Exercise 21.1

1. (a) $y(x) = 3/5 + 2/25\ (x\text{-}1) + ½\ (-4/125)\ (x\text{-}1)^2$
and $y(2) = 3/5 + 2/25\ (2\text{-}1) + ½\ (-4/125)\ (2\text{-}1)^2 = 0.664$, while the true value is 4/6 = 0.667. Thus the error is approximately 0.003.

(b) $y(x) = 7.389 + 14.778\ (x\text{-}1) + ½\ 29.556\ (x\text{-}1)^2$
and $y(2) = 7.389 + 14.778 + ½\ 29.556 = 36.945$, while the true value is 54.598. Thus the error is a (large) 17.653.

(c) $y(x) = 1.099 + 0.667\ (x\text{-}1) + ½\ (.222)\ (x\text{-}1)^2$
and $y(2) = 1.099 + 0.667 + ½\ (.222) = 1.877$, while the true value is 1.792. Thus the error is 0.085.

2. (a) $y(x) = 3/5 + 2/25\ (x\text{-}1) + ½\ (-4/125)\ (x\text{-}1)^2 + 1/6\ (12/625)\ (x\text{-}1)^3$
and $y(2) = 3/5 + 2/25\ (2\text{-}1) + ½\ (-4/125)\ (2\text{-}1)^2 + 1/6\ (12/625) = 0.6672$,and the error is zero to 3 decimal places.

(b) $y(x) = 2 + \ln(2)\ 2\ (x\text{-}1) + ½\ (\ln(2))^2\ 2\ (x\text{-}1)^2 + 1/6\ (\ln(2))^3\ 2\ (x\text{-}1)^3$
and $y(2) = 2 + \ln(2)\ 2 + ½\ (\ln(2))^2\ 2 + 1/6\ (\ln(2))^3\ 2 = 3.978$, while the true value is 4 and the error is thus 0.022.

(c) $y(x) = 1.099 + 1.432\ (x\text{-}1) + ½\ 5/9\ (x\text{-}1)^2 – 1/6\ (7/27)\ (x\text{-}1)^3$,
and $y(2) = 1.099 + 1.432 + ½\ 5/9 – 1/6\ (7/27) = 2.765$, while the true value is 2.772 and the error is -0.007.

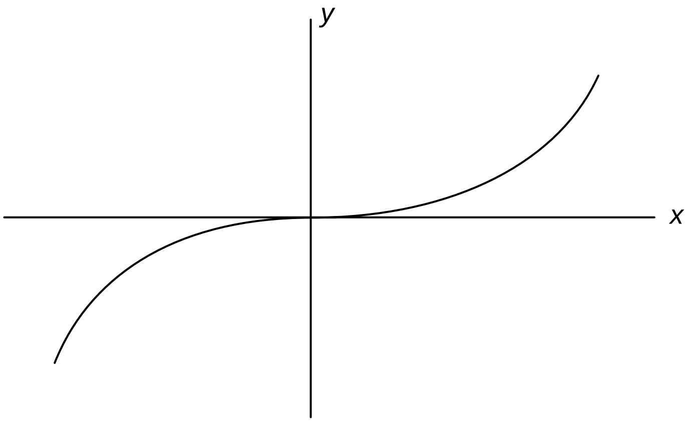
3. (a) $x = 5$, $y = 6$ and minimum
(b) $x = 10$, $y = 4$ and maximum
(c) $X = 6$, $Y = 2$ and minimum
(d) $x = 2$, inflection
(e) $x = -1$, minimum.

4. There are three solutions to the two first-order conditions. These are
(i) $x = 0$, $y = 1$
(ii) $x = -2$, $y = 3$
(iii) $x = -1$, $y = 3/2$
The first two are saddlepoints; the last is a minimum.

5. All apart from the first and the fourth are concave for $x \geq 0$.

6. True

7. The function $y = x^3$ has these properties; see sketch below.



8. First order condition for an interior solution is $1 - 2(x-a) = 0$. Thus maximising $x = ½ + a$ if $a \geq -½$ and $x = 0$ otherwise. For the second part, maximising $x = ½ + a$ if $a \geq ½$ and $x = 1$ otherwise.

Progress exercise 21.2

1.
$1 - y - 2x = 0$ and
$-x + c - 2y = 0$
Solution is max if $-2<0$ and $(-2)(-2) - (-1)(-1) > 0$: these conditions clearly hold.
Solution is $y = (2c-1)/3$, $x = (2-c)/3$

2. Tax revenue $R = t(100 - p - t)$. First order condition is $100 - p - 2t = 0$. This implies $t^* = (100 - p)/2$. Hence $dt^*/dp = -½ < 0$. Also R is convex in p, so that since $dR/dp = -t$, we have $d^2R/dp^2 = -dt/dp > 0$ and so $dt/dp < 0$.

3. (a) $TR = -2Q_1^2 - Q_2^2 + 3Q_1Q_2 + 50Q_1 + 30Q_2$
(b) $\Pi = -2Q_1^2 - Q_2^2 + 2Q_1Q_2 + 40Q_1 + 20Q_2$
(c) $Q_1 = 30$; $Q_2 = 40$.
(d) $\Pi = 1000$.
(e) Second derivative of profit with respect to Q_1 is $-4<0$; also
$\frac{\partial^2 \Pi}{\partial Q_1^2}\frac{\partial^2 \Pi}{\partial Q_2^2} - \left(\frac{\partial^2 \Pi}{\partial Q_1 \, \partial Q_2}\right) = 8 - 4 > 0$. Thus second order conditions for a maximum hold.

