Progress exercise 7.1

(a) $\frac{dy}{dx} = -2$. As this is negative for all x, the function is always negatively sloped (= monotonic decreasing).

(b) $\frac{dy}{dx} = 3x^2 + 1$. This is positive for all x, so function is monotonic increasing.

(c) Function is decreasing when $\frac{dy}{dx} = -6x + 4 < 0$, which requires $x > \frac{2}{3}$.

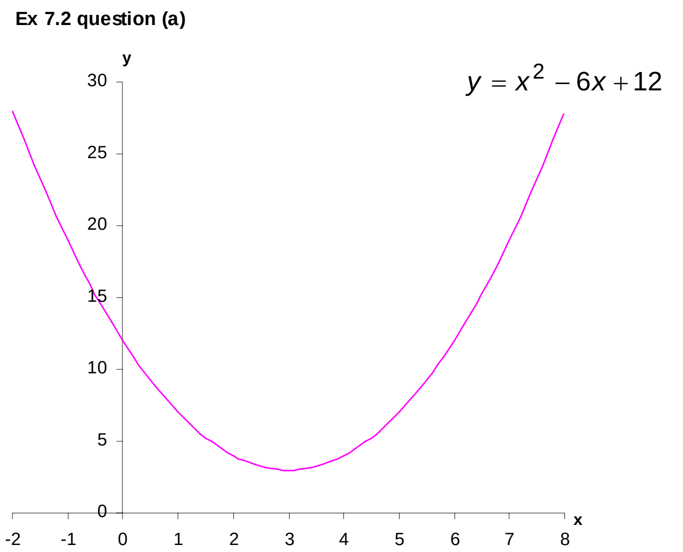
(d) $\frac{dy}{dx} = 3x^2 + 12x + 12 = 3(x+2)^2$. This is always positive. Thus when $x > -2$, $\frac{dy}{dx} > 0$ and fn is increasing.

(e) $\frac{dy}{dx} = 6x^2 + 42x + 60 = 6(x+5)(x+2)$. So $\frac{dy}{dx} > 0$ when $(x + 5)$ and $(x + 2)$ have the same sign (that is, both positive or both negative). They are both positive when $x > -2$ and both negative when $x < -5$.

Progress exercise 7.2

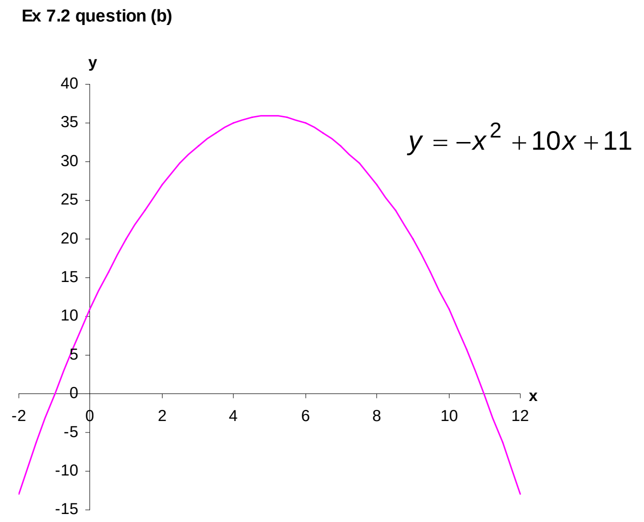
(a)

$\frac{dy}{dx} = 2x - 6$. For a maximum or minimum we require $\frac{dy}{dx} = 0 \Rightarrow 2x - 6 = 0 \Rightarrow x = 3$. Thus there is a stationary point (SP) at $x = 3$. We have $\frac{d^2y}{dx^2} = \frac{d}{dx}(2x-6) = 2$. This is positive for all values of x, including $x = 3$, so this stationary point (SP) is a minimum of y.



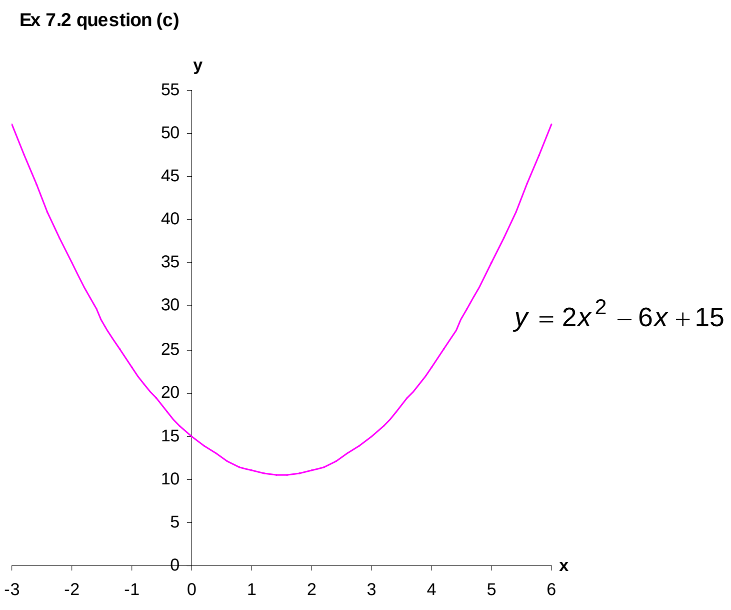
7.2 (b)

$\frac{dy}{dx} = -2x + 10 = 0 \Rightarrow 2x = 10 \Rightarrow x = 5$. There is a stationary point (SP) at $x = 5$. We have $\frac{d^2y}{dx^2} = -2$. This is negative when $x = 5$ so this SP is a maximum of y.



7.2 (c)

$\frac{dy}{dx} = 4x - 6 = 0 \Rightarrow x = \frac{3}{2}$. So SP at $x = \frac{3}{2}$. Also, $\frac{d^2y}{dx^2} = 4$. This is positive at $x = \frac{3}{2}$, so this SP is a minimum.



