Progress exercise 12.1

1. (a) Without using calculator:

(1) If x = ln 30, then by definition e^x = 30. Since e is (very roughly) 3, and 3^3 = 27, x must be a little more than 3 ; say 3.5.

(2) Ln 900 = ln (30 x 30) = ln[$(30)^2$] = 2 ln 30 . From (i), 2 ln 30 must be about 2 x 3.5 = 7.

(3) If x = ln 10, e^x = 10 and since e equals roughly 3 and 3^2 = 9, x must be about 2½.

(4) Ln (0.1) = ln $\left(\frac{1}{10}\right)$ = ln 1 – ln 10 = 0 - 2½ approx (using (iii) above). So answer to (4) is same as answer to (3) but with sign reversed.

(5) If x = ln 2.71828, then by definition e^x = 2.71828. But we know that e = 2.71828 (to 5 dp) so we have $e^x = e$. So x must equal 1.

(6) Since e = 2.71828 (to 5 dp), $e^3 = (2.71828)^3$. Since 3^3 = 27, $(2.71828)^3$ must be a little less than 27; say 23.

(7) $e^1 = e$ = 2.71828 approx.

(8) If x = ln 1, then e^x = 1 by definition. So x = 0.

(9) If x = ln e, $e^x = e$. So x = 1 .

(10) e^o = 1.

(b) Using calculator (to 3 dp):

(1) 3.401; (2) 6.802; (3) 2.303; (4) –2.303; (5) 1 (key strokes SHIFT or INV or 2nd F (depending on make of calculator), then ln/1); (6) 20.086 (key strokes shift/ln/3); (7) 2.71828 (to 5 dp) (key strokes shift/ln/1); (8) 0 (key strokes ln/1; (9) 1 (key stokes shift/ln/2.71828); (10) 1 (key strokes shift/lin/0)

2. (a) See Figure 12.2 in book.

(b) In Figure 12.2 the x's are the natural logs of the y's, so for example, 3 = ln 20.081. Therefore ln 20 must be very slightly less than 3, say 2.95. From question 1 (a)(4) above we know that ln $(\frac{1}{x})$ = - ln x, so

$\ln 0.05 \equiv \ln(\frac{1}{20}) = -\ln 20 = -2.95$. Or, from Figure 12.2 we can read off ln (0.05) as 3 (approx).

3. (a) See Figure 12.8.

(b) If x = ln (2.5), then e^x = 2.5 by definition. Since e^1 = 2.718 approx., which is more than 2.5, x must be a bit less than 1; say 0.85. This is confirmed by Figure 12.9. (Calculator gives 0.916)

If x = ln (0.4), then e^x = 0.4. Since e^o = 1, which is a lot more than 0.4, x must be quite a lot less than 0; say -0.85. (Calculator gives –0.916)

Note that 0.4 = $\frac{1}{2.5}$, and therefore ln 0.4 = –ln 2.5.

(c) If $y = \ln x$, then by definition the inverse function is $x = e^y$. If we interchange the labels on the axes of the graph of $y = e^x$ (Figure 12.2) it becomes the graph $x = e^y$.

4. (a) Apart from a change of scale on x axis (all values divided by 10), this graph is identical to that of $y = e^{0.5x}$ (see Figure 12.4)

(b) See Figure 12.5.

(c) Same as $y = e^x$ but with all labels on x axes divided by 2 and all labels on y axes multiplied by 100.

(d) See Figure 12.8

Progress exercise 12.2

1. (a) Annual jumps so $y = a(1 + r)^x$ with a = 100, r = 0.03 . After 1 year, y = $100(1.03)^1$ = 103. After 25 years $y = 100(1.03)^{25}$ = 209.4 (using ^ key on calculator – or y^x key on some calculators).

Graph is identical in shape to figure 12.9(b) but with slightly smaller steps (3% instead of 4% growth per year).

(b) Continuous growth so $y = ae^{rx}$ with a = 100, r = 0.03. After 1 year y = $100e^{0.03}$ = 103.045 (using ^ key). After 25 years $y = 100e^{0.03(25)}$ = $100e^{0.75}$ = 211.7

Note compounding effect; y grows by 111.7%, compared with 75% (25 year at 3% per year) if growth simple rather than compound growth.

Graph is same as figure 12.9(a) but turns up slightly more slowly as growth only 3% in this case.