4. (a) $TR = -2Q_1^2 - Q_2^2 + 3Q_1Q_2 + (50 - t)Q_1 + 30Q_2$
(b) $\Pi = -2Q_1^2 - Q_2^2 + 2Q_1Q_2 + (40 - t)Q_1 + 20Q_2$
(c) $Q_1 = 30 - t/2$; $Q_2 = 40 - t/2$: both output levels decline by the same amount ½ dt.

(d) $\frac{\mathrm{d}\Pi}{\mathrm{d}t} = \frac{\partial \Pi}{\partial t} = -Q_1*$, by applying the envelope property that profit is maximised at the optimal output levels.

(e) same as question 3.

5. (a) $(\sqrt{3},1)$; $r = 2$, $\theta = \tan^{-1}\left(\frac{1}{\sqrt{3}}\right) = \frac{\pi}{6}$

(b) $(0,1)$; $r = 1$, $\theta = \frac{\pi}{2}$

(c) $(-1,1)$; $r = \sqrt{2}$, $\theta = \frac{3\pi}{4}$.

Progress exercise 21.3

1. (a) $(\sqrt{3},1)$; $r = 2$, $\theta = \tan^{-1}\left(\frac{1}{\sqrt{3}}\right) = \frac{\pi}{6}$

(b) $(0,1)$; $r = 1$, $\theta = \frac{\pi}{2}$

(c) $(-1,1)$; $r = \sqrt{2}$, $\theta = \frac{3\pi}{4}$.

2. (a) $Y_t = A(2)^t + B(0.2)^t$. Divergent.

(b) $Y_t = A\left(\frac{1}{2}\right)^t + B\left(-\frac{1}{2}\right)^t$. Converges to 0.

(c) Auxiliary equation is: $4m^2 - 2m + 1 = 0 \Rightarrow m = \frac{1}{4} \pm \sqrt{-\frac{3}{16}}$, so $\alpha = \frac{1}{4}, \beta = \frac{\sqrt{3}}{4}$.

Polar form is $\left(\frac{1}{2}, \frac{\pi}{3}\right)$. Solution is: $Y_t = \left(\frac{1}{2}\right)^t \left\{A\cos(\frac{\pi t}{3}) + B\sin(\frac{\pi t}{3})\right\}$.

Convergent to zero (with oscillations)

3. (a) $Y_t = A(2)^t + B(0.2)^t - 1$. Divergent

(b) $Y_t = A\left(\frac{1}{2}\right)^t + B\left(-\frac{1}{2}\right)^t + \frac{1}{3}$. Converges to $\frac{1}{3}$.

(c) $Y_t = \frac{1}{2} + (A + Bt)(5)^t$. Divergent.

(d) $Y_t = 2 + \left(\frac{1}{2}\right)^t \left\{A\cos(\frac{\pi t}{3}) + B\sin(\frac{\pi t}{3})\right\}$. Damped oscillations about 2

4. (b) $A = 1/2, B = -5/6$, (c) $A = 0, B = 1/10$, (d) $A = 0, B = 4/\sqrt{3}$.

5. Here '$\rightarrow$ 'denotes 'converges towards'

(a) $Y_t = (0.8)Y_{t-1} + 300$, $Y_t = -1000(0.8)^t + 1500 \rightarrow 1500$, (no oscillations)

(b) $Y_t = (0.5)Y_{t-1} + 300$, $Y_t = -500(0.5)^t + 1000 \rightarrow 1000$, (no oscillations)

(c) $Y_t = 1.5Y_{t-1} - 500$, $Y_t = -500(1.5)^t + 1000 \rightarrow 1000$, diverges.

(d) $Y_t - 1.4Y_{t-1} + 0.6Y_{t-2} = 200$. The characteristic equation is

$m^2 - 1.4m + 0.6 = 0$ which has solutions $m = 0.7 \pm \sqrt{-0.11}$, so

$\alpha = 0.7,\ \beta = \sqrt{0.11} \approx 0.33167, \quad r = 0.7746, \quad \theta = 0.4425$; Particular Solution = 1000.

General Solution is $1000 + (0.7746)^t \left\{A\cos(0.4425t) + B\sin(0.4425t)\right\}$, which converges to 1000.

6. Characteristic equation is $m^2 - m - 1 = 0$, and the roots are given by

$$m_1 = \frac{1+\sqrt{1+4}}{2} \quad \text{and} \quad m_2 = \frac{1-\sqrt{1+4}}{2}$$

The solution then is: $Y_t = A_1 m_1^t + A_2 m_2^t$

Here m_2 disappears as t becomes larger since it is less than one in absolute value. Hence Y_t converges to $A_1 m_1^t$, and Y_t/Y_{t-1} to $m_1 = ½ (1+\sqrt{5}) = 1.618034$. When $Y_0 = 0$ and $Y_1 = 1$ we have $0 = A_1+A_2$, and $1 = A_1 m_1+A_2 m_2$, so that $A_1 = 1/(m_1- m_2) = 1/\sqrt{5}$, $A_2 = -1/\sqrt{5}$.