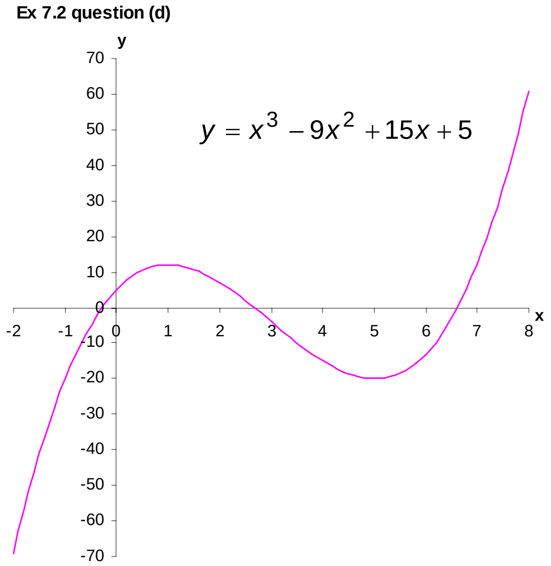
7.2 (d)

$\frac{dy}{dx} = 3x^2 - 18x + 15 = 0 \quad \Rightarrow \quad 3(x^2 - 6x + 5) = 0$

$\Rightarrow \; 3(x-5)(x-1) = 0 \quad \Rightarrow x = 5$ or 1. (We have two SPs because $\frac{dy}{dx}$ is a quadratic equation with 2 roots.) Also, $\frac{d^2y}{dx^2} = 6x - 18 = 6(x-3)$.

When $x = 5$, $\frac{d^2y}{dx^2} = 6(5-3) > 0$. So this *SP* is a minimum.

When $x = 1$, $\frac{d^2y}{dx^2} = 6(1-3) < 0$. So this *SP* is a maximum.



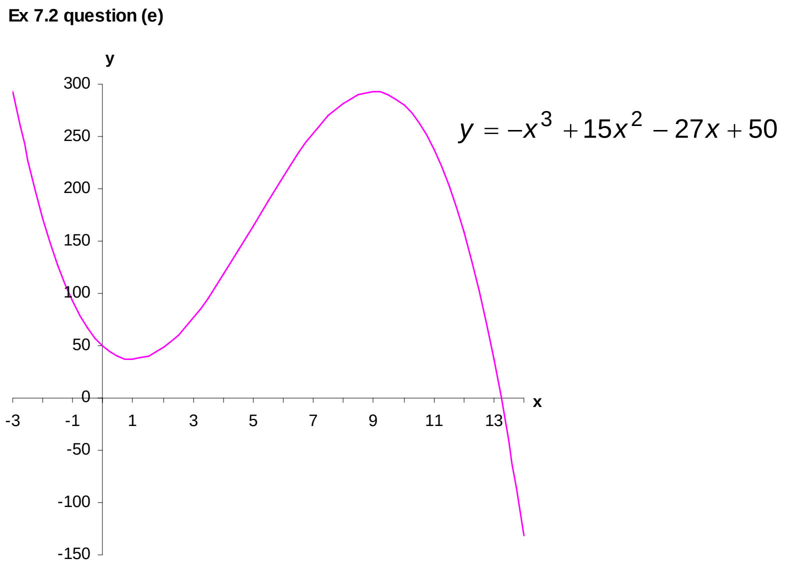
7.2 (e)

$\frac{dy}{dx} = -3x^2 + 30x - 27 = 0 \Rightarrow 3x^2 - 30x + 27 = 0$

$\Rightarrow 3(x^2 - 10x + 9) = 3(x-9)(x-1) = 0$. So we have *SP*s at $x = 9$ and $x = 1$.

$\frac{d^2y}{dx^2} = -6x + 30 = -6(x-5)$. When $x = 9$, $\frac{d^2y}{dx^2} = -6(9-5) < 0$ so maximum at $x = 9$

When $x = 1$, $\frac{d^2y}{dx^2} = -6(1-5) > 0$ so minimum at $x = 1$.



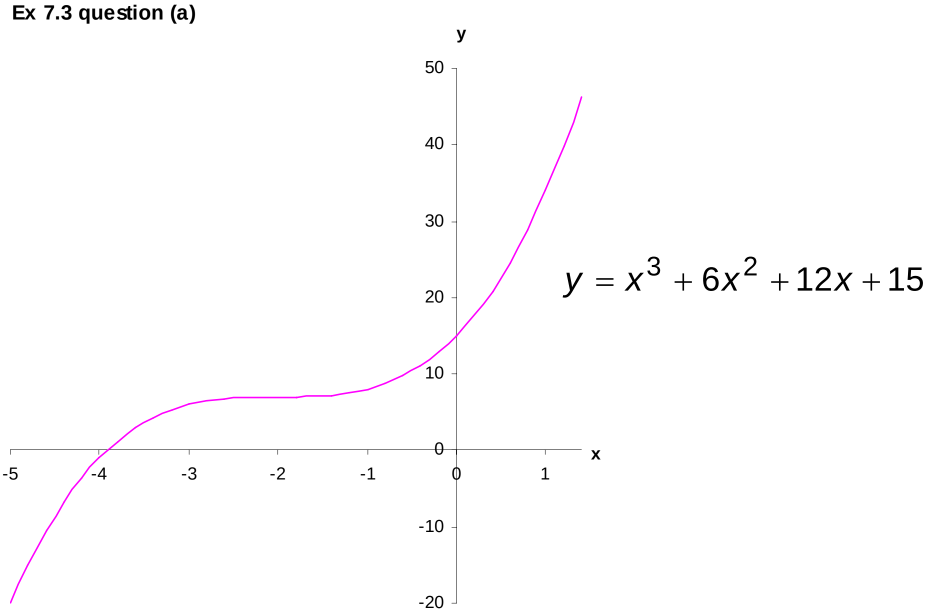
Progress exercise 7.3
(a)

