Progress exercise 15.1

1. The partial derivatives are: $\frac{\partial z}{\partial x} = 60 - 12x - 4y$ and $\frac{\partial z}{\partial y} = 34 - 4x - 6y$.

Setting these equal to zero and solving simultaneously gives a stationary point

at $x = 4$, $y = 3$. The second derivatives are: $\frac{\partial^2 z}{\partial x^2} = -12$; $\frac{\partial^2 z}{\partial y^2} = -6$; $\frac{\partial^2 z}{\partial x \partial y} = -4$; and

$\frac{\partial^2 z}{\partial y \partial x} = -4$. Since at $x = 4$, $y = 3$, $\frac{\partial^2 z}{\partial x^2}$ and $\frac{\partial^2 z}{\partial y^2}$ are both negative, and

$\frac{\partial^2 z}{\partial x^2} \cdot \frac{\partial^2 z}{\partial y^2} > \frac{\partial^2 z}{\partial x \partial y} \cdot \frac{\partial^2 z}{\partial y \partial x}$, the SP is a maximum.

2. The simultaneous equations are: $\frac{\partial z}{\partial x} = 8x - y - 3x^2 = 0$, and $\frac{\partial z}{\partial y} = -x + 2y = 0$,

with solutions $x = y = 0$ and $x = \frac{5}{2}$, $y = \frac{5}{4}$. The second derivatives are: $\frac{\partial^2 z}{\partial x^2} = 8 - 6x$,

$\frac{\partial^2 z}{\partial y^2} = 2$, and $\frac{\partial^2 z}{\partial x \partial y} = \frac{\partial^2 z}{\partial y \partial x} = -1$. At $x = y = 0$, $\frac{\partial^2 z}{\partial x^2}$ and $\frac{\partial^2 z}{\partial y^2}$ are both positive and

$\frac{\partial^2 z}{\partial x^2} \cdot \frac{\partial^2 z}{\partial y^2} = (8)(2) = 16$, while $\frac{\partial^2 z}{\partial x \partial y} \cdot \frac{\partial^2 z}{\partial y \partial x} = (-1)(-1) = 1$, so

$\frac{\partial^2 z}{\partial x^2} \cdot \frac{\partial^2 z}{\partial y^2} > \frac{\partial^2 z}{\partial x \partial y} \cdot \frac{\partial^2 z}{\partial y \partial x}$ and this point is therefore a minimum.

At $x = \frac{5}{2}$, $y = \frac{5}{4}$, $\frac{\partial^2 z}{\partial x^2} = 8 - 6\left(\frac{5}{2}\right) = -7$ and $\frac{\partial^2 z}{\partial y^2} = 2$ (opposite signs). And

$\frac{\partial^2 z}{\partial x^2} \cdot \frac{\partial^2 z}{\partial y^2} < \frac{\partial^2 z}{\partial x \partial y} \cdot \frac{\partial^2 z}{\partial y \partial x}$. So a saddle point.

3. (a) Using same method as (1) and (2) above, minimum at $x = 5$, $y = 6$

(b) Using same method as (1) and (2) above, maximum at $x = 10$, $y = 4$

(c) Using same method as (1) and (2) above, minimum at $X = 6$, $Y = 2$

(d) $f_x = 3x^2 - 3 = 0 \Rightarrow x = 1$ or -1; $f_y = 3y^2 - 12 = 0 \Rightarrow y = 2$ or -2;

$f_{xx} = 6x$; $f_{yy} = 6y$; $f_{yx} = 0$; $f_{xy} = 0$. This example is unusual in that f_x and therefore f_{xx} are functions only of x; and f_y and therefore f_{yy} are functions only of y.