2. Continuous growth so $y = ae^{rx}$ with a = 1(say), so y = 2 (doubling from 1 to 2), r = 0.03, x unknown. So $2 = e^{0.03x}$

$\Rightarrow \ln 2 = \ln(e^{0.03x}) = 0.03x \ln e$

$= 0.03x$ (since $\ln e = 1$)

so $x = \frac{\ln 2}{0.03} = \frac{0.6931}{0.03} = 23.105$ years

If $r = 0.23$ with x unknown, we have $2 = e^{r23}$

$\Rightarrow \ln 2 = 23r$

so $r = \frac{\ln 2}{23} = \frac{0.6931}{23} = 0.03$

Comparing the two problems we see that in both, $rx = 0.6931$. So when a variable doubles, the product of the growth rate, r , and the number of years, x , is always 0.6931. (The rule of 69).

Progress exercise 12.3

1. (a) $y = ae^{rx} = 100e^{0.05x}$. Graph is like $y = e^{0.5x}$ in Figure 12.4 with labels on x axes multiplied by 10, as x now needs to be 10 times as large for a given y; and labels on y axes multiplied by 100. (Note that y intercept then becomes y = 100 instead of y = 1).

(b) $y = 100e^{-0.05x}$. Graph is like Figure 12.5 but with labels on x axis multiplied by 20, and on y axis by 100.

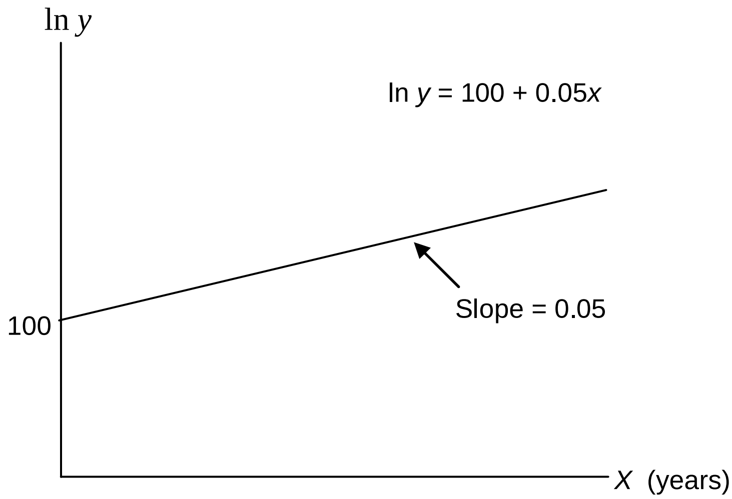
(c) In (a) $y = 100e^{0.05x}$, so taking nat. logs on both sides gives

$\ln y = \ln 100 + \ln(e^{0.05x})$

$= \ln 100 + 0.05x \ln e$

$= \ln 100 + 0.05x$ (since $\ln e = 1$)

Sketch graph next page:



In (b), following the same steps we get

$\ln y = \ln 100 - 0.05x$
Graph: as above, but with negative slope of –0.05

2. $y = 100e^{0.05x}$. Graph is like figure 12.2 in book, but with labels on x axis multiplied by 20, and on y axis by 100. Thus for example after 20 years we have $y = 100e^{0.05x} = 100e^{1} = 271.828$.

After 30 years ($x = 30$) $y = 100e^{1.5} = 100\ (4.482) = 448.2$ (using "inv" and "ln" keys).

Absolute growth: $\Delta y = 448.2 - 100 = 348.2$

Proportionate growth is $\dfrac{\Delta y}{y} = \dfrac{348.2}{100} = 3.482$

Percentage growth is $\dfrac{\Delta y}{y} \text{ x } 100 = 348.2\%$

3. $y = 100e^{-0.05x}$. Graph is like figure 12.5 in book, but with labels on x axis multiplied by 20, and on y axis by 100.

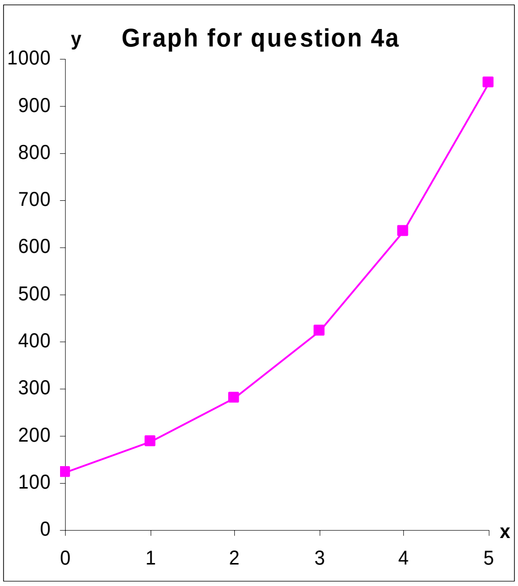
After 30 years, $y = 100e^{-0.05(30)} = 100e^{-1.5} = 100\ (0.223) = 22.3$

$\Delta y = 22.3 - 100 = -77.7$

Proportionate growth = $\dfrac{\Delta y}{y} = -\dfrac{77.7}{100} = -0.777$

% growth = -0.777 x 100 = - 77.7% (that is, the variable has decreased by 77.7% and thus is only 22.3% of its initial value). Note that when a variable is declining (negative growth) the compounding effect *reduces* the decline. In this case, if growth were simple rather than compound, a variable decrease by 5% a year for 30 years would decline by –5 × 30 = –150%, so its level would fall from 100 to –150. With compounding, it falls from 100 to 22.3.