$\frac{dy}{dx} = 3x^2 + 12x + 12 = 0 \Rightarrow 3(x^2 + 4x + 4) = 0 \Rightarrow 3(x+2)(x+2) = 0 \Rightarrow x = -2.$

So we have a SP when $x = -2$. We have $\frac{d^2y}{dx^2} = 6x + 12 = 6(x+2)$. When $x = -2$,

$\frac{d^2y}{dx^2} = 6(-2+2) = 0$. So this SP may be a point of inflection. We have $\frac{d^3y}{dx^3} = 6$. This is positive when $x = -2$. So at $x = -2$, we have $\frac{dy}{dx} = 0$, $\frac{d^2y}{dx^2} = 0$, $\frac{d^3y}{dx^3} \neq 0$

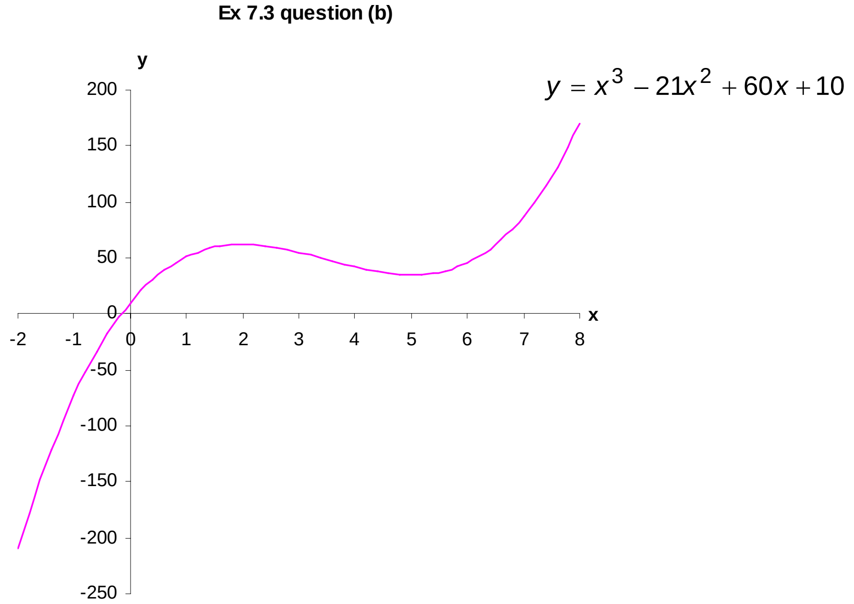
So $x = -2$ is a point of inflection.



7.3 (b)

$\frac{dy}{dx} = 6x^2 - 42x + 60 = 0 \Rightarrow 6(x-5)(x-2) = 0 \Rightarrow x = 5$ or 2. (Two SPs). We have $\frac{d^2y}{dx^2} = 12x - 42 = 6(2x-7)$. When $x = 5$, $\frac{d^2y}{dx^2} > 0$ so this SP is a minimum.

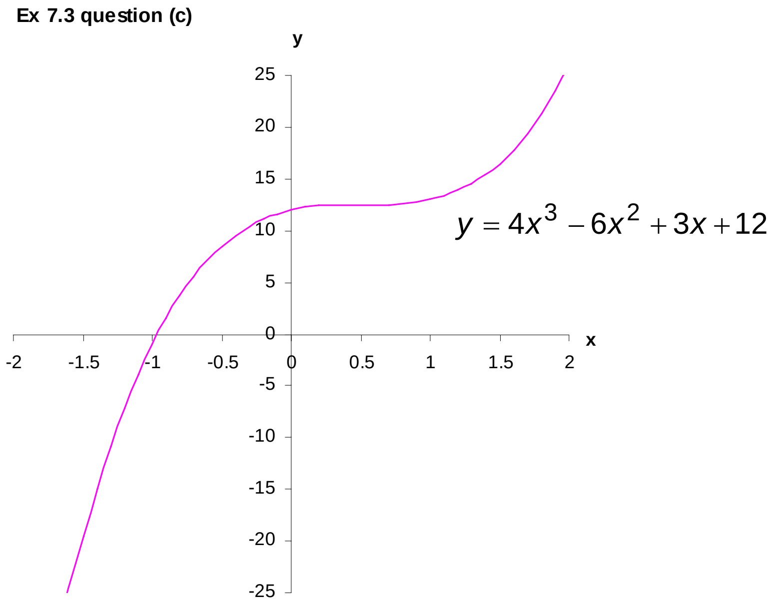
When $x = 2$, $\frac{d^2y}{dx^2} < 0$ so this SP is a maximum.



7.3 (c)

$\frac{dy}{dx} = 12x^2 - 12x + 3 = 0 \Rightarrow 12\left(x^2 - x + \frac{1}{4}\right) = 0 \Rightarrow 12\left(x - \frac{1}{2}\right)\left(x - \frac{1}{2}\right) = 0$

$\Rightarrow x = \frac{1}{2}$ (SP). We have $\frac{d^2y}{dx^2} = 24x - 12 = 12(2x - 1)$. When $x = \frac{1}{2}$, $\frac{d^2y}{dx^2} = 0$. So this SP may be a point of inflection. We have $\frac{d^3y}{dx^3} = 24 \neq 0$ at $x = \frac{1}{2}$. So this is confirmed as a point of inflection.

