Progress exercise 16.1

1. (i) (a) Direct substitution method: $z = 4x^2 - 200x + 3200$

$\Rightarrow \quad \dfrac{dz}{dx} = 8x - 200 = 0 \quad \Rightarrow \quad x = 25; \quad \dfrac{d^2z}{dx^2} = 8 > 0$. So minimum of z when

$x = 25$, $y = 40 - x = 15$. (Note, substitution of constraint into objective function ensures constraint is satisfied.)

(b) Equating slopes method: Slope of an iso-z section, by implicit differentiation, is

$-\dfrac{\frac{\partial z}{\partial x}}{\frac{\partial z}{\partial y}} = -\dfrac{2x - y}{-x + 4y}$. Slope of constraint $y = 40 - x$ is $\dfrac{dy}{dx} = -1$. Equating these:

$-\dfrac{2x - y}{-x + 4y} = -1 \quad \Rightarrow \quad x = \frac{5}{3}y$. Substituting into constraint,

$y = 40 - \frac{5}{3}y \quad \Rightarrow \quad y = 15 \quad \Rightarrow \quad x = \frac{5}{3}y = 25$, as before.

(c) The shape of $z = x^2 - xy + 2y^2$ is, roughly speaking, a cone with its vertex at the origin. So the iso-z sections are concave when viewed from the origin. The tangency of the iso-z contour to the constraint is therefore like the tangency at W in figure 16.6 in the book . So $x = 25$, $y = 15$ is a constrained *minimum* of z. That is, it is the lowest value of z with the constraint satisfied. This is supported (though this evidence is not conclusive) by looking at another point on the constraint, say $y = 16$, $x = 24$, which gives $z = 704\ (> 700)$. This is a point like J or T in figure 16.6.

(ii) (a) Using method of (i)(a) above, $y = 20$, $x = 36$, giving $z = 9744$.

(b) Using method of (i)(b) above, same answer is obtained.

(c) Shape is similar to (i) above. The constrained SP seems to be a minimum, since when $x = 35$ and $y = 21$, $z = 9751$ (> 9744).

(iii) (a) Using method of (i) above, $y = 8$, $x = 15$, giving $z = 245$.

(b) Using method of (i)(b) above, same answer is obtained.

(c) The function $z = (x-1)^2 + (y-1)^2$ has an *unconstrained* minimum at $x = y = 1$, with $z = 0$. Its shape is roughly conical with its vertex at $x = y = 1$. The constrained SP seems to be a minimum, since when $x = 16$ and $y = 6$, $z = 275$ (> 245).

Progress exercise 16.2

1. (i) Lagrangean expression is: $V = x^2 - xy + 2y^2 + \lambda(40 - x - y)$, so taking partial derivatives and setting them equal to zero gives:

$V_x = 2x - y - \lambda = 0$ (1)

$V_y = -x + 4y - \lambda = 0$ (2)

$V_\lambda = 40 - x - y = 0$ (3)

From (1), $2x - y = \lambda$. From (2), $-x + 4y = \lambda$. Therefore $2x - y = -x + 4y \quad \Rightarrow \quad x = \frac{5}{3}y$. Therefore, in (3) $40 - \frac{5}{3}y - y = 0 \quad \Rightarrow \quad y = 15$, so $x = \frac{5}{3}y = 25$. So $z = (25)^2 - 25(15) + 4(15)^2 = 700$. Substituting $x = 25$, $y = 15$ into, say, equation (1) we get $2(25) - 15 - \lambda = 0 \quad \Rightarrow \quad \lambda = 35$.

If we re-solve equations (1) to (3) with a constraint of $y = 41 - x$ instead of $y = 40 - x$, we get $y = 15\frac{3}{8}$, $x = 25\frac{5}{8}$, $z = 735.4375$. So the increase in *z* is 35.4375, which is very close to the value of λ, 35.

(ii) Lagrangean expression is: $V = 4x^2 + 3xy + 6y^2 + \lambda(56 - x - y)$, so:

$V_x = 4x + 3y - \lambda = 0$ (1)

$V_y = 3x + 12y - \lambda = 0$ (2)

$V_\lambda = 56 - x - y = 0$ (3)

Solving these simultaneously gives the same solution as in exercise 16.1.

(ii) Lagrangean expression is: $V = (x-1)^2 + (y-1)^2 + \lambda(38 - 2x - y)$, so:

$V_x = 2(x-1) - 2\lambda = 0$ (1)

$V_y = 2(y-1) - \lambda = 0$ (2)

$V_\lambda = 38 - 2x - y = 0$ (3)

Solving these simulatneously gives the same solution as in exercise 16.1.