So we have 4 cases in which f_x and f_y are both zero:

When $x = 1$ and	$y = 2$	Here	$f_{xx} > 0$,	$f_{yy} > 0$,	$f_{xx} \cdot f_{yy} > f_{yx} \cdot f_{xy}$	$\Rightarrow$	Minimum
When $x = 1$ and	$y = -2$	Here	$f_{xx} > 0$,	$f_{yy} < 0$,	$f_{xx} \cdot f_{yy} < f_{yx} \cdot f_{xy}$	$\Rightarrow$	Saddle point
When $x = -1$ and	$y = 2$	Here	$f_{xx} < 0$,	$f_{yy} > 0$,	$f_{xx} \cdot f_{yy} < f_{yx} \cdot f_{xy}$	$\Rightarrow$	Saddle point
When $x = -1$ and	$y = -2$	Here	$f_{xx} < 0$,	$f_{yy} < 0$,	$f_{xx} \cdot f_{yy} > f_{yx} \cdot f_{xy}$	$\Rightarrow$	Maximum

Progress exercise 15.2

1. (a) $dz = (2x + y)\,dx + (x + 2y)\,dy$

(b) $dz = \left(0.25x^{-0.75}y^{0.5}\right)dx + \left(0.5x^{0.25}y^{-0.5}\right)dy$

(c) $dz = \left(\alpha x^{\alpha-1}y^{\beta}\right)dx + \left(\beta x^{\alpha}y^{\beta-1}\right)dy$

(d) $\dfrac{\partial z}{\partial x} = 0.5\left(x^2 + y^3\right)^{-0.5}(2x) = x\left(x^2 + y^3\right)^{-0.5}$;

$$\frac{\partial z}{\partial y} = 0.5\left(x^2 + y^2\right)^{-0.5}\left(3y^2\right) = \tfrac{3}{2}y^2\left(x^2 + y^2\right)^{-0.5}$$

So $dz = \left[x\left(x^2 + y^3\right)^{-0.5}\right]dx + \left[\tfrac{3}{2}y^2\left(x^2 + y^3\right)^{-0.5}\right]dy$

2. (a) If lawn has sides *x* and *y*, area *A* is: $A = xy$ with $\frac{\partial A}{\partial x} = y$ and $\frac{\partial A}{\partial y} = x$. So total differential is: $\mathrm{d}A = y\ \mathrm{d}x + x\ \mathrm{d}y$. Here $x = 10$, $y = 5$ and $\mathrm{d}x = \mathrm{d}y = 1$, so: $\mathrm{d}A = 5(1) + 10(1) = 15$. So, if I use the total differential to make the calculation, I conclude that I must buy 15 square metres of turf.

The true value is $A_1 - A_0$ where $A_0 = \text{initial area}$, $A_1 = \text{new area}$. So $A_0 = 10 \times 5 = 50$, $A_1 = 11 \times 6 = 66$ so $A_1 - A_0 = 16$ square metres. So using the total differential I would buy 15 square metres when I actually need 16.

(b) Diagram is identical to figure 15.12 with x_0 = 10, y_0 = 5 and d*x* = d*y* = 1. Error = area C = 1 metre × 1 metre = 1 square metre.

3. Total differential is: $\mathrm{d}z = 2x\ \mathrm{d}x + 2y\ \mathrm{d}y$. Here $x = 5$, $y = 10$ and $\mathrm{d}x = \mathrm{d}y = 0.1$, so: $\mathrm{d}z = 10(0.1) + 20(0.1) = 3$. The true change is $z_1 - z_0 = \left(x_1^2 + y_1^2\right) - \left(x_0^2 + y_0^2\right)$ where $x_0 = 5$, $y_0 = 10$, $x_1 = 5.1$, $y_1 = 10.1$. So $z_1 - z_0 = (5.1)^2 + (10.1)^2 - \left(5^2 + 10^2\right) = 3.02$. So error is 0.02 which, as a percentage of the true value, is $\frac{0.02}{3.02}(100) = 0.662\%$

Progress exercise 15.3

1.(a) From $z = x^3 + y^2$, total differential is $\mathrm{d}z = 3x^2\ \mathrm{d}x + 2y\ \mathrm{d}y$. Divide through by d*x* $\Rightarrow$ $\frac{\mathrm{d}z}{\mathrm{d}x} = 3x^2 + 2y\frac{\mathrm{d}y}{\mathrm{d}x}$. Given $y = x^2$ so $\frac{\mathrm{d}y}{\mathrm{d}x} = 2x$. Using this to substitute 2*x* in place of $\frac{\mathrm{d}y}{\mathrm{d}x}$, we get: $\frac{\mathrm{d}z}{\mathrm{d}x} = 3x^2 + 2y\frac{\mathrm{d}y}{\mathrm{d}x} = 3x^2 + 2y(2x) = 3x^2 + 4xy$.