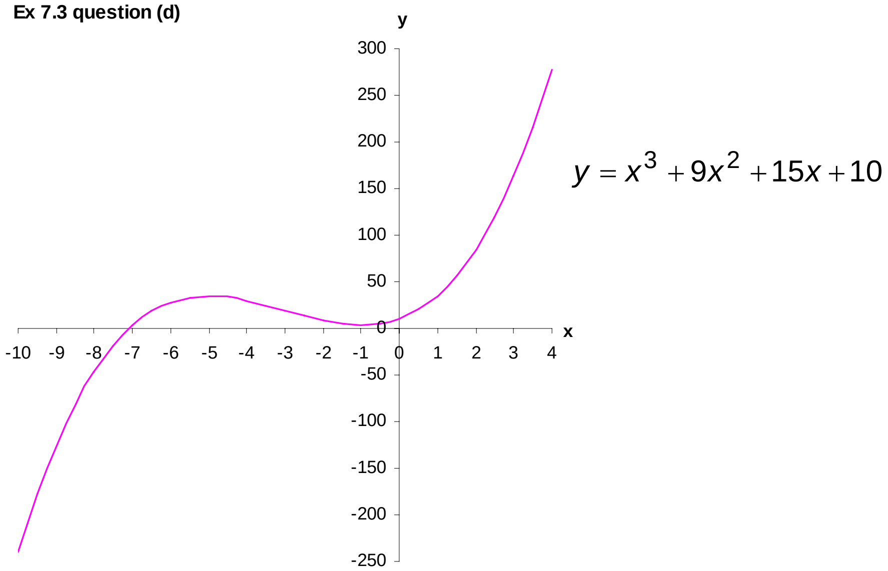


7.3 (d)

$\frac{dy}{dx} = 3x^2 + 18x + 15 = 0 \quad \Rightarrow \quad 3(x+5)(x+1) = 0 \Rightarrow x = -5 \text{ or } -1.$

$\frac{d^2y}{dx^2} = 6x + 18 = 6(x+3)$; $\frac{d^2y}{dx^2} < 0$ at x=-5 So this is a maximum

$\frac{d^2y}{dx^2} > 0$ at $x = -1$. So this is a minimum

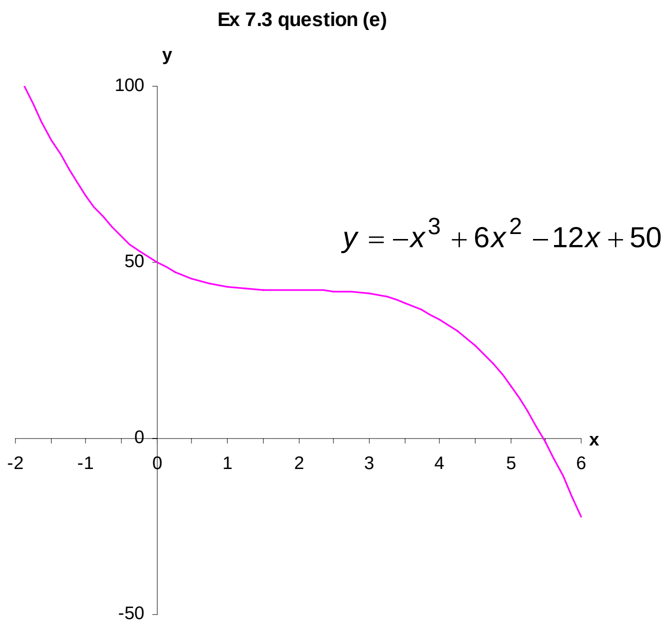


7.3 (e)

$\frac{dy}{dx} = -3x^2 + 12x - 12 = 0 \quad \Rightarrow \quad -3(x - 4x + 4) = 0 \quad \Rightarrow$

$-3(x-2)(x-2) = 0$ So $x = 2$ is a SP

When $x = 2$, $\frac{d^2y}{dx^2} = -6x + 12 = 0$ and $\frac{d^3y}{dx^3} = -6 \neq 0$, so $x = 2$ is a pt of inflection.



Progress exercise 7.4

(a)

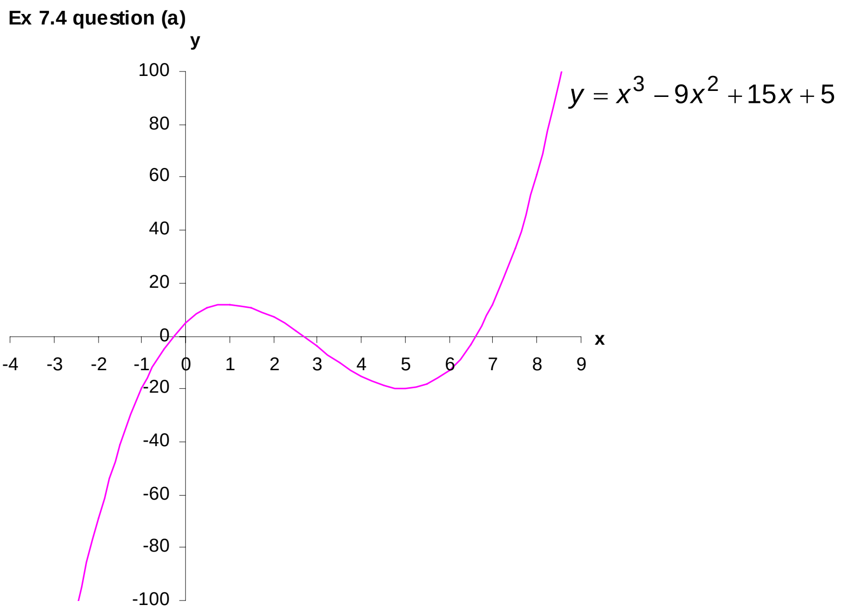
$\frac{dy}{dx} = 3x^2 - 18x + 15 = 0 \Rightarrow 3(x-5)(x-1) = 0 \Rightarrow x = 5$ or $x = 1$ are SPs.

$\frac{d^2y}{dx^2} = 6x - 18 = 6(x-3)$. So $\frac{d^2y}{dx^2} > 0$ at x = 5 ⇒ minimum.

$\frac{d^2y}{dx^2} < 0$ at x = 1 ⇒ maximum.