4. (a) Graph:

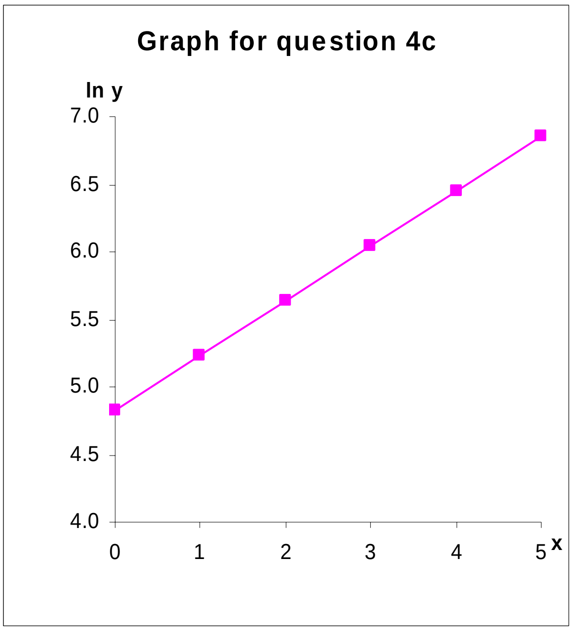


(b) Graph becomes steeper as x increases, so *absolute* growth of y is increasing, but it is impossible to say with confidence what is happening to proportionate or percentage growth.

(c) Graph next page.

Table of values:

x	ln y
0	4.828
1	5.234
2	5.639
3	6.045
4	6.450
5	6.856



(d) Graph (b) is clearly linear, which tells us that the absolute growth of log y is constant. Because the *absolute* growth of the log of any variable equals the *proportionate* growth of the variable itself, this tells us that the growth rate of y is, in fact, constant (neither increasing nor declining).

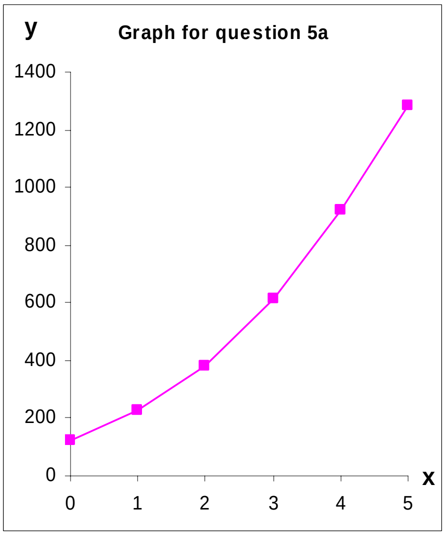
(e) The above inference is confirmed, as we have

$$\frac{187.5}{125} = \frac{281.25}{187.5} = \frac{421.88}{281.25} = \frac{632.81}{421.88} = \frac{949.22}{632.81} = 1.5$$

So growth is 50% in every year.

(f) With a log scale on the vertical axis, the slope of the graph gives us the proportionate growth rate of the variable in question.

5. (a) see graph below.

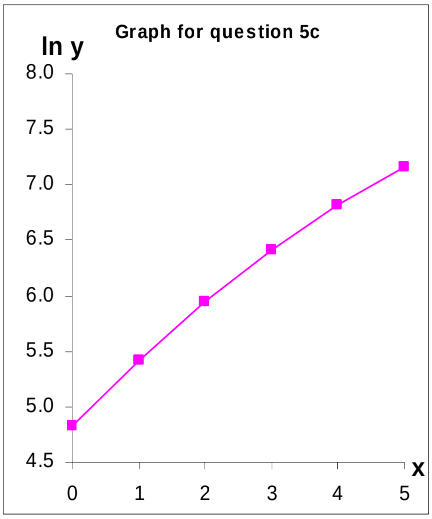


(b) Graph becomes steeper as *x* increases, but as in previous question we can't be sure what is happening to the proportionate growth rate.