Progress exercise 16.3

1. (a) The problem is to minimise $TC = wL + rK = L + 4r$, subject to

$Q = K^{0.5}L^{0.5} = 100$. Lagrangean equation is therefore

$V = L + 4K + \lambda\left(K^{0.5}L^{0.5} - 100\right)$

$V_L = 1 + 0.5\lambda K^{0.5}L^{-0.5} = 0$ (1)

$V_K = 4 + 0.5\lambda K^{-0.5}L^{0.5} = 0$ (2)

$V_\lambda = K^{0.5}L^{0.5} - 100 = 0$ (3)

From (1), after multiplying both sides by 4: $2\lambda K^{0.5}L^{-0.5} = -4$

From (2): $0.5\lambda K^{-0.5}L^{0.5} = -4$. Combining these, $2\lambda K^{0.5}L^{-0.5} = 0.5\lambda K^{-0.5}L^{0.5}$

Divide both sides by right hand side, $\Rightarrow \dfrac{2\lambda K^{0.5}L^{-0.5}}{0.5\lambda K^{-0.5}L^{0.5}} = 1 \Rightarrow 4KL^{-1} = 1$

So $L = 4K$. Substituting this into equation (3):

$K^{0.5}\left(4K\right)^{0.5} = 100 \Rightarrow 2K = 100 \Rightarrow K = 50; L = 4K = 200$

Minimised TC is then $TC = L + 4K = 200 + 4(50) = 400$

(b) Since $MPL \equiv \dfrac{\partial Q}{\partial L} = 0.5K^{0.5}L^{-0.5}$ and $MPK \equiv \dfrac{\partial Q}{\partial K} = 0.5K^{-0.5}L^{0.5}$, in equations (1) and (2) we have: $w + \lambda MPL = 0 \Rightarrow MPL = -\dfrac{1}{\lambda}w$; and;

$r + \lambda MPK = 0 \Rightarrow MPK = -\dfrac{1}{\lambda}r$

Dividing the first equation by the second gives: $\dfrac{MPL}{MPK} = \dfrac{w}{r}$, as the question requires us to show.

(c) If we re-work (a) above with $Q = 200$, equations (1) and (2) are unchanged so we get $L = 4K$ as before. Equation (3) is now: $K^{0.5}L^{0.5} - 200 = 0$; and when we substitute $L = 4K$ into this we get: $\left(4K\right)^{0.5}K^{0.5} = 200 \Rightarrow$

$2K = 200 \Rightarrow K = 100, L = 4K = 400$

As quantity of both inputs has exactly doubled, and input prices are unchanged, it is obvious that TC is also doubled. And as output has also doubled, it is clear that average cost, $\frac{TC}{Q}$, is unchanged. (Specifically, when $Q = 100$,

$AC = \frac{TC}{Q} = \frac{L + 4K}{Q} = \frac{400}{100} = 4$; when $Q = 200$, $AC = \frac{800}{200} = 4$.) So average cost is constant, at least for *Q* = 100 and *Q* = 200.

(d) In this particular example, we have from (b) above

$\frac{MPL}{MPK} = \frac{0.5K^{0.5}L^{-0.5}}{0.5K^{-0.5}L^{0.5}} = K^{0.5-(-0.5)}L^{-0.5-0.5} = \frac{K}{L}$. We also know that for cost minimisation we require $\frac{MPL}{MPK} = \frac{w}{r}$. Combining these, $\frac{K}{L} = \frac{w}{r} \quad \Rightarrow \quad rK = wL$.

Thus the cost of machines, *rK*, equals the wage bill, *wL*, so each will account for 50% of total cost. In this example, this is true for all values of *w* and *r*.

2. (a) Lagrangean equation is: $V = 3.2L + 7.5K + \lambda\left(K^{0.75}L^{0.5} - 40\right)$. So:

$V_L = 3.2 + 0.5\lambda K^{0.75}L^{-0.5} = 0$ (1)

$V_K = 7.5 + 0.75\lambda K^{-0.25}L^{0.5} = 0$ (2)

$V_\lambda = K^{0.75}L^{0.5} - 40 = 0$ (3)

From (1) and (2), we can get $L = \frac{25}{16}K$. Substituting this into (3) gives:

$K^{0.75}\left(\frac{25}{16}K\right)^{0.5} = 40 \Rightarrow K^{1.25} = \left(\frac{16}{25}\right)^{0.5} 40 = 32 \Rightarrow K = (32)^{\frac{1}{1.25}} = 16$. So

$L = \frac{25}{16}K = 25$. Minimum total cost is then $TC = 3.2L + 7.5K = 400$, as required.