Point of inflection requires $\frac{d^2y}{dx^2} = 0$ and $\frac{d^3y}{dx^3} \neq 0$. In this example,

$\frac{d^2y}{dx^2} = 6(x-3) = 0 \Rightarrow x = 3$. Also $\frac{d^3y}{dx^3} = 6$. Thus at x = 3, we have $\frac{d^2y}{dx^2}$ = 0 and $\frac{d^3y}{dx^3} \neq 0$ so this is a point of inflection.



7.4 (b)

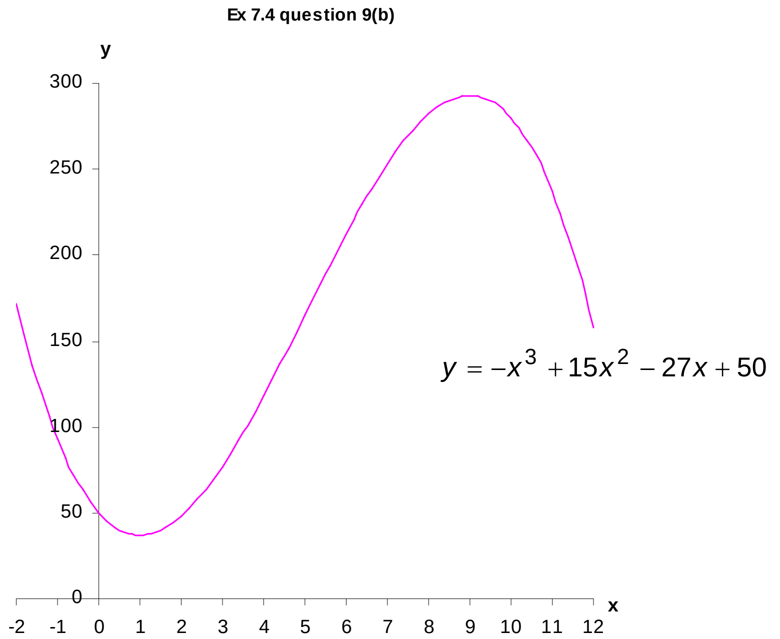
$\frac{dy}{dx} = -3x^2 + 30x - 27 = 0 \quad \Rightarrow \quad -3(x^2 - 10x + 9) = 0 \quad \Rightarrow$

$-3(x-9)(x-1) = 0 \Rightarrow x = 9$ or 1. We have $\frac{d^2y}{dx^2} = -6x + 30 = -6(x-5)$.

So $\frac{d^2y}{dx^2} < 0$ at x = 9, and this SP is a maximum; also $\frac{d^2y}{dx^2} > 0$ at x = 1 so this SP is a minimum.

Point of inflection requires $\frac{d^2y}{dx^2} = 0$ and $\frac{d^3y}{dx^3} \neq 0$. In this case,

$\frac{d^2y}{dx^2} = -6(x-5) = 0 \Rightarrow x = 5$. At x = 5, $\frac{d^3y}{dx^3} = -6 \neq 0$, so there is a point of inflection at x = 5.

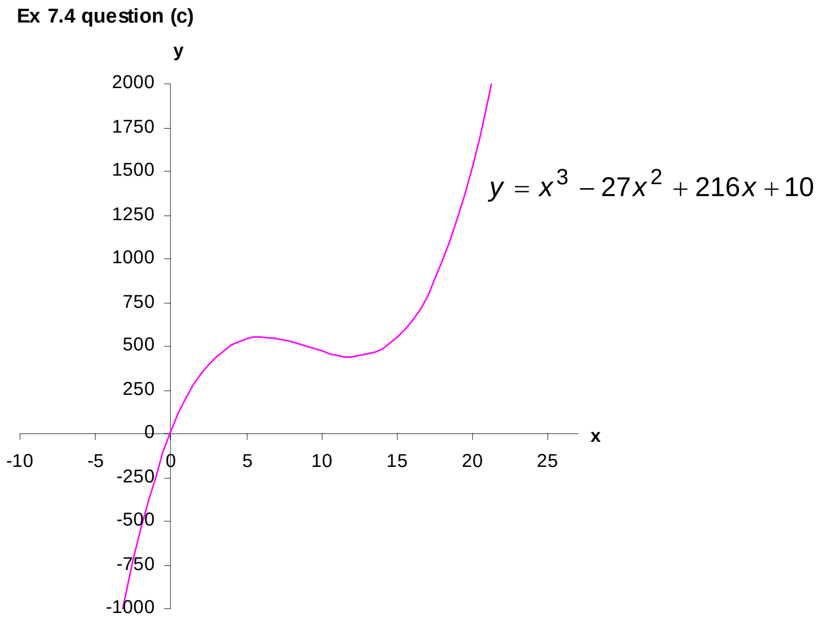


7.4 (c)

$\frac{dy}{dx} = 3x^2 - 54x + 216 = 0 \;\Rightarrow\; 3(x^2 - 18x + 72) = 0 \;\Rightarrow$

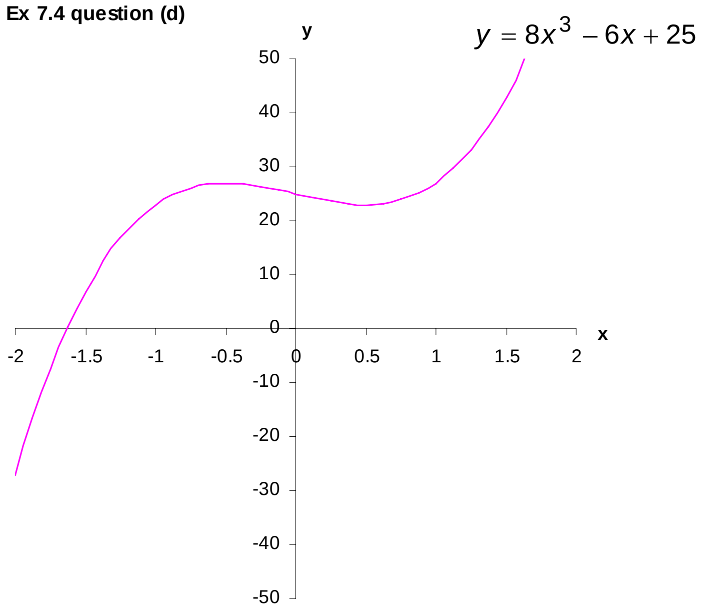
$3(x - 6)(x - 12) = 0 \;\Rightarrow\; x = 6$ or 12.

