WebAppendix 7: Expectations and Budget Constraints over Many Periods

This appendix provides some discussion of expectations as well as a generalization of the two-period analysis of intertemporal budget constraints presented in the main text (Chapter 7).

A7.1 Alternative hypotheses about how agents think about the future

We have seen that macroeconomics is to a great extent about the views held by households and firms of the future. Thus, knowing what economic agents do in the present is not only a function of the past, but also of the future. While they do not know the future with certainty, it is unreasonable to assume that expectations of the future are completely independent of what actually does happen. The most likely outcome would appear to be that economic agents do think about the future and attempt to get their expectations right, but may not always be on target.

Thus, when thinking about expectations formally, it is necessary to incorporate uncertainty or randomness explicitly. We consider an agent's view about a random (uncertain) variable x_{t+1} from the perspective of period t and denote the expectation of its value in time $t + 1$ as ${}_t x_{t+1}$.

The rational expectations hypothesis asserts that the difference between a variable's expectation and the value that actually occurs, called its realization, are unpredictable:

$$ {}_t x_{t+1} - x_{t+1} = \varepsilon_{t+1}, $$

where the forecast error ε_{t+1} is purely random and unpredictable. It is sometimes called *white noise*. The strongest form of the rational expectations hypothesis would equate ε_{t+1} to the "true" underlying fundamental uncertainty in an economic model. A weaker version would assert that the expected value of the forecast errors based on available information are equal to zero.[1] Lacking a model, one might simply assume that agents' mistakes are statistically uncorrelated with information available at time t, including previous expectational errors made. If ε_{t+1} equals zero each period, we have the case of *perfect foresight*. Macroeconomists frequently assume rational expectations as a disciplining device which prevents them from making arbitrary assumptions. In this spirit, the perfect foresight variant is adopted in the textbook as a simplification.

[1] A more formal description is that ${}_{t-1}x_t$ is the mathematical expectation of x_t based on all information available at time t –1, denoted as I_{t-1}: this is written ${}_t x_{t+1} = E(x_t \mid I_{t-1})$.

The adaptive expectations *hypothesis* assumes that agents act to gradually correct their mistakes over time, but do so in a relatively mechanical way. According to this view, if they underestimated an economic variable last time, they raise their expectation; if they overestimated it, they reduce their forecast:

$${}_t x_{t+1} - {}_{t-1}x_t = \alpha(x_t - {}_{t-1}x_t)$$

where α measures the extent to which agents adjust their expectations for past mistakes ($0<\alpha<1$). While making sense, this trial-and-error process implies that agents accumulate knowledge only from their past experience; moreover, they do not learn from their mistakes. *Static* expectations, in which agents do not revise their expectations at all, is the special case $\alpha = 0$, whereas *myopic* expectations ($\alpha = 1$) implies that agents' forecasts are simply equal to last period's realized value.

An alternative assumption related to adaptive expectations is that agents form ***extrapolative expectations*** of the future as a continuation of past trends:

$${}_t x_{t+1} = x_t + \beta(x_t - x_{t-1}).$$

The parameter β summarizes the extent to which agents expect past trends to continue in the future. It can be thought of as a variant of adaptive expectations when expectations are myopic (${}_t x_{t+1} = x_t$) but with $\alpha \neq 1$.

In the textbook and in what follows we will assume that people know the future with complete certainty, which, while a brave assumption, is an important starting point for any intertemporal analysis. More realism is always possible, but at the cost of more complication with limited additional wisdom. Such analyses are left to more advanced textbooks.

A7.2 Households without firms

Let us first consider a household which has access to borrowing and lending at the same, constant real interest rate (think of a small open economy). The analysis then moves to the valuation of the firm, using that given rate of interest. Next, ownership of the firms is transferred to the households. Finally, the government sector and the foreign sector are brought in.

In Section 5.3 of the textbook, the household receives income Y_1 and Y_2 in the first and second periods of life, respectively, and consumes C_1 and C_2. Suppose it begins the first period with tradable wealth B_1 which pays real interest rate r per period if $B_1>0$, or "costs" r per period as a loan ($B_1<0$). Since lending and borrowing in the textbook example had no purpose beyond the second period (Crusoe is rescued), the intertemporal budget restriction is simply (5.4) of the main text:

(A7.1) $$C_1 + \frac{C_2}{1+r} = B_1 + Y_1 + \frac{Y_2}{1+r},$$

with the usual interpretation that the present value of resources equals the present value of consumption. Strictly speaking, (A7.1) could be formulated as a weak

inequality, since the consumer can always simply throw resources away; we must rule out, however, that the consumer tries to live beyond his means in present value terms.