Finally, using $y = x^2$ to substitute for y, this becomes: $\frac{\mathrm{d}z}{\mathrm{d}x} = 3x^2 + 4x^3$

(b) $dz = \left(6x + 4y^3\right) dx + 12xy^2\ dy$. Divide through by $dx \Rightarrow$

$\frac{dz}{dx} = 6x + 4y^3 + 12xy^2 \frac{dy}{dx}$. Given: $y = (x+1)^2 \Rightarrow \frac{dy}{dx} = 2(x+1)$. So

$\frac{dz}{dx} = 6x + 4y^3 + 24xy^2(x+1)$. Using $y = (x+1)^2$ to substitute for y, this

becomes: $\frac{dz}{dx} = 6x + 4(x+1)^6 + 24x(x+1)^5$

(c) $dz = u\ dv + v\ du$. Divide through by $dx \Rightarrow \frac{dz}{dx} = u\frac{dv}{dx} + v\frac{du}{dx}$. Given

$u = 3x + 2$, $\frac{du}{dx} = 3$; and $v = 3x^2$ so $\frac{dv}{dx} = 6x$. So: $\frac{dz}{dx} = u(6x) + v(3)$

$= (3x+2)(6x) + 3x^2(3) \quad = 18x^2 + 12x + 9x^2 = 27x^2 + 12x \quad = 3x(9x+4)$.

2. (a) Given: $TR = pq$, total differental is: $dTR = p\ dq + q\ dp$. Divide through by

$dq \quad \Rightarrow \frac{dTR}{dq} = p + q\frac{dp}{dq}$. Given $p = 50e^{-0.5q}$,

$\frac{dp}{dq} = -0.5\left(50e^{-0.5q}\right) = -0.5p$ (since $p = 50e^{-0.5q}$). So $\frac{dTR}{dq} = p - 0.5pq$

$= p(1 - 0.5q)$.

(b) By direct substitution, $TR = pq = q\left(50e^{-0.5q}\right)$. Since we now have TR as a function of q alone, we can simply differentiate, using product rule, and find $\frac{dTR}{dq}$ as: $\frac{dTR}{dq} = q\left((-0.5)50e^{-0.5q}\right) + \left(50e^{-0.5q}\right)(1)$. Since $p = 50e^{-0.5q}$, we have $\frac{dTR}{dq} = -0.5pq + p = p(1-0.5q)$, as before. Here total derivative, $\frac{dTR}{dq} \equiv$ marginal revenue.

(c) From $MR \equiv \frac{dTR}{dq} = p + q\frac{dp}{dq}$ we see that MR has two components: (i) the change in revenue from selling 1 more unit, which obviously equals p; (ii) the change in revenue from the price reduction necessary to sell 1 more unit, which equals $q\dfrac{dp}{dq}$ (= the price reduction × initial quantity sold, and is negative since $\dfrac{dp}{dq} < 0$)

(d) $\dfrac{dTR}{dq} \equiv$ marginal revenue.

3. (a) Let $z = 2x - 3y$. Then $dz = 2\,dx - 3\,dy$ but $z = 0$ so $dz = 0$ $\Rightarrow$

$2\,dx - 3\,dy = 0$. This re-arranges to give: $\dfrac{dy}{dx} = \dfrac{2}{3}$.

(b) Same method as (a) above gives: $(2x + y)\,dx + (y + 2x)\,dy = 0$, from which

$$\frac{dy}{dx} = -\frac{2x + y}{x + 2y}$$

(c) $\left(\alpha x^{\alpha-1}y^{\beta}\right)dx + \left(\beta x^{\alpha}y^{\beta-1}\right)dy = 0$, so $\dfrac{dy}{dx} = -\dfrac{\alpha x^{\alpha-1}y^{\beta}}{\beta x^{\alpha}y^{\beta-1}} = -\dfrac{\alpha}{\beta}\dfrac{y}{x}$

(d) $0.5\left(x^2 + 3y\right)^{-0.5}(2x)\,dx + 0.5\left(x^2 + 3y\right)^{-0.5}(3)\,dy = 0$, so $\dfrac{dy}{dx} = -\tfrac{2}{3}x$

4. (a) Given $z = x^3 + y^2$ with differential: $dz = 3x^2\,dx + 2y\,dy$. But along any iso-z section, z is constant by definition so $dz = 0$. So: $3x^2\,dx + 2y\,dy = 0$, from which: $\dfrac{dy}{dx} = -\dfrac{3x^2}{2y} =$ slope of any iso-z section