$\frac{d^2y}{dx^2} = 6x - 54 = 6(x - 9) \Rightarrow \frac{d^2y}{dx^2} < 0$ at $x = 6$, so maximum. Similarly $\frac{d^2y}{dx^2} > 0$ at $x = 12$, so minimum. For point of inflection, set $\frac{d^2y}{dx^2} = 6x - 54 = 0$, with solution $x = 9$. Then $\frac{d^3y}{dx^3} = 6 \neq 0$ at $x = 9$, so point of inflection at $x = 9$ is confirmed.



7.4 (d)

$\frac{dy}{dx} = 24x^2 - 6 = 0 \Rightarrow x^2 = \frac{6}{24} = \frac{1}{4} \Rightarrow x = \pm\frac{1}{2}$. We have $\frac{d^2y}{dx^2} = 48x > 0$ at $x = \frac{1}{2}$ so minimum; and $\frac{d^2y}{dx^2} = 48x < 0$ at $x = -\frac{1}{2}$ so maximum. Also $\frac{d^2y}{dx^2} = 48x = 0$ when $x = 0$, and $\frac{d^3y}{dx^3} = 48 \neq 0$ when $x = 0$. So $x = 0$ is a point of inflection.

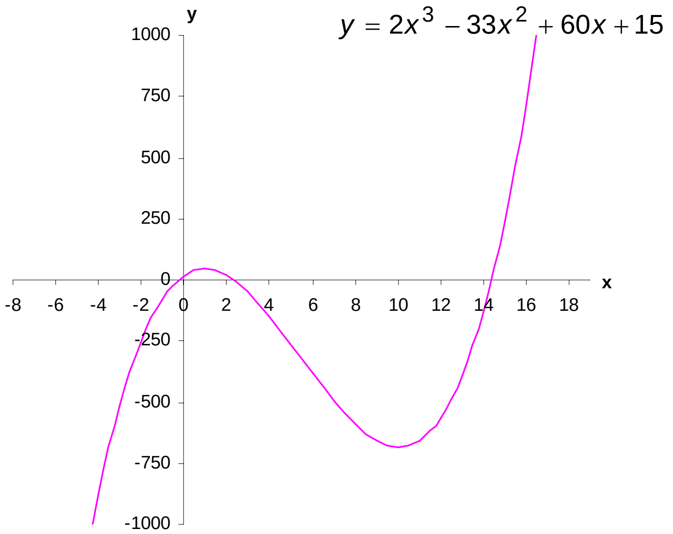


7.4 (e)

$\frac{dy}{dx} = 6x^2 - 66x + 60 = 0 \Rightarrow 6(x^2 - 11x + 10) = 0 \Rightarrow$

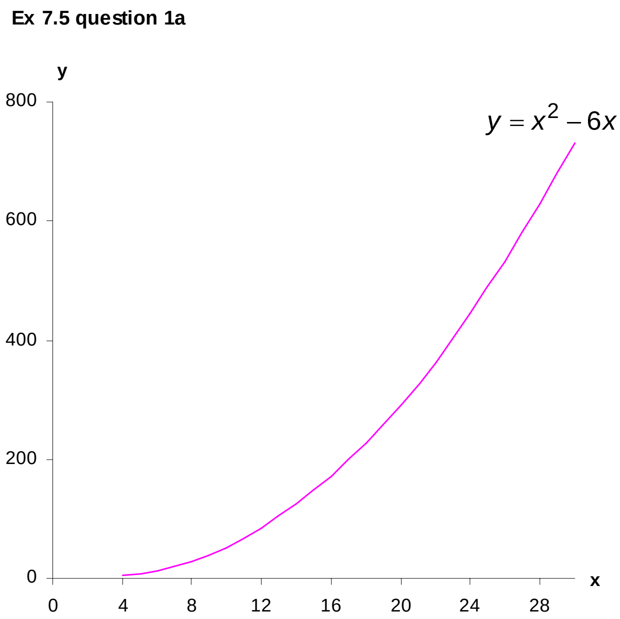
$6(x - 10)(x - 1) = 0 \Rightarrow x = 10$ or 1. We have $\frac{d^2y}{dx^2} = 12x - 66 > 0$ at $x = 10$ so minimum. And $\frac{d^2y}{dx^2} = 12x - 66 < 0$ at $x = 1$ so maximum. Point of inflection when $\frac{d^2y}{dx^2} = 12x - 66 = 0 \Rightarrow x = \frac{66}{12} = 5\frac{1}{2}$. As $\frac{d^3y}{dx^3} = 12 \neq 0$ at $x = 5\frac{1}{2}$ this is a point of inflection.