(c) See graph below.

Table of values:

x	ln y
0	4.828314
1	5.4161
2	5.946729
3	6.416732
4	6.822197
5	7.15867



(d) In the graph for (c) above we see that the slope of the graph is decreasing, so we can infer that proportionate growth rate of *y* is decreasing.

(e) This is confirmed when we check the actual growth rates, which are 80%, 70%, 60%, 50% and 40% in year 1, 2, 3, 4 and 5 respectively.

6. Let G = level of government expenditure, with initial level G_o. Let Y = GDP with initial level Y_o .

(a) Assuming continuous growth, $G = G_o = G_o e^{0.03x}$ and $Y = Y^o e^{0.02x}$ where x = time.

Therefore $\frac{G}{Y} = \frac{G_o e^{0.03x}}{Y_o e^{0.02x}} = \frac{G_o}{Y_o} e^{(0.03-0.02)x} = \frac{G_o}{Y_o} e^{0.01x}$

(Remark: This tells us that the ratio of government expenditure to GDP is growing at 1% per year.)

We are also given $\frac{G_o}{Y_o} = 0.4$, so

$\frac{G}{Y} = 0.04e^{0.01x}$

Setting $\frac{G}{Y} = 0.5$, we have $0.5 = 0.4e^{0.01x}$

$\Rightarrow \frac{G}{Y} = \frac{0.5}{0.4} = e^{0.01x}$

Taking logs on both sides:

$\ln 1.25 = 0.01x$ (since $\ln e = 1$)

$\Rightarrow x = \frac{\ln 1.25}{0.01} = \frac{0.02231}{0.01} = 22.31$ years

(Remark: Note a tempting error. If $\frac{G}{Y}$ is now 40% and is growing at 1% per year, it is tempting to infer that in 10 years' time $\frac{G}{Y}$ will be 50%. In fact it takes 22 years, because 1% of 40% is 0.04, not 1, so after 1 year G/Y increases from 40 to 40.04%, not to 41% - and so on.)

(b) Growth in annual jumps.

$G = G_o(1 + 0.03)^x$; $Y = Y_o(1 + 0.02)^x$; $\frac{G_o}{Y_o} = 0.4$

$$\Rightarrow \frac{G}{Y} = \frac{G_o(1.03)^x}{Y_o(1.02)^x} = (0.04)\frac{(1.03)^x}{(1.02)^x} = 0.5$$

$\Rightarrow \log[(1.03)^x] - \log[(1.02)^x] = \log 1.25$

$x[\log(1.03)] - \log(1.02)] = \log 1.25$

$$x = \frac{\log 1.25}{\log(1.03) - \log(1.02)} = 22.87$$

7. (a) Let Y_o^s and Y_o^d be GDP of Stagnatia and Dynamica in 1995.

We have $\frac{Y_o^s}{Y_o^d} = 2$ (given).

If Stagnatia's GDP is growing continuously at annual rate of α, its GDP in year x is given by $Y^S = Y_0^S e^{\alpha x}$ where x = time.

Similarly Dynamica's GDP in year x is given by $Y^D = Y_0^D e^{\beta x}$ where β is Dynamica's growth rate.

So the ratio of GDPs in year x is $\frac{Y^s}{Y^d} = \frac{Y_o{}^S e^{\alpha x}}{Y_o{}^d e^{bx}} = 2e^{(\alpha-\beta)x}$

We also know that in 2005 (x = 10), the ratio of GDPs was 1.5. So with x = 10, we have

$$\frac{Y^s}{Y^d} = 2e^{(\alpha-\beta)10} = 1.5$$

$\Rightarrow$ $e^{(\alpha-\beta)10}$ = 0.75. Taking logs on both sides:

$10(\alpha - \beta) = \ln 0.75$ (since $\ln e = 1$)

$$\alpha - \beta = \frac{\ln 0.75}{10} = = -0.02877 = -\ 2.877\%$$

So $\beta - \alpha$ = 2.877%; that is, Dynamica's growth rate is 2.877% faster than Stagnatia's.

(b) From (a) we have $\frac{Y^s}{Y^d} = 2e^{(\alpha-\beta)x} = 2e^{(-0.02877)x}$

If we set $Y^s = Y^d$, we have $1 = 2e^{(-0.02877)x}$

=> $\ln(½) = \ln[e^{-(0.02877)x}] = (-0.02877)x$

$$\text{So} \quad x = \frac{\ln(\frac{1}{2})}{-0.02877} = 24.0927$$

So $Y^s = Y^d$ 24 years after $Y^s = 2Y^d$, which was 1990. So Y^s will equal Y^d in 2014.