(b) Using same method as (a): $dz = \left(6x + 4y^3\right)dx + 12xy^2\,dy = 0$, from which:

$$\frac{dy}{dx} = -\frac{6x + 4y^3}{12xy^2}$$

(c) $dz = u\,dv + v\,du = 0$, from which: $\dfrac{dv}{du} = -\dfrac{v}{u}$

Progress exercise 15.4

1. (a) $\frac{\partial Q}{\partial L} \equiv MPL = 0.5K^{0.5}L^{-0.5}$; and $\frac{\partial Q}{\partial K} \equiv MPK = 0.5K^{-0.5}L^{0.5}$. Both are always positive because *K* and *L* are positive and a positive number raised to any power is positive. Graphs of *MPL* and *MPK* have the same shape as figures 14.19(a) (sic).

(b) (i) If $Q = 10$, we have $10 = K^{0.5}L^{0.5}$. This is the equation of the $Q = 10$ isoquant, as an implicit function. To obtain an explicit function, raise both sides to power 2, giving: $100 = KL \quad \Rightarrow \quad K = \frac{100}{L}$ or $L = \frac{100}{K}$

(ii) If $Q = 100$, we obtain in the same way: $K = \frac{10{,}000}{L}$ or $L = \frac{10{,}000}{K}$

(c) Total differential is $\mathrm{d}Q \equiv \frac{\partial Q}{\partial K}\mathrm{d}K + \frac{\partial Q}{\partial L}\mathrm{d}L$. Along any isoquant $\mathrm{d}Q = 0$ so

$$\frac{\partial Q}{\partial K}\mathrm{d}K = -\frac{\partial Q}{\partial L}\mathrm{d}L \quad \Rightarrow \quad \frac{\mathrm{d}K}{\mathrm{d}L} = -\frac{\frac{\partial Q}{\partial L}}{\frac{\partial Q}{\partial K}} = -\frac{0.5K^{0.5}L^{-0.5}}{0.5K^{-0.5}L^{0.5}} = -K^{0.5-(-0.5)}L^{-0.5-0.5} = -\frac{K}{L}$$

(using (a) above). The slope is negative provided *K* and *L* are positive. By definition, $\frac{\partial Q}{\partial L} = MPL$ and $\frac{\partial Q}{\partial K} = MPK$.

(d) The graphs are very similar to figure 15.14.

(e) To stay on the same isoquant when *L* changes by a small amount $\mathrm{d}L$, we require the differential $\mathrm{d}Q = MPL\,\mathrm{d}L + MPK\,\mathrm{d}K = 0$. We can re-arrange this as $\mathrm{d}K = -\frac{MPL}{MPK}\mathrm{d}L$, which gives us the change in *K*, d*K*, necessary to offset $\mathrm{d}L$ and leave *Q* unchanged. For a given $\mathrm{d}L$, $\mathrm{d}K$ is greater, the larger the *MPL* (for if *MPL* is large, $\mathrm{d}L$ has a large effect on output); and the smaller is *MPK* (for if *MPK* is small, $\mathrm{d}K$ must be large to achieve the necessary change in output.

2. (a) $MPL \equiv \dfrac{\partial Q}{\partial L} = 4K - 2L$. Assuming K and L both positive, *MPL* > 0 when $\frac{K}{L} > \frac{1}{2}$.

Similarly, $MPK \equiv \dfrac{\partial Q}{\partial K} = 4L - 2K > 0$ when $\frac{K}{L} < 2$. So both MPL and MPK are positive when both of these inequalities are satisfied; that is, when $\frac{1}{2} < \frac{K}{L} < 2$

The graph of *MPL* is a straight line with slope -2 and intercept on *MPL* (vertical) axis of $4K$, where *K* is a given constant. And similarly for the graph of *MPK*.

(b) From 1(c) above, we know that the slope of any isoquant is

$$\frac{\mathrm{d}K}{\mathrm{d}L} = -\frac{MPL}{MPK} = -\frac{4K - 2L}{4L - 2K}$$ in this example.

Any isoquant is negatively sloped if *MPK* and *MPL* are both positive. From (a) this is true when $\frac{1}{2} < \frac{K}{L} < 2$.