Ex 7.4 question (e)



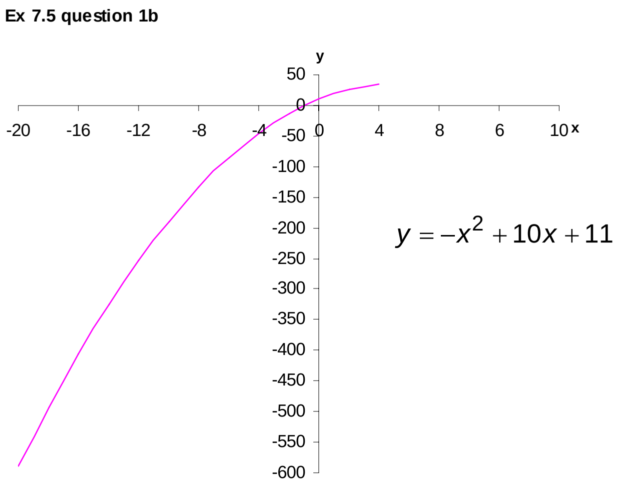
Progress exercise 7.5
Question 1 (a)

$\frac{dy}{dx} = 2x - 6 > 0$ when $x \geq 4$. Also $\frac{d^2y}{dx^2} = 2 > 0$ when $x \geq 4$. So slope is positive and increasing (that is, convex from below) when $x \geq 4$.



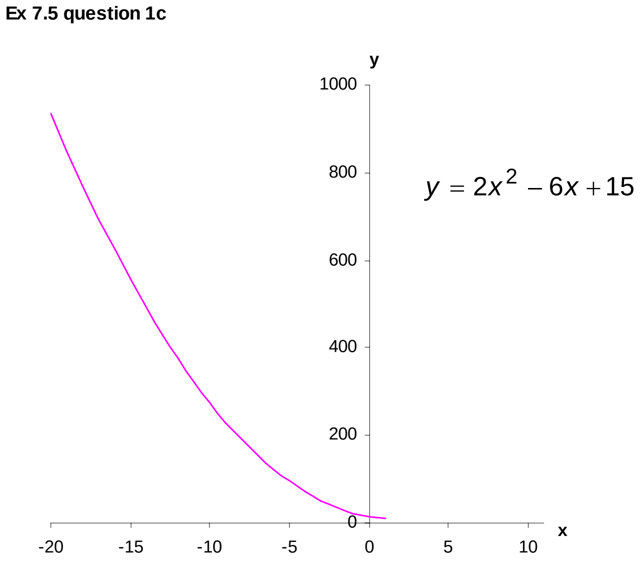
Progress exercise 7.5
Question 1 (b)

$\frac{dy}{dx} = -2x + 10 = -2(x-5) > 0$ when $x \le 4$. Also $\frac{d^2y}{dx^2} = -2 < 0$ when $x \le 4$.
So when $x \le 4$, slope is positive but decreasing ; that is, concave.



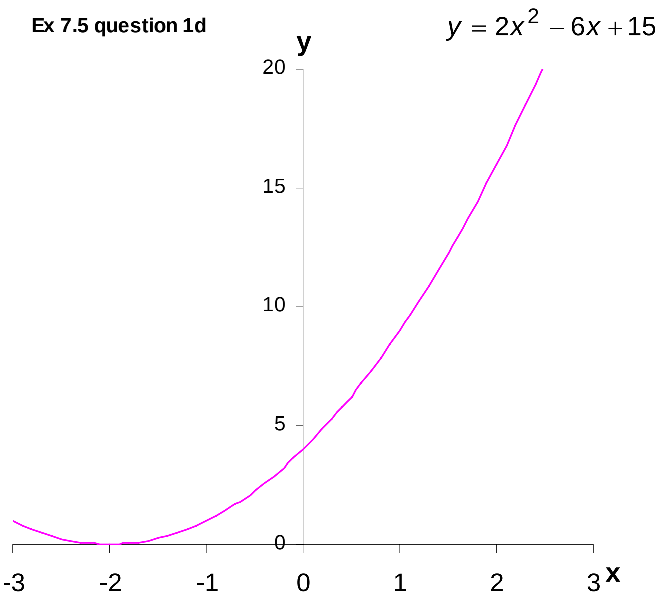
Progress exercise 7.5
Question 1 (c)

$\frac{dy}{dx} = 4x - 6 = 2(2x - 3) < 0$ when $x < 1.5$. Also $\frac{d^2y}{dx^2} = 4 > 0$ when $x < 1.5$.
So when $x < 1.5$, slope is negative and increasing; that is, convex.



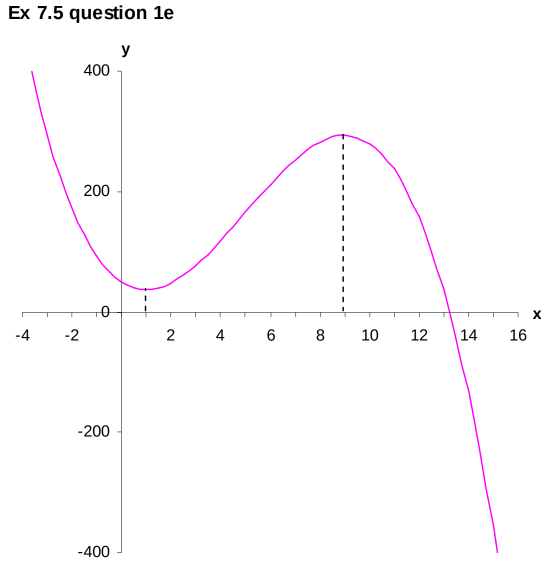
Progress exercise 7.5
Question 1 (d)

$\frac{dy}{dx} = 2x + 4 = 2(x+2) > 0$ when $x > -2$. Also $\frac{d^2y}{dx^2} = 2 > 0$ when $x > -2$. So when $x > -2$, slope is positive and increasing; that is, convex.



Progress exercise 7.5
Question 1 (e)

$\frac{dy}{dx} = -3x^2 + 30x - 27 = -3(x-9)(x-1)$. Also $\frac{d^2y}{dx^2} = -6x + 30 = -6(x-5)$. So when $x > 9$, $(x-9)$ and $(x-1)$ are both positive so $\frac{dy}{dx} < 0$; and $\frac{d^2y}{dx^2} < 0$. So slope is negative and decreasing; that is, concave. When $x < 1$, $(x-9)$ and $(x-1)$ are both negative so $\frac{dy}{dx} <$ 0; but $\frac{d^2y}{dx^2} > 0$. So slope is negative and increasing; that is, convex.