(b) From equations (1) and (2) above, we can get: $\frac{0.5K^{0.75}L^{-0.5}}{0.75K^{-0.25}L^{0.5}} = \frac{3.2}{7.5}$. Here the left hand side is $\frac{MPL}{MPK}$ = (absolute) slope of any isoquant. Right hand side is $\frac{w}{r}$ = (absolute) slope of any isocost line. So cost minimisation requires these slopes be equal (that is, a tangency between linear iso-cost line and convex isoquant).

(c) In (a) above we found $L = \frac{25}{16}K$, using only equations (1) and (2). As the level of output affects only equation (3) above, cost minimisation requires $L = \frac{25}{16}K$ whatever the output.

3. (a) Lagrangean equation is: $V = K^{0.4}L^{0.5} + \lambda(3L + 4K - 270)$. So:

$V_L = 0.5K^{0.4}L^{-0.5} + 3\lambda = 0$ (1)

$V_K = 0.4K^{-0.6}L^{0.5} + 4\lambda = 0$ (2)

$V_\lambda = 3L + 4K - 270 = 0$ (3)

Dividing (1) by (2)

$$\Rightarrow \quad \frac{0.5K^{0.4}L^{-0.5}}{0.4K^{-0.6}L^{0.5}} = \frac{-3\lambda}{-4\lambda} = \frac{3}{4} \quad \Rightarrow \quad K^{0.4-(-0.6)}L^{-0.5-0.5} = \frac{3}{4}\cdot\frac{4}{5} \quad \Rightarrow$$

$$\frac{K}{L} = \frac{3}{5} \quad \Rightarrow \quad K = \tfrac{3}{5}L$$

Substituting this into equation (3) $\Rightarrow$ $3L + 4\left(\tfrac{3}{5}L\right) = 270$ $\Rightarrow$ $L = 50$. So

$K = \tfrac{3}{5}L = 30$

With $Q = (30)^{0.4}(50)^{0.5} = 3.898(7.071) = 27.56$

(b) They are methodologically identical. In order to achieve maximum output from a given budget, it is a logical necessity that each and every unit of output be produced at minimum cost.

Progress exercise 16.4

1. (a) Problem is to maximise $\Pi = PQ - (wL + rK)$ subject to constraint $K^{0.4}L^{0.5} - Q = 0$. Lagrangean equation is: $V = PQ - (wL + rK) + \lambda\left(K^{0.4}L^{0.5} - Q\right)$ $= (20)Q - 5L - 4K + \lambda\left(K^{0.4}L^{0.5} - Q\right)$. Partial derivatives are:

$V_Q = 20 - \lambda = 0$ (1)

$V_L = -5 + \lambda 0.5K^{0.4}L^{-0.5} = 0$ (2)

$V_K = -4 + \lambda 0.4K^{-0.6}L^{0.5} = 0$ (3)

$V_\lambda = K^{0.4}L^{0.5} - Q = 0$ (4)

From (1), $\lambda = 20$. Substitute this into (2) and (3). Then, from (2):

$10K^{0.4}L^{-0.5} = 5 \quad \Rightarrow \quad 2K^{0.4} = L^{0.5} \quad \Rightarrow \quad L = 4K^{0.8}$. Substitute this into (3) (with $\lambda = P = 20$) $\Rightarrow \; -4 + 8K^{-0.6}\left(4K^{0.8}\right)^{0.5} = 0 \;\Rightarrow\; 16K^{-0.6}K^{0.4} = 4 \;\Rightarrow\; K^{-0.2} = \frac{1}{4}$

$\Rightarrow \; K^{0.2} = 4 \quad \Rightarrow \quad K = 4^5 = 1024$. So $L = 4K^{0.8} = 4(256) = 1024$

Substitute $K = L = 1024$ into (4), we get: $(1024)^{0.4}(1024)^{0.5} = Q = 512$

So profits are $\Pi = PQ - (wL + rK) \;= 20(512) - 5(1024) - 4(1024) = 1024$.