Now we consider the case of a household which lives several periods and we will chain the period-by-period budget constraints together across time into an overall intertemporal resource constraint. Now financial wealth at the beginning of the second period is what has been saved during the first period plus the interest served on that saving:

(A7.2) $$B_2 = (1+r)(B_1 + Y_1 - C_1).$$

This equation will also be true in period 3:

(A7.3) $$B_3 = (1+r)(B_2 + Y_2 - C_2).$$

Substituting (A7.2) in (A7.3) yields:

(A7.4) $$B_3 = (1+r)^2(B_1 + Y_1 - C_1) + (1+r)(Y_2 - C_2).$$

Suppose the household lives T periods. The financial assets or liabilities of the family at the beginning of T+1 can be computed by repeating the previous operation until period T+1 is reached:

(A7.5) $$B_{T+1} = (1+r)^T(B_1 + Y_1 - C_1) + \ldots + (1+r)(Y_T - C_T).$$

Now rearrange (A7.5) and divide both sides by $(1+r)^T$ to obtain

(A7.6) $$\frac{B_{T+1}}{(1+r)^T} = B_1 + Y_1 - C_1 + \frac{Y_2 - C_2}{1+r} + \ldots + \frac{Y_T - C_T}{(1+r)^{T-1}}$$
$$= B_1 + \sum_{t=1}^{T} \frac{Y_t - C_t}{(1+r)^{t-1}}.$$

In words, (A7.6) expresses the present value of the financial "legacy" (debt or inheritance) from the perspective of period 1. This is called a present value; all values are discounted back to period 1. The end of horizon present value of the family's financial wealth is found to be the initial wealth B_1 plus the present value of all saving and borrowing in the intervening periods.

We can now think of what the budget constraint means. Remember that at time t, B_t can take positive or negative values: $B_t>0$ corresponds to a creditor position (accumulating savings at the beginning of t) while $B_t<0$ implies indebtedness. One definition of the budget constraint is that the family is not allowed to pass debt on to

their children or to others. This would imply that $\frac{B_{T+1}}{(1+r)^T} \geq 0$, i.e. the stock of assets at the end must be non-negative in present value terms (and thus in current value terms!). At the same time, why would a family throw financial assets away, when they might be consumed? This would speak for requiring that the stock of assets at the end not to be positive. If B_{T+1} is neither negative nor positive, it can only zero, so $B_{T+1}/(1+r)^T = 0$.
Imposing this condition allows us to rewrite (A7.6) as

$$B_1 = \sum_{t=1}^{T} \frac{C_t - Y_t}{(1+r)^{t-1}}.$$

or

(A7.7) $$\sum_{t=1}^{T} \frac{C_t}{(1+r)^{t-1}} = B_1 + \sum_{t=1}^{T} \frac{Y_t}{(1+r)^{t-1}} = \Omega,$$

where, as in the textbook, Ω denotes total household wealth (financial assets plus income). When the budget constraint is satisfied, we find that the present value of consumption is equal to wealth. Equivalently, the current assets of the family B_1 represent the total amount by which present and future consumption may exceed present and future income in present value terms.

This analysis applies to the case of fixed and finite T. It is also interesting to consider the decision of a "dynasty" or family with an infinite life. (When discounting with positive interest rates, the infinite horizon case isn't very different from a case of a few decades of life!). In the case of infinite horizons, however, a more relaxed requirement will also work: the asset or liability position in T+1, B_{T+1} may not grow faster than the interest rate. As $(1+r)^t$, the discount factor applied to resources received in t, grows exponentially over time, this means requiring that the term B_{T+1} grows more slowly than $(1+r)^t$, in which case we have the requirement that the present value of the family's financial assets be zero as T goes to infinity, $\lim_{T\to\infty} \frac{B_{T+1}}{(1+r)^T} = 0$.[2] Then, the budget constraint can be rewritten as follows:

[2] This requirement is equivalent to imposing a weaker condition on the family with a finite life – that the present discounted value of B_{T+1} may not exceed some critical value, and then letting T go to infinity. The critical value may increase over time, but not faster than the interest rate.