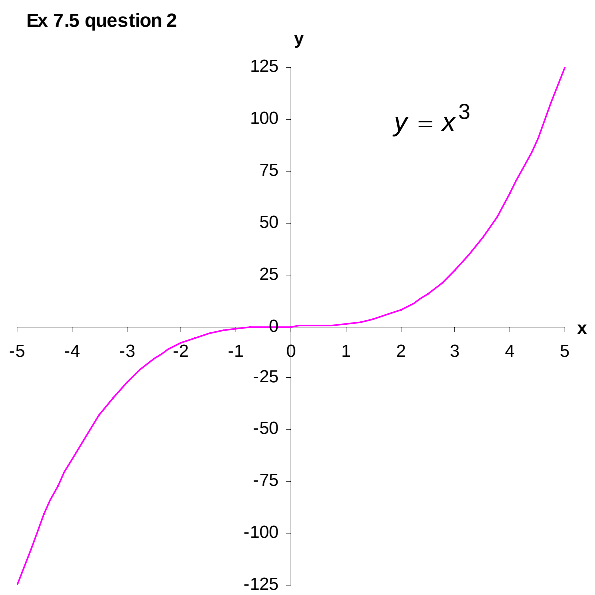


Progress exercise 7.5
Question 2

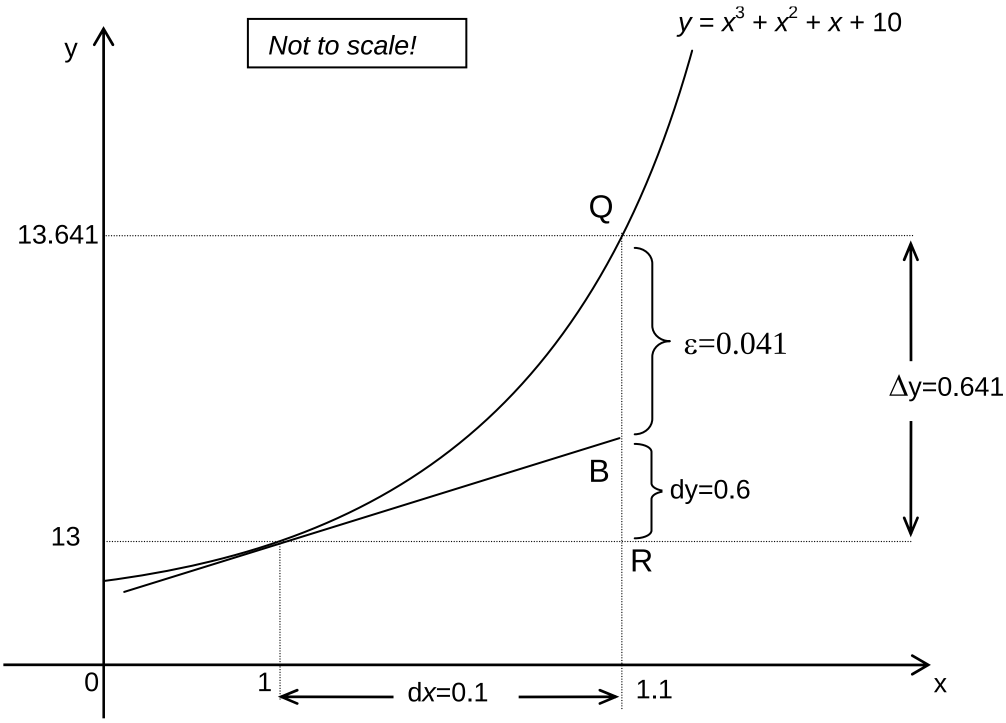
Point of inflection: $\frac{dy}{dx} = 3x^2 = 0$ when $x = 0$, so $x = 0$ is a SP. Also $\frac{d^2y}{dx^2} = 6x = 0$ when $x = 0$, and $\frac{d^3y}{dx^3} = 6 \neq 0$ when $x = 0$. So $x = 0$ is a SP (since $\frac{dy}{dx} = 0$) and is also a point of inflection since $\frac{d^2y}{dx^2} = 0$ and $\frac{d^3y}{dx^3} \neq 0$

Convexity/concavity: when $x < 0$, $\frac{dy}{dx} = 3x^2 > 0$ and $\frac{d^2y}{dx^2} = 6x < 0$, therefore concave.

When $x > 0$, $\frac{dy}{dx} > 0$ and $\frac{d^2y}{dx^2} > 0$, therefore convex.



Progress exercise 7.5
Question 3
When $x = 1$, $y = 13$; when $x = 1.1$, $y = 13.641$. So true change in y is $\Delta y = 13.641 - 13 = 0.641$. Using differential formula, $\mathrm{d}y = \frac{\mathrm{d}y}{\mathrm{d}x} \cdot \mathrm{d}x = \left(3x^2 + 2x + 1\right) \mathrm{d}x$. So when $x = 1$ and $\mathrm{d}x = 0.1$, $\frac{\mathrm{d}y}{\mathrm{d}x} = \left(3x^2 + 2x + 1\right) = 3 + 2 + 1 = 6$. So $\mathrm{d}y = \frac{\mathrm{d}y}{\mathrm{d}x}\mathrm{d}x = 6(0.1) = 0.6$. So absolute error, which we can define as estimated change, minus actual change, is 0.6 – 0.641 = -0.041. As a percentage of the actual change, this error is $\frac{-0.041}{0.641} \times 100 = -6.40\%$ to 2 decimal places.



Distance RB is given by the differential, $\mathrm{d}y = \frac{\mathrm{d}y}{\mathrm{d}x}\mathrm{d}x$. This is an approximation to the true change in y, given by the distance RQ. The error, ε , is the distance BQ.
(For the sake of clarity, this figure is drawn without regard to scale.)