(b) If P rose from 20 to 22, re-solving gives: $K = L = 2655.992$; $Q = 1206.863$; $\Pi = 2647.058$. Thus the 10% rise in P (from 20 to 22) induces an increase in output of 135% (from 512 to 1206.863) so (arc) supply elasticity is $\frac{135}{10} = 13.5$; highly elastic.

(c) From equation (2) in (1)(a) above, we have: $\lambda 0.5K^{0.4}L^{-0.5} = 5$ and similarly from equation (3): $\lambda 0.4K^{-0.6}L^{0.5} = 4$. Dividing the first equation by the second:

$\frac{0.5}{0.4} \cdot \frac{K^{0.4}L^{-0.5}}{K^{-0.6}L^{0.5}} = \frac{5}{4}$, which simplifies to $K = L$

Diagrammatically, if we draw the isoquants with (as usual) K on the vertical and L on the horizontal axis, then draw a line from the origin with slope of 1 (or 45 degrees); then where this line cuts the isoquants gives the cost-minimising combination of K and L; that is, $K = L$.

(d) We have: $TC = wL + rK = 5L + 4K$. From (c) above we have $K = L$ when costs are minimised. Combining these, $TC = wL + rL = (w + r)L = (w + r)K$

Also, with $K = L$ the production function can be written as $Q = K^{0.4}K^{0.5} = K^{0.9}$

from which $K = Q^{\frac{1}{0.9}} = Q^{1.111}$. Substituting this into the TC function gives

$TC = (w + r)Q^{1.111} = 9Q^{1.111}$

The graph of this becomes steeper as Q increases, because the exponent of Q is greater than 1. We can confirm this by finding marginal cost as:

$MC \equiv \frac{dTC}{dQ} = (1.111)9Q^{0.111}$. Here we see that MC increases as Q increases;

more rigorously, if we take the second derivative $\frac{d^2TC}{dQ^2}$ we find it is positive.

Also average cost, $AC \equiv \frac{TC}{Q} = \frac{9Q^{1.111}}{Q} = 9Q^{0.111}$, increases as Q increases.

(Note that $MC > AC$ at every level of output, because

$MC = (1.111)9Q^{0.111} > AC = 9Q^{0.111}$. This is expected, because we know that when AC is rising, we must have $MC > AC$.

The marginal cost curve is upward sloping; whereas marginal revenue is constant (and equal to price) irrespective of output, so is a horizontal line. The intersection of the upward sloping MC curve and the horizontal MR curve gives the most profitable output.

Progress exercise 16.5

1. (a) The problem is to maximise $U = X^2Y^3$ subject to budget constraint

$B = P_X X + P_Y Y$; in this case, $120 = 4X + 3Y$. The Lagrangean equation is:

$V = X^2Y^3 + \lambda[4X + 3Y - 120]$, so:

$V_X = 2XY^3 + 4\lambda = 0$ (1)

$V_Y = 3Y^2X^2 + 3\lambda = 0$ (2)

$V_\lambda = 4X + 3Y - 120 = 0$ (3)

Dividing equation (1) by equation (2), after slight rearrangement:

$\frac{2XY^3}{3Y^2X^2} = \frac{-4\lambda}{-3\lambda} = \frac{4}{3}$. This simplifies to: $Y = 2X$. Substituting this into equation

(3) we get: $4X + 3(2X) = 120 \quad \Rightarrow \quad X = 12$ so $Y = 2X = 24$

(b) The graph is identical to figure 16.14, with X_0 = 12, Y_0 = 24,

B_0 = $120 = 4X + 3Y$, and the maximised level of utility, $U_0 = (12)^2(24)^3$.

(c) (i) $X = 12$, so $P_X X = 4(12) = 48$. Similarly, $Y = 24$, so $P_Y Y = 3(24) = 72$

(ii) $\frac{P_X X}{B} = \frac{48}{120} = 40\%$; $\frac{P_Y Y}{B} = \frac{72}{120} = 60\%$

(d) If $P_X = 6$, equation (1) becomes $2XY^3 + 6\lambda = 0$. Following same steps as in (a) we get:

$\frac{Y}{X} = \frac{6}{3} \cdot \frac{3}{2} = 3$, so $Y = 3X$. Thus in equation (3):

$6X + 3(3X) = 120 \quad \Rightarrow \quad X = 8$, so $Y = 3X = 24$. Thus consumption of *X* falls from 12 to 8, but consumption of *Y* is unchanged. (Unchanged consumption of *Y* following a change in P_X is not generally to be expected, but is a feature of the Cobb-Douglas utility function.)

