Exercise WS15.1

1. Show that the function $z = 30x + 50y - 3y^2 - 3x^2 - 4xy + 200$ has a single stationary point (SP) at $x = -1,\ y = 9$ and that this SP is a maximum.

2. Show that $z = 30.005x^2 - xy + 50y^2 - 0.25x^3$ has a minimum at x = y = 0 and a saddle point at x = 80, y = 0.8.

3. Find the stationary points of the following functions, and determine whether each is a maximum, minimum or saddle point.

   (a) $z = 3x^2 + 2y^2 - 15x + 12y + 100$

   (b) $z = -x^2 + 2.5y^2 + 10xy - 16x - 30y$

Exercise WS15.2

1. Find the total differential (dz) of the following functions:

   (a) $z = (x+1)^2 + (y^2 - 2)^3$

   (b) $z = x^2 + y^3 + \dfrac{2x}{3y^2}$

   (c) $z = \dfrac{x^2 + 2y}{x^3 - y^2}$

   (d) $z = x^\alpha \left(\dfrac{y}{x}\right)^{1-\alpha}$ (where α is a parameter)

2. I want to lay a concrete path in my garden which I calculate will require $3\frac{1}{3}$ cubic yards of concrete. However, when I telephone a local supplier of ready-mixed concrete he tells me that he supplies concrete only by the cubic metre. (Note: 1 yard = 36 inches; assume 1 metre = 39 inches). Using the total differential:

   (a) Calculate how much concrete will be left over if I order $3\frac{1}{3}$ cubic metres.

   (b) Whether, if I order 3 cubic metres, this will be enough.

3. (a) If $z = x^2 + 3y$, suppose x increases from 10 to 10.1, and y increases from 5 to 5.1. What percentage error occurs if we use the total differential, dz, to calculate the resulting change in z?

   (b) If $z = x^{\frac{1}{3}} y^{\frac{2}{3}}$, suppose initially x = y = 1000. Then x increases by 5% and y increases by 10%. What percentage error occurs if we use the total differential, dz, to calculate the resulting change in z?

Exercise WS15.3

1. In each of the following, use the differential to find the specified derivative. Then check your answer by direction substitution.

   (a) $z = 2x^3 + xy$, where $x = y^2 + 1$; find $\frac{dz}{dy}$.

   (b) $z = \frac{x^2}{2u - 1}$, where $u = (1 - x)^2$; find $\frac{dz}{dx}$.

2. Use the function of a function rule to find the total differential, dz, where $z = u^{\frac{1}{2}}$ and $u = \frac{y}{x}$.

3. Use the differential (dz) to find the derivative ($\frac{dy}{dx}$) of each of the following implicit functions:

   (a) $x^2 + 2xy^2 + y^3 = 0$

   (b) $0 = (x^2 + y^3)^{\frac{1}{3}}$

   (c) $100 - x^{\frac{1}{2}} y^{\frac{3}{4}} = 0$

   (d) $\frac{2x + 1}{x^2 + y^3} = 0$

4. For each of the functions in question 1 above, use the differential to find the slope of an iso-z section.

Exercise WS15.4

1. A firm's production function is $Q = K^{0.5}L^{0.5}$

   (a) Find the marginal products of capital and labour. Are they always positive? Sketch their graphs.

   (b) Find the equations of the isoquants for (i) 10 units, and (ii) 15 units of output. In each case, express the isoquant both as an implicit function and also with K as an explicit function of L.

   (c) By implicit differentiation, find the slope of any isoquant and show that this slope is given by the ratio of the marginal products of capital and labour. Is the slope always negative?

   (d) Use the information from (b) and (c) above to sketch the isoquants.

   (e) Suppose a firm is producing on the Q = 10 isoquant using L_0 units of labour and K_0 units of capital. Then it decides to employ one more unit of labour while keeping output constant. Use the differential, dQ, to calculate the required reduction in the capital input. Compare this with the reduction calculated from the production function itself, if $K_0 = L_0 = 10$. Explain with the aid of a diagram why the two answers differ.

2. A firm's production function is $Q = 15KL - 4K^2 - 5L^2 + 6K + 4L$

   (a) Find the marginal products of capital and labour. Are they always positive? Sketch their graphs.

   (b) Use the differential of the production function to find the slope of any isoquant and show that this slope is given by the ratio of the marginal products of capital and labour. Is the slope always negative?

   (c) Use the information from (a) and (b) above to sketch some typical isoquants. Is their shape plausible?

   (d) Write down the equation of a typical short run production function; say, for $K = 10$. What determines its slope? What is true at its maximum value? Sketch its graph.

3. Consider the utility function $U = 10XY - 3X^2 - 2Y^2 + 40X + 50Y$

   (a) Find the marginal utilities of the goods *X* and *Y*. Are they always positive? Sketch their graphs.

   (b) By implicit differentiation, find the slope of any indifference curve and show that it is given by the ratio of marginal utilities of the two goods. Are the indifference curves always negatively sloped? Sketch their graphs.

4. Suppose there are two individuals, Ann and Bernard. Ann's utility function is $U_{\text{A}} = XY$, while Bernard's is $U_{\text{B}} = \ln(XY)$.

   (a) Find the marginal utilities of the two goods for Ann and Bernard, and sketch them. What does this tell us about their tastes?

   (b) Show by implicit differentiation that, for any given combination of *X* and *Y*, Ann's indifference curve has the same slope as Bernard's. What does this imply about their tastes?