WebAppendix 8: Consumption, Investment, and Tobin's q

Using figures and economic reasoning, the textbook showed how households consume and firms invest optimally. This appendix provides the corresponding mathematical treatment. It retains the two-period simplification. For a multiperiod perspective, the reader is referred to more advanced textbooks, (e.g. Romer (2005) or Heijdra and van der Ploeg (2002)).

A8.1. Consumption

Figure 8.2 shows indifference curves along which the consumer has a constant level of utility. Formally, this is represented by a utility function defined over consumption in both periods: $U(C_1, C_2)$. A simple way of describing the choice between consumption today and consumption tomorrow is to consider that tastes remain identical over time, so that the utility of consuming C_1 in the first period is measured in the same way as the utility of consuming C_2 in the second period. This means that we use the same one-period utility function $u(.)$ to measure each period's consumption. But we also recognize that people are impatient so, seen from today's perspective, the same quantity of consumption tomorrow yields less utility than it does today. Total utility is given by:

(A8.1)
$$U(C_1, C_2) = u(C_1) + \frac{u(C_2)}{1+\rho},$$

where ρ, which discounts tomorrow's utility, is called the subjective rate of time preference. We also assume $u' > 0$, i.e. more consumption raises utility, $u'' < 0$, i.e. this gain in utility declines for higher levels of consumption (u' is called marginal utility and $u'' < 0$ asserts that marginal utility is declining in the level of consumption).

Denote exogenous income in period t by Y_t and total wealth by $\Omega = Y_1 + Y_2/(1 + r)$, so the budget constraint is:

(A8.2)
$$C_1 + \frac{C_2}{1+r} = \Omega.$$

Substituting C_1 from (A8.2) into (A8.1), we get $U(\frac{C_2}{1+r}, C_2)$. We maximize this function with respect to C_2 and, using (A8.2), we obtain the first-order condition

(A8.3)
$$(1+\rho)\frac{u'(C_1)}{u'(C_2)} = 1 + r.$$

The left hand-side of (A8.3) measures the household's marginal rate of substitution of goods in period 2 for goods in period 1. The optimality condition sets this marginal rate of substitution equal to the interest factor $1 + r$, which measures the return from saving and can be thought of as the economic rate of transformation).

Equation (A8.3) is a central result.[1] Since $u'' < 0$, u' is a decreasing function of C. If r rises, $u'(C_1)$ must increase relatively to $u'(C_2)$, so C_1 must fall relatively to C_2. Intuitively, a higher r means that saving is more rewarding – we say that the price of period 2 consumption falls relative to that of period 1 consumption, $1/(1 + r)$ measures this relative price. Similarly, an increase in impatience (ρ increases) produces the opposite effect: the utility cost of waiting rises so C_1 is increased relatively to C_2. An interesting implication of (A8.3) is that when $r = \rho$, consumption is constant $C_1 = C_2$. This result is called consumption-smoothing. It occurs because $r = \rho$ means that saving, i.e. the return to postponing consumption (measured by the real interest rate) is exactly compensated by impatience.

Note that (A8.3) only tells us how C_1 and C_2 relate to each other, not their absolute levels. In order to derive these levels, we need to use the budget constraint (A8.2). The result is tied to the particular form of the utility function $u(.)$, which we have left general without any particular functional form. Rather, we ask what is the effect on consumption today of marginal changes in wealth and the interest rate. To do so, we substitute (A8.2) in (A8.3) and differentiate (A8.3) to find:

(A8.4)
$$[(1+\rho)u''(C_1) + (1+r)^2 u''(C_2)]dC_1 = (1+r)^2 u''(C_2)d\Omega + [u'(C_2) + (1+r)(\Omega - C_1)u''(C_2)]dr,$$

In order to see the effect of a change $d\Omega$ in wealth, we set $dr = 0$ in (A8.4) and solve for $dC_1/d\Omega$:

(A8.5)
$$\left.\frac{dC_1}{d\Omega}\right|_{dr=0} = \frac{(1+r)^2 u''(C_2)}{(1+\rho)u''(C_1)+(1+r)^2 u''(C_2)},$$

which is unambiguously positive (remember: $u'' < 0$). A corresponding expression can be derived for C_2, which also rises. This is the result shown in Fig. 8.5.

In a similar way, the effect of an increase in the interest rate on C_1 can be derived from (A8.4), setting instead $d\Omega=0$:

---

[1] It is called the Euler equation. Leonhard Euler was a Swiss mathematician, born in 1707 in Basel and died in St Petersburg in 1783.