(c) See figure below, which shows a typical isoquant, for *Q* = 108. The graph confirms (b) above, that the marginal products are not always positive. If either marginal product is negative, $\dfrac{\mathrm{d}K}{\mathrm{d}L} = -\dfrac{MPL}{MPK}$ is positive and the isoquant is positively sloped. The isoquant is horizontal when $\dfrac{\mathrm{d}K}{\mathrm{d}L} = -\dfrac{4K - 2L}{4L - 2K}$ = 0, which is true when the numerator equals zero. With *Q* = 108, this requires *K* = 6, *L* = 12. The isoquant is vertical when the denominator equals zero, which, with *Q* = 108, requires *K* = 12, *L* = 6.

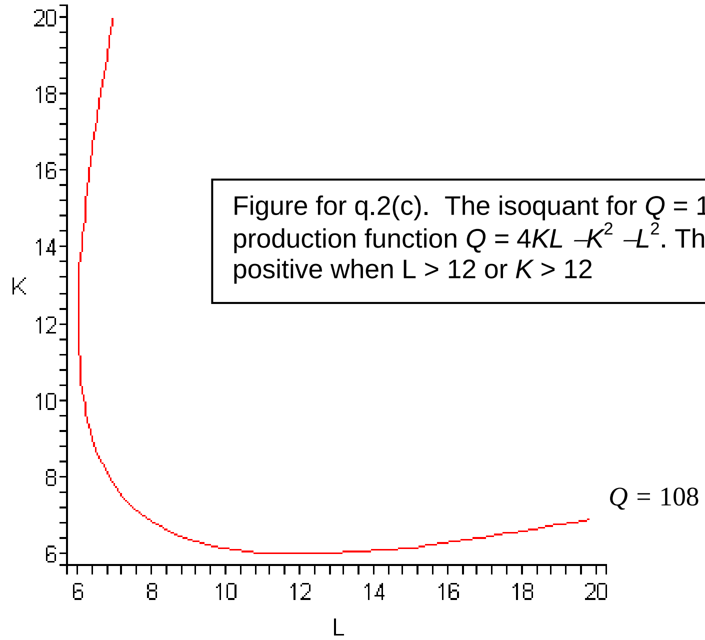

Figure for q.2(c). The isoquant for $Q = 108$ from the production function $Q = 4KL - K^2 - L^2$. The slope is positive when L > 12 or $K > 12$

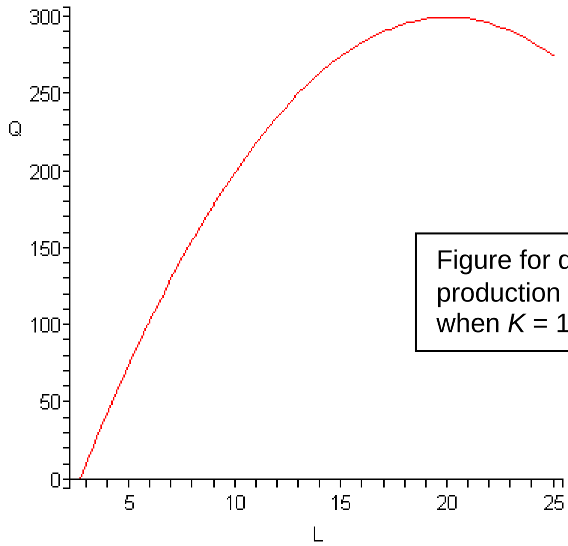

Figure for q. 2(d). The short run production function $Q = 40L - 100 - L^2$ when $K = 10$.

(d) If we fix *K* at, say, $K = 10$, the short run production function is $Q = 40L - 100 - L^2$ (see figure above). The intercepts on the *L* axis are where $40L - 100 - L^2 = 0$. This quadratic has approximate solutions $L = 2.7$ or 37.3. Q is maximised when *MPL* = 40 - 2*L* = 0; that is, when *Q* = 20.

Generalising from this, if we fix *K* at K_0, the short run production function is $Q = 4K_0L - K_0^2 - L^2$. The slope equals $\frac{\partial Q}{\partial L} = MPL = 4K_0 - 2L$. The maximum of *Q* is thus where $4K_0 - 2L = 0$. If *L* is increased beyond $L = 2K_0$, *MPL* becomes negative and total output falls.