(A7.8) $$\sum_{t=1}^{\infty}\frac{C_t}{(1+r)^{t-1}} = B_1 + \sum_{t=1}^{\infty}\frac{Y_t}{(1+r)^{t-1}} = \Omega$$

(A7.8) looks exactly like (A7.7), except that we look at consumption and income streams over an infinite horizon, and the interpretation is exactly the same: the present value of consumption is equal to wealth, i.e. initial assets plus the present value of income.

A7. Firms

We now look at the budget constraint of firms, and consider a slightly different case than in the textbook. The assumption that Robinson can gather coconuts from the trees for free, described in the text as the endowments Y_1 and Y_2, is dropped. Instead, we assume that output has to be produced, using capital and labour, which is represented by the production function $Y_t = F(K_{t-1}, L_t)$. The output is produced by a firm that not only employs capital inherited from the previous period K_{t-1}, but also labor L_t in the current period, which it pays real wage w_t. The firm can also invest some of its resources to raise its capital stock. In period t, the firm's cash flow Π_t is defined as its profits (income less wage costs) *less* expenditures for investment I_t:

$$\Pi_t = F(K_{t-1}) - I_t - w_t L_t \,.$$

We can now define the firm's value, the price V as the amount that a purchaser would be willing to pay for a firm which is planning to hire L_t, L_{t+1}, L_{t+2}, ... and to invest I_t, I_{t+1}, I_{t+2},…, given an initial capital stock to hire K_{t-1}, and a constant rate of interest r. It is important to stress that this is a *valuation* exercise; profit maximization is discussed in detail in Chapter 8 and WebAppendix 8. Today's value of the firm, V_1, can be decomposed as this period's cash flow plus the present value of the firm next period:

(A7.9) $$V_1 = F(K_{t-1}) - I_t - w_t L_t + \frac{V_2}{1+r}$$

Note that this definition does not assume that the firm maximizes its profits, although profit maximization is a central aspect of the theory of the firm and of investment. (Obviously the owners of firms will – in their own interests – try to maximize profits and increase their own wealth). Iterating on (A7.9) in a fashion similar to the household budget constraint results in the valuation of a firm which exists for T periods:

(A7.10) $$V_1 = \sum_{t=1}^{T}\frac{\Pi_t}{(1+r)^{t-1}} + \frac{V_{T+1}}{(1+r)^{T}}$$

The value of the firm is equal to the present discounted value of present (period 1) and future (periods 2, 3,…, T) profits, plus the disposal value of the firm in period T+1. Let us call a firm with negative present value *bankrupt*. If bankruptcy is never possible in any period, it follows from (A7.10) that $\frac{V_{T+1}}{(1+r)^T} \geq 0$. At the same time it would seem unreasonable to allow a firm to reach "the end" of its existence with positive present value, since this would mean that the owners of the firm will effectively forego access to the resources locked in the firm. Applying the *transversality condition* $\lim_{T\to\infty} \frac{V_{T+1}}{(1+r)^T} = 0$ guarantees that resources are not wasted. In this case, the value of the firm in period 1 is the present discounted value of its profits:

$$V_1 = \sum_{t=1}^{\infty} \frac{\Pi_t}{(1+r)^{t-1}}.$$

A7.4 The Consolidated Private Sector

To clarify the role of firm ownership in the extreme, the representative household is assumed to own the representative firm in its entirety, meaning that all income from business operations is paid to the households either as labor income wL or as profits, which each period is equal to cash flow Π plus investment I. We therefore exclude the possibility of foreigners owning the firms. To simplify the bookkeeping further, we further simplify by requiring that firms borrow and lend abroad only, and do so at rate r. From the national income identity, total output is divided between consumption C and investment I:

(A7.10) $$w_tL_t + \Pi_t + I_t = Y_t = F(K_{t-1}) = C_t + I_t$$

For simplicity, we assume in this case that capital depreciates completely each period. For the household, the present value of consumption cannot exceed the present discounted value of available resources—the sum of labor income w_tL_t and net cash flow of the firm. In present value terms, this can be written as

