$$

According to the simple theory of heat, the flow of heat energy is proportional to the gradient of temperature where the proportionality constant is the heat conductivity, K, yielding

$$\text{net heat energy flow} = K\left(\frac{\mathrm{d}T}{\mathrm{d}x}\right). \tag{1.77}$$

Equating now eqn (1.76) with (1.77) we obtain

$$K = \frac{1}{2}N_\mathrm{e}v_\mathrm{th}\ell k. \tag{1.78}$$

We may now relate the heat conductivity to the electrical conductivity as follows

$$\frac{K}{\sigma} = \frac{\frac{1}{2}N_\mathrm{e}V_\mathrm{th}\ell k}{N_\mathrm{e}(e^2\tau)/m} = \frac{3}{2}\left(\frac{k}{e}\right)^2 T \tag{1.79}$$

where the relation $\ell = v_\mathrm{th}\tau$ (neglect v_D in eqn (1.11)) has been used. The functional relationship is exactly the same: the ratio of the two conductivities is indeed proportional to T as was stipulated by the empirical formula. But what is the value of the constant? Inserting the values of e and k into eqn (7.79), we obtain for the Wiedemann–Franz constant a value of $1.22 \times 10^{-8}\,\mathrm{W\,S^{-1}\,K^{-2}}$, about a factor 2 smaller than the experimental value. This was regarded at the time as extremely good and as a justification of the electron as a particle model. Well, the factor of 2 was bothersome and the separate variation of the two conductivities with temperature did not fit very well either. All that was perhaps acceptable, but a closer look at the specific heat of metals versus insulators revealed that something was seriously wrong.

Up to now we have talked only about the electronic contribution to the specific heat and quoted it as being $\frac{3}{2}\mathrm{N_e k}$, but classically the lattice will also contribute a term* $3Nk$ where N is the density of atoms. Thus, we should expect an alkali metal (in which $N_\mathrm{e} = N$) to have a 50% greater specific heat than an insulator having the same number of lattice atoms because of the electronic contribution. These expectations are, however, wrong. It turns out that metals and insulators have about the same specific heat. Our model fails again to explain the experimentally observed value. What shall we do? Modify our model. But how? Up to now the modifications have been fairly obvious. The 'wrong sign' of the Hall voltage could be explained by introducing positive carriers, and when cyclotron resonance measurements showed that the mass of an electron in a solid was different from the 'free' electron mass, we simply said: 'all right, the electron's mass is not a constant. How should we modify our model now?' There seems to be no simple way of doing so. An entirely new start is needed.

Metals behave as if the free electrons make practically no contribution to the specific heat.

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* Valid at room temperature but fails at low temperatures.

There is no quick fix for this real dilemma. We have to go quite deeply into wave theory and quantum mechanics. Finally, all is revealed in Chapter 6.3 when we find that electrons do make a quantifiable contribution to specific heat, which turns out to be very small.

Exercises

1.1. A 10 mm cube of germanium passes a current of 6.4 mA when 10 m V is applied between two of its parallel faces. Assuming that the charge carriers are electrons that have a mobility of 0.39 $\mathrm{m^2V^{-1}\,s^{-1}}$, calculate the density of carriers. What is their collision time if the electron's effective mass in germanium is $0.12\,m_0$ where m_0 is the free electron mass?

1.2. An electromagnetic wave of free space wavelength 0.5 mm propagates through a piece of indium antimonide that is placed in an axial magnetic field. There is resonant absorption of the electromagnetic wave at a magnetic field, $B = 0.323$ wb $\mathrm{m^{-2}}$.

(i) What is the effective mass of the particle in question?
(ii) Assume that the collision time is 15 times longer (true for electrons around liquid nitrogen temperatures) than in germanium in the previous example. Calculate the mobility.
(iii) Is the resonance sharp? What is your criterion?

1.3. If both electrons and holes are present the conductivities, add. This is because under the effect of an applied electric field the holes and electrons flow in opposite directions, and a negative charge moving in the (say) $+z$-direction is equivalent to a positive charge moving in the $-z$-direction.

Assume that in a certain semiconductor the ratio of electronic mobility, μ_e, to hole mobility, μ_h, is equal to 10, the density of holes is $N_h = 10^{20}\mathrm{m^{-3}}$, and the density of electrons is $N_e = 10^{19}\mathrm{m^{-3}}$. The measured conductivity is 0.455 $\mathrm{ohm^{-1}m^{-1}}$. Calculate the mobilities.