3. (a) Differential is $\mathrm{d}U = \frac{\partial U}{\partial X}\mathrm{d}X + \frac{\partial U}{\partial Y}\mathrm{d}Y$. Along any isoquant, $\mathrm{d}U = 0$ so

$\frac{\mathrm{d}Y}{\mathrm{d}X} = -\frac{\frac{\partial U}{\partial X}}{\frac{\partial U}{\partial Y}} = -\frac{MU_x}{MU_y} = -\frac{0.75X^{-0.25}Y^{1.5}}{1.5X^{0.75}Y^{0.5}} = -\frac{1}{2}\frac{Y}{X}$. This gives the slope of any indifference curve at any point.

[As a check on this result, in Exercise 14.4, question 3(a)(ii), we found that when *U* = 27, $Y = \frac{9}{X^{0.5}}$. Substituting $Y = \frac{9}{X^{0.5}}$ into our expression above for $\frac{\mathrm{d}Y}{\mathrm{d}X}$, we obtain $\frac{\mathrm{d}Y}{\mathrm{d}X} = -\frac{1}{2}\frac{Y}{X} = -\frac{4.5}{X^{1.5}}$, which equals the value of $\frac{\mathrm{d}Y}{\mathrm{d}X}$ that we found in Ex. 14.4, question 3(a)(ii).]

4. (a) For this utility function, by implicit differentiation, slope of any indifference curve is: $\frac{\mathrm{d}Y}{\mathrm{d}X} = -\frac{\frac{\partial U}{\partial X}}{\frac{\partial U}{\partial Y}} \equiv -\frac{MU_x}{MU_y} = -\frac{0.5(X+1)^{-0.5}(Y+2)^{0.5}}{0.5(X+1)^{0.5}(Y+2)^{-0.5}} = -\frac{Y+2}{X+1}$

(b) Equation of indifference curve for $U = \overline{U}$ is $\overline{U} = (X+1)^{0.5}(Y+2)^{0.5}$. After squaring both sides and a little re-arrangement, this becomes: $Y = \frac{\overline{U}^2}{X+1} - 2$.

The first derivative is: $\frac{\mathrm{d}Y}{\mathrm{d}X} = -\overline{U}^2(X+1)^{-2} = \frac{-\overline{U}^2}{(X+1)^2}$, which is negative (provided $X \neq -1$) for all U including U = 6. The second derivative is:

$\frac{d^2Y}{dX^2} = \bar{U}^2(X+1)^{-3} = \frac{\bar{U}^2}{(X+1)^3}$ which is positive for all U including U = 6. So the indifference curve is downward sloping and convex.

(c) Given $U = (X+a)^{\alpha}(Y+b)^{\beta}$, from (a) above:

$$\frac{dY}{dX} \equiv -\frac{MU_x}{MU_y} = -\frac{\alpha(X+a)^{\alpha-1}(Y+b)^{\beta}}{\beta(X+a)^{\alpha}(Y+b)^{\beta-1}} = -\frac{\alpha}{\beta}\frac{Y+b}{X+a}$$

5. Since Larry's utility function is the same as that in 4(c) above, we know that the slope of one of Larry's indifference curves is: $\frac{dY}{dX} = -\frac{\alpha}{\beta}\frac{Y+b}{X+a}$ (equation 1).

Given Milly's utility function $U_M = \left[(X+a)^{\alpha}(Y+b)^{\beta}\right]^2 = (X+a)^{2\alpha}(Y+b)^{2\beta}$,

by implicit differentiation the slope of one of Milly's indifference curves is:

$$\frac{dY}{dX} = -\frac{2\alpha(X+a)^{2\alpha-1}(Y+b)^{2\beta}}{2\beta(X+a)^{2\alpha}(Y+b)^{2\beta-1}} = -\frac{\alpha}{\beta}\frac{Y+b}{X+a}$$ (equation (2)).

Comparing equations (1) and (2) we see that for any given values of *X* and *Y*, Larry's and Milly's indifference curves have the same slope at *any* point. This means that Milly's indifference curves must be identical to Larry's. Only the utility levels differ; the utility that Milly derives from any given bundle of *X* and *Y* equals the square of Larry's utility level from an identical bundle. But this is of no significance, since the units in which utility is measured are arbitrary.