(e) New expenditures are $P_X X = 6 \times 8 = 48$ and $P_Y Y = 3 \times 24 = 72$. Expenditure shares are unchanged at $\frac{48}{120} = 40\%$ and $\frac{72}{120} = 60\%$.

(f) Following an increase in the price of *X*, the quantity purchased falls so as to leave total expenditure, $P_X X$, unchanged. This is a property of a demand function with a constant own-price elasticity equal to -1.

2. (a) The Lagrangean is now: $V = X^\alpha Y^\beta + \lambda(P_X X + P_Y Y - B)$. So:

$$V_X = \alpha X^{\alpha-1} Y^\beta + \lambda P_X = 0 \qquad (1)$$

$$V_Y = \beta X^\alpha Y^{\beta-1} + \lambda P_Y = 0 \qquad (2)$$

$$V_\lambda = P_X X + P_Y Y - B = 0 \qquad (3)$$

Dividing (1) by (2), preceded by a little re-arrangement, gives:

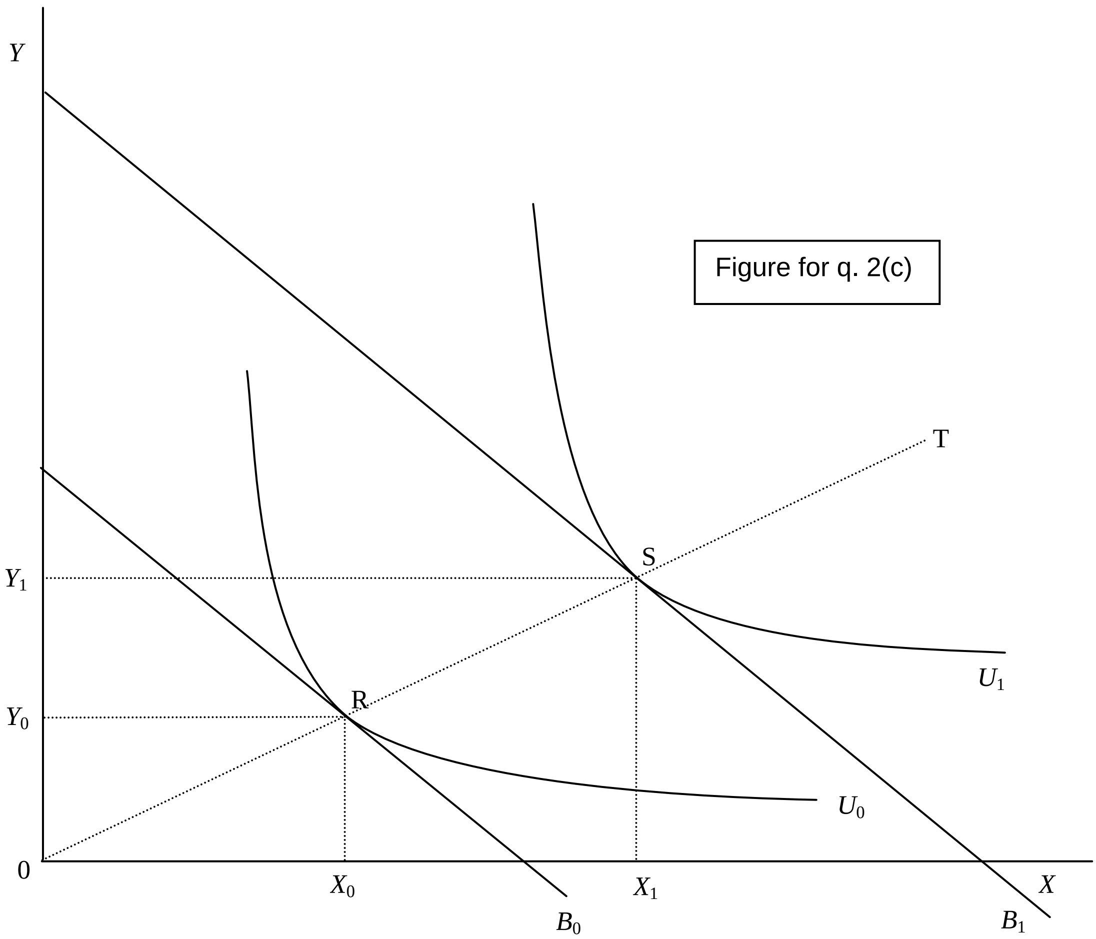
$$\frac{\alpha X^{a-1} Y^\beta}{\beta X^\alpha Y^{\beta-1}} = \frac{P_X}{P_Y} \qquad (4)$$

The left hand side of this equation is the ratio of the marginal utility of *X* to the marginal utility of *Y*; that is, the consumer's marginal rate of substitution between *X* and *Y*. This must equal the ratio of prices (the right hand side) as a necessary condition for utility maximisation. (It is not however a sufficient condition for utility maximisation, since it is also necessary that equation (3) be satisfied, and also that the relevant 2nd order conditions are satisfied – which we did not consider in the book.)