1.4. Measurements on sodium have provided the following data: resistivity 4.7×10^{-8} ohm m, Hall coefficient $-2.5 \times 10^{-10}\,\mathrm{m^3\,C^{-1}}$, critical wavelength of transparency 210 nm, and density 971 kg $\mathrm{m^{-3}}$.

Calculate (i) the density of electrons, (ii) the mobility, (iii) the effective mass, (iv) the collision time, (v) the number of electrons per atom available for conduction.

Electric conduction in sodium is caused by electrons. The number of atoms in a kg mole is 6.02×10^{26} and the atomic weight of sodium is 23.

1.5. For an electromagnetic wave propagating in sodium plot the real and imaginary part of the wave number k as a function of frequency (use a logarithmic scale) from 10^6 to 10^{16} Hz.

Determine the penetration depth for 10^6, 10^{15}, and 2×10^{15} Hz.

Use the conductivity and the collision time as obtained from example 1.4.

1.6. A cuboid of Ge has contacts over all of its 2 mm × 1 mm ends and point contacts approximately half way along its 5 mm length, at the centre of the 5 mm × 1 mm faces. A magnetic field can be applied parallel to this face. A current of 5 mA is passed between the end contacts when a voltage of 310 mV is applied. This generates a voltage across the point contacts of 3.2 mV with no magnetic field and 8.0 mV when a field of 0.16 T is applied.

(i) Suggest why an apparent Hall voltage is observed with no magnetic field.
(ii) Using the corrected Hall voltage find the carrier density in the Ge sample.
(iii) Estimate the conductivity of the Ge.
(iv) What is the mobility of the carriers?
(v) Is it a p or n type semiconductor?

1.7. The Hall effect (see Fig. 1.3) is measured in a semiconductor sample in which both electrons and holes are present. Under the effect of the magnetic field both carriers are deflected in the same transverse direction. Obviously, no electric field can stop simultaneously both the electrons and the holes, hence whatever the Hall voltage there will always be carrier motion in the transverse direction. Does this mean that there will be an indefinite accumulation of electrons and holes on the surface of one of the boundaries? If not, why not?

1.8. Derive an expression for the Hall coefficient R_H [still defined by eqn (1.20)] when both electrons and holes are present.

The experimentally determined Hall coefficient is found to be negative. Can you conclude that electrons are the dominant charge carriers?

(Hint: Write down the equation of motion (neglect inertia) for both holes and electrons in vectorial form. Resolve the equations in the longitudinal (z-axis in Fig. 1.3) and in the transverse (x-axis in Fig. 1.3) directions. Neglect the product of transverse velocity with the magnetic field. Find the transverse velocities for electrons and holes. Find the transverse current, and finally find the transverse field from the condition that the transverse current is zero.)

1.9. An electromagnetic wave is incident from Medium 1 upon Medium 2 as shown in Figs 1.4 and 1.5. Derive expressions for the reflected and transmitted power. Show that the transmitted electromagnetic power is finite when $\omega > \omega_\mathrm{p}$ and zero when $\omega < \omega_\mathrm{p}$.

[Hint: Solve Maxwell's equations separately in both media. Determine the constants by matching the electric and magnetic fields at the boundary. The power in the wave (per unit surface) is given by the Poynting vector.]

1.10. An electromagnetic wave is incident upon a medium of width d, as shown in Fig. 1.6. Derive expressions for the reflected and transmitted power. Calculate the transmitted power for the cases $d = 0.25\,\mu\mathrm{m}$ and $d = 2.5\,\mu\mathrm{m}$ when $\omega = 6.28 \times 10^{15}\,\mathrm{rad\,s^{-1}}, \omega_\mathrm{p} = 9 \times 10^{15}\,\mathrm{rad\,s^{-1}}$ (take $\epsilon = \epsilon_0$ and $\mu = \mu_0$).

1.11. In a medium containing free charges the total current density may be written as $\mathbf{J}_\mathrm{total} = \mathbf{J} - \mathrm{i}\omega\epsilon\mathcal{E}$, where $\mathbf{J}$ is the particle current density, $\mathcal{E}$ electric field, ϵ dielectric constant, and ω frequency of excitation. For convenience, the above expression is often written in the form $\mathbf{J}_\mathrm{total} = -\mathrm{i}\omega\bar{\bar{\epsilon}}_\mathrm{eqv}\mathcal{E}$, defining thereby an equivalent dielectric tensor $\bar{\bar{\epsilon}}_\mathrm{eqv}$. Determine $\bar{\bar{\epsilon}}_\mathrm{eqv}$ for a fully ionized electron-ion plasma to which a constant magnetic field B_0 is applied in the z-direction.