(A8.6) $$\left.\frac{dC_1}{dr}\right|_{d\Omega=0} = \frac{u'(C_2)+(1+r)(\Omega-C_1)u''(C_2)}{(1+\rho)u''(C_1)+(1+r)^2u''(C_2)}.$$

Unlike (A8.5), expression (A8.6), cannot be signed unambiguously. One can however rewrite the right-hand side of (A8.6) as:

(A8.7) $$\frac{u'(C_2)}{(1+\rho)u''(C_1)+(1+r)^2u''(C_2)}+\frac{(1+r)(\Omega-C_1)u''(C_2)}{(1+\rho)u''(C_1)+(1+r)^2u''(C_2)}.$$

Multiply numerator and denominator of the first term by $(1+r)^2u''(C_2)$ and use (A8.5) to rewrite (A8.7) as:

(A8.8) $$\frac{u'(C_2)}{(1+r)^2u''(C_2)}\frac{dC_1}{d\Omega}+\frac{(\Omega-C_1)}{(1+r)}\frac{dC_1}{d\Omega}.$$

Finally, define the elasticity of intertemporal substitution as[2]

(A8.9) $$\sigma=\frac{-u'(C_2)}{u''(C_2)C_2}.$$

Using this definition, rewrite (A8.8) as

(A8.10) $$\left.\frac{dC_1}{dr}\right|_{d\Omega=0}=\left[\frac{(\Omega-C_1)}{(1+r)}-\frac{\sigma C_2}{(1+r)^2}\right]\frac{dC_1}{d\Omega}=\frac{1-\sigma}{1+r}(\Omega-C_1)\frac{dC_1}{d\Omega}$$

The budget constraint imposes that $C_1 \leq \Omega$, $dC_1/dr\,|_{d\Omega=0} < 0$ for $\sigma > 1$, and $dC_1/dr|_{d\Omega=0} > 0$ for $\sigma < 1$. Holding total wealth Ω constant, the effect of an interest rate increase depends on the elasticity of substitution; it is negative for a high elasticity of substitution ($\sigma > 1$) and positive for a low value ($\sigma < 1$).[3] This corresponds to the discussion in Section 8.2.4 in the main text. On the other hand, the overall effect on consumption consists of (A8.10) plus the wealth effect of an interest rate increase, which is always negative (since $d\Omega/dr < 0$); so

(A8.11) $$\frac{dC_1}{dr}=\left.\frac{dC_1}{dr}\right|_{d\Omega=0}+\frac{dC_1}{d\Omega}\frac{d\Omega}{dr}.$$

---

[2] The parameter σ is the inverse of the elasticity of marginal utility of consumption $-[u''(C_2)C_2]/u'(C_2)$.

[3] In the case of unit elasticity—the case of log utility, or $U(C) = \log(C)$—the net effect is zero.

(?) (−)

A4.1 Investment without adjustment costs

In order to study optimal investment, we proceed in two steps. First, we ignore adjustment costs. We assume that the household owns the firm and that there is no capital to begin with (in the metaphor of Robinson Crusoe, there are no trees on the island). The key decision is: how much investment today (I_1) (coconuts to plant). This will give rise to productive capital tomorrow $K_2 = I_1$. If Y_i continues to represent period i endowment (labor income, for example), the budget constraint is:

(A8.12) $$C_1 + \frac{C_1}{1+r} = Y_1 + \frac{Y_2}{1+r} + \frac{F(K_2)}{1+r} - I_1 .$$

The household chooses I_1 in order to maximize wealth, i.e. the right-hand side of (A8.12). The first-order condition[4] for investment is:

(A8.13) $$\frac{F'(K_2)}{1+r} = 1 ,$$

which states that the marginal product of capital $F'(K_2)$ is equal to the interest rate factor $(1 + r)$.[5]

A4.2 Investment with previous capital and adjustment costs: Tobin's q

Now we consider the case where there exist adjustment costs, providing a formalization of the arguments behind Fig. 8.7. We also allow for some initial capital K_1 and we recognize that capital depreciates over time so $K_2 = I_1 + (1-\delta)K_1$ where δ is the rate of depreciation.

Adjustment costs arise because the act of investment itself is costly, above and beyond the fact that capital has to be financed out of savings. To formalize this idea,

---

[4] The second-order condition $F''(K_2)/(1 + r) < 0$ is assumed to be satisfied.