(A7.11) $$\sum_{t=1}^{\infty} \frac{C_t}{(1+r)^{t-1}} = B_1 + \sum_{t=1}^{\infty} \frac{w_tL_t + \Pi_t}{(1+r)^{t-1}}$$

$$= B_1 + \sum_{t=1}^{\infty} \frac{w_tL_t + F(K_{t-1}) - I_t - w_tL_t}{(1+r)^{t-1}}$$

$$= B_1 + \sum_{t=1}^{\infty} \frac{F(K_{t-1}) - I_t}{(1+r)^{t-1}} = B_1 + V_1 = \Omega.$$

In words, the present value of consumption is equal to initial financial wealth plus the value of the firm. This left-hand side of (A7.11) corresponds to Ω, which was defined in the text for the two-period model in equation (5.9) and derived in footnote 11. If the household receives additional investment periodic income from abroad ($\Gamma_t > 0$) or pays interest or investment income ($\Gamma_t < 0$), then the formula could be modified to read

(A7.11´)
$$\sum_{t=1}^{\infty} \frac{C_t}{(1+r)^{t-1}} = B_1 + \sum_{t=1}^{\infty} \frac{F(K_{t-1}) - I_t}{(1+r)^{t-1}} + \sum_{t=1}^{\infty} \frac{\Gamma_t}{(1+r)^{t-1}}$$
$$= B_1 + V_1 + \sum_{t=1}^{\infty} \frac{\Gamma_t}{(1+r)^{t-1}} = \Omega .$$

A7.5 Government

Extending the two-period budget constraint of the government sector presented in the text is relatively straightforward. To start with, governments have indebtedness equal to D_1, so the equation describing the evolution of government debt is:

(A7.12)
$$D_2 = (1+r_G)(D_1 + G_1 - T_1).$$

Recall that T represents net taxes on households (taxes minus transfers) and G is current purchases of output by the government in the form of goods and services. Recursive substitution of (A7.12) to period T+1 yields

(A7.13)
$$\frac{D_{T+1}}{(1+r_G)^T} = D_1 + \sum_{t=1}^{T} \frac{G_t - T_t}{(1+r_G)^{t-1}} .$$

Analogous to the discussion in previous sections, capital markets are likely to impose a condition that governments are not allowed to roll over their net public debt forever, and which requires the present value of taxes to equal the initial debt and the present value of spending. Mathematically, this requirement can be expressed as $\lim_{T\to\infty} \frac{D_{T+1}}{(1+r_G)^T} = 0$, meaning that (A7.13) becomes the infinite–horizon equivalent of (5.11):

(A7.14)
$$D_1 = \sum_{t=1}^{\infty} \frac{T_t - G_t}{(1+r_G)^{t-1}} .$$

In words, a solvent government must eventually run primary surpluses (i.e. $T>G$) over time in present value terms, if it has debt to start with.[3]

A7.6 The Public and Private Sectors Consolidated[4]

Consolidating public and private sectors requires a bit of work if all possible financial relationships are considered in detail. To simplify the accounting, we look at the case in which the private sector does not lend and borrow to the government directly. Thus, all financial assets (liabilities) of the private sector the liabilities (assets) vis-à-vis the foreign sector. Similarly, the government is assumed to borrow or save at rate r_G but does so only vis-à-vis the foreign sector. After appropriate consolidation, results presented below will also hold when more complicated financial linkages are allowed.

Modify the private sector budget constraint to account for taxes T_t in each period. Now the private sector's budget constraint (A7.11) becomes

(A7.15) $$\sum_{t=1}^{\infty}\frac{C_t}{(1+r)^{t-1}} = B_1 + \sum_{t=1}^{\infty}\frac{w_t L_t - T_t + \Pi_t}{(1+r)^{t-1}} = B_1 + V_1 - \sum_{t=1}^{\infty}\frac{T_t}{(1+r)^{t-1}}$$

As in the text, private wealth is reduced by the present value of taxes, with discounting occurring at the private rate.[5]