(b) The relevant diagram is again figure 16.14.

(c) Equation (4) simplifies to: $\frac{\alpha}{\beta}\frac{Y}{X}=\frac{P_X}{P_Y} \Rightarrow \frac{Y}{X}=\frac{\beta}{\alpha}\frac{P_X}{P_Y}$ (5)

So the ratio of *Y* consumption to *X* consumption depends only on price ratio $\frac{P_X}{P_Y}$ (given the parameter values α and β). The ratio is independent of the budget, *B*. See figure below



When the budget increases from B_0 to B_1 the equilibrium shifts from R to S, but the proportion in which the two goods are consumed is unchanged. This is because R and S lie on a ray from the origin, so $\frac{Y_0}{X_0}=\frac{Y_1}{X_1}$. The slope of the ray 0T gives the proportion in which the two goods are consumed, and is given by $\frac{Y}{X}=\frac{\beta}{\alpha}\frac{P_X}{P_Y}$ (equation (5) above).

(d) From equation (5) above, $Y = \frac{\beta}{\alpha}\frac{P_X}{P_Y}$. Substitute this into equation (3):

$$\Rightarrow \quad B = P_X X + P_Y\left(\frac{\beta}{\alpha}\frac{P_X}{P_Y}\right)X = P_X X\left(1+\frac{\beta}{\alpha}\right) = P_X X\left(\frac{\alpha+\beta}{\alpha}\right)$$

$$\Rightarrow \quad X = \frac{\alpha}{\alpha+\beta}\frac{B}{P_X}. \text{ Similarly } Y = \frac{\beta}{\alpha+\beta}\frac{B}{P_Y}.$$

(e) From the two equations immediately above we can easily get:

$\frac{P_X X}{B} = \frac{\alpha}{\alpha+\beta}$ and $\frac{P_Y Y}{B} = \frac{\beta}{\alpha+\beta}$. Here $\frac{P_X X}{B}$ gives the proportion of budget spent on X, and similarly $\frac{P_Y Y}{B}$ gives the proportion of the budget spent on Y.

(f) From (d) we have the demand function for X as: $X = \frac{\alpha}{\alpha+\beta}\frac{B}{P_X}$. The graph of this relationship between X and P_X (with α, β, and B as parameters) is a rectangular hyperbola. Total expenditure on X is $P_X X = \frac{\alpha}{\alpha+\beta}B$. The right hand side of this is constant, unless B changes (or the consumer's tastes, reflected in α and β, change). So if P_X doubles, X will halve and vice versa. The same is true of the demand function for Y, which is: $Y = \frac{\beta}{\alpha+\beta}\frac{B}{P_Y}$.

(g) Again, using (e) above, we have the expenditure shares as: $\frac{P_X X}{B} = \frac{\alpha}{\alpha+\beta}$ and $\frac{P_Y Y}{B} = \frac{\beta}{\alpha+\beta}$. Since α and β are parameters, if B doubles or halves then $P_X X$ and $P_Y Y$ must also double or halve to continue to satisfy these equations. Since P_X and P_Y are constants, this necessitates that X and Y, the quantities, must double or halve.

(h) If B, P_X and P_Y increase to 2B, $2P_X$ and $2P_Y$ then the right hand side of the equation $\frac{\alpha X^{a-1}Y^{\beta}}{\beta X^{\alpha}Y^{\beta-1}} = \frac{P_X}{P_Y}$ obtained in (a) above becomes $\frac{2P_X}{2P_Y}$ and since the 2's cancel this means that the equation is left unchanged. Equation (3) becomes $V_\lambda = 2P_X X + 2P_Y Y - 2B = 2(P_X X + P_Y Y - B) = 0$ which has the same solution as the original equation (3). So doubling B, P_X and P_Y does not change the equilibrium values of X and Y.