[5] The standard microeconomic result is $F'(K) = r$. Why the difference? Remember that the second period is the "end of the world", so that there is no salvage value for the capital stock. The general result is $F'(K) = r + \delta$, where δ is the rate of capital depreciation; the marginal product must be sufficiently great to compensate for the cost r and depreciation δ of invested capital. Here the end-of-the-world assumption means $\delta = 1$. See Box 8.6 for more discussion of this point.

let the installation costs of capital that $\Phi' > 0$, $\Phi'' > 0$ (installation costs are convex, meaning they increase more than proportionately to the level of investment, see Fig. 8.7). Further assume that $\Phi(0) = \Phi'(0) = 0$ (no costs are incurred when no investment is undertaken). Now the budget constraint for the household which undertakes the investment is:

(A8.14) $$C_1 + \frac{C_1}{1+r} = Y_1 + \frac{Y_2}{1+r} + \frac{F\left(K_1(1-\delta)+I_1\right)}{1+r} - I_1 - I_1\Phi\left[\frac{I_1}{K_1}\right].$$

Total wealth is equal to the present discounted value of income ($Y_1 + Y_2/(1+r)$ plus the value of the firm. The firm's value is the return from sales of production $-F(K_1)$ this period and $F(K_2)$ next period, the latter has to be discounted – less the cost of investment, direct and indirect.

The household now chooses I_1 in order to maximize its wealth, i.e. the right-hand side of (A8.14).[6] . The first-order condition[7] for investment is:

(A8.15) $$\frac{F'\left(K_2\right)}{1+r} = 1 + \Phi\left[\frac{I_1}{K_1}\right] + \left[\frac{I_1}{K_1}\right]\Phi'\left[\frac{I_1}{K_1}\right].$$

With the interpretation that the present discounted value of the marginal product of capital at the optimum (the marginal product of investment) is equal to the marginal cost of investment. Note that the right hand of (A8.15) is strictly greater than in (A8.13) as long as $I_1>0$. Since F" < 0, it means that the optimal capital stock K_2 is lower in the presence of adjustment stocks, which is logical.

Now, we define the function $\Psi(I_1/K_1)$ as

(A8.16) $$\Psi\left[\frac{I_1}{K_1}\right] = \Phi\left[\frac{I_1}{K_1}\right] + \left[\frac{I_1}{K_1}\right]\Phi'\left[\frac{I_1}{K_1}\right].$$

Given our earlier assumptions $\Phi'(I_1/K_1) > 0$ and $\Phi''(I_1/K_1) > 0$, the function $\Psi(I_1/K_1)$ is increasing. In addition, $\Psi(0) = 0$.

Next, we define a new variable:

---

[6] Maximizing wealth is the first step. Given maximized wealth, the second step is to maximize utility given wealth, i.e. the consumption decision discussed in section A8.1.

[7] The second-order condition $F''(K_2)/(1 + r) - (2/K_1)\phi'(I_1/K_1)-(I_1/K_1)\phi''(I_1/K_1) < 0$ is satisfied since F and ϕ are assumed to be concave and I,K are greater or equal 0.

(A8.17) $$q_1 = \frac{F'(K_2)}{1+r}$$

This variable – called Tobin's q – is the present value of next period's marginal productivity of capital. It measures how much is gained by investing one more unit of capital. If $q_1 > 1$, the investment pays off more than what is spent; in the opposite case when $q_1 < 1$, investing one unit of capital delivers less value than that invested.

Using the definitions (A8.16) and (A8.17), the first order condition (A8.15) becomes $\Psi(I_1/K_1) = q_1^{-1}$, which can be inverted to give

(A8.18) $$\frac{I_1}{K_1} = \Psi^{-1}(q_1 - 1),$$

where Ψ^{-1} is the inverse function of Ψ. Since Ψ is increasing in I_1/K_1, Ψ^{-1} is increasing in q_1 and since $\Psi(0)=0$, it is also the case that $\Psi^{-1}(0)=0$.

The investment function (A8.18) summarizes the results shown in Fig. 8.17. It says that investment – as a share of existing capital – increases with Tobin's q. More precisely, no investment occurs when $q_1 = 1$ and the firm invests when $q_1 > 1$ while it disinvests when $q_1<1$. Note that when $q_1>1$, (A8.17) shows that it is optimal to accumulate capital K_2 until $F'(K_2) > 1 + r$ while in the absence of adjustment costs it is optimal to choose K_2 such that $F'(K_2) = 1 + r$. The reason is that adjustment costs lower the returns from investment. In a more general case with more than two periods and a resale market for capital, q will always have to be greater than unity so that firms have an incentive to invest to replace capital as it becomes obsolete or depreciates. Under these conditions, more investment today means less investment is needed tomorrow, and this will give firms a stronger incentive to bunch investment, but will also lead to a lower long-run capital stock.