What happens when households see through the veil, that is, that present and future tax liabilities are seen as financing the government intertemporal budget constraint? In words, they will understand that government borrowing at a lower interest rate makes them better off, while government lending at a lower interest rate will make them worse off. First rewrite the government's budget constraint as

$$D_1 = \sum_{t=1}^{\infty}\frac{T_t - G_t}{(1+r_G)^{t-1}} = \sum_{t=1}^{\infty}\left[\frac{(T_t - G_t)(1+r)^{t-1}}{(1+r_G)^{t-1}(1+r)^{t-1}} + \frac{(T_t - G_t)(1+r_G)^{t-1}}{(1+r_G)^{t-1}(1+r)^{t-1}} - \frac{(T_t - G_t)(1+r_G)^{t-1}}{(1+r_G)^{t-1}(1+r)^{t-1}}\right]$$

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[3] This condition may seem unreasonable at first glance, because governments rarely run budget surpluses; yet it only refers to primary surpluses, which means the surplus *after interest payments have been added back.* For more discussion, see Chapter 17 of the textbook.

[4] This section covers mathematically more advanced and difficult material.

[5] These are lump sum taxes and will not affect agent's behavior, as the discussion in Section 5.3.3 explains. In contrast, distortionary taxes are likely to change the size of the total resource cake and will thereby affect the budget constraint even if Ricardian equivalence holds.

$$= \sum_{t=1}^{\infty} \frac{\left[(1+r)^{t-1} - (1+r_G)^{t-1}\right]}{(1+r_G)^{t-1}(1+r)^{t-1}}(T_t - G_t) + \sum_{t=1}^{\infty} \frac{T_t - G_t}{(1+r)^{t-1}}$$

$$= \sum_{t=1}^{\infty} \left[\frac{1}{(1+r_G)^{t-1}} - \frac{1}{(1+r)^{t-1}}\right](T_t - G_t) + \sum_{t=1}^{\infty} \frac{T_t - G_t}{(1+r)^{t-1}}$$

Present and future primary surpluses that the government runs can be rewritten as the present value of the primary surpluses financed or invested at the private interest rate r, plus the present discounted value of the government's access to an alternative interest rate r_G. Thus, if $r_G<r$ (the government borrows and lends at a lower interest rate), any given path of future primary surpluses ($T>G$) can be used by the public sector to service a higher initial level of indebtedness, in the amount of the first term of the last expression.

The present value of tax liabilities from the household's perspective is given by $\sum_{t=1}^{\infty} \frac{T_t}{(1+r)^{t-1}}$. Using the previous equation, we can write

$$\sum_{t=1}^{\infty} \frac{T_t}{(1+r)^{t-1}} = D_1 + \sum_{t=1}^{\infty} \frac{G_t}{(1+r)^{t-1}} + \sum_{t=1}^{\infty} \left[\frac{1}{(1+r_G)^{t-1}} - \frac{1}{(1+r)^{t-1}}\right](G_t - T_t)$$

Now insert this into (A7.15) and rearrange:

(A7.15)

$$\sum_{t=1}^{\infty} \frac{C_t}{(1+r)^{t-1}} = B_1 - D_1 + V_1 - \sum_{t=1}^{\infty} \frac{G_t}{(1+r)^{t-1}} + \sum_{t=1}^{\infty} \left[\frac{1}{(1+r_G)^{t-1}} - \frac{1}{(1+r)^{t-1}}\right](T_t - G_t)$$

As in the two-period case, the government reduces private wealth in the first instance by the amount equal to the present value of lost resources it "spends", i.e. by G_t for t=1,2,… In addition, the government reduces private wealth to the extent that it does "worse" than the private sector could have done on its own; specifically, if it had borrowed money at a higher rate ($r_G>r$) and ran future primary budget surpluses, or had lent money at a lower rate ($r_G<r$) and was expected to run deficits in the future. When private sector and government borrow and lend at the same rate ($r = r_G$),

(A7.15) $$\sum_{t=1}^{\infty} \frac{C_t}{(1+r)^{t-1}} = B_1 - D_1 + V_1 - \sum_{t=1}^{\infty} \frac{G_t}{(1+r)^{t-1}}$$

This is the Ricardian equivalence result: the path of taxes is irrelevant for the value of household wealth. Since there is only one ‘representative agent’ in the economy, and since there are no other forms of wealth in this example, $B_1 - D_1$ is also the nation’s inherited external investment position (net foreign assets/liabilities), which is the sum of private assets vis-à-vis the rest of the world (B_1) and government assets vis-à-vis the rest of the world ($-D_1$).