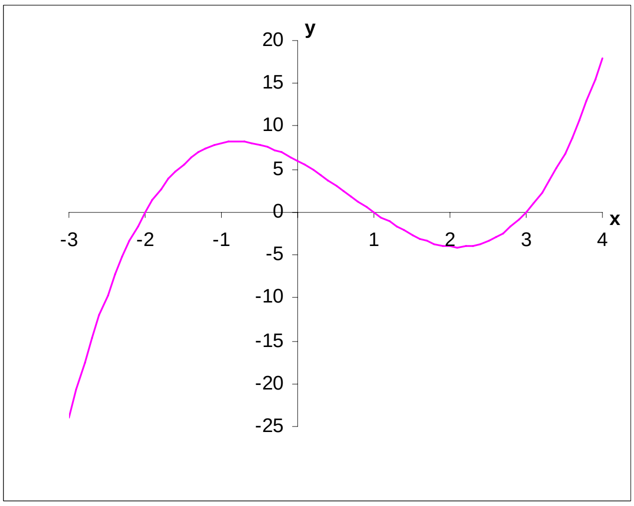
Progress exercise 5.1

1. Plot the graphs of the following cubic functions. Do not aim at a high degree of accuracy, but instead try to draw a sketch which is accurate enough to identify the main features of the graph, especially turning points and intercepts on the *x*- and *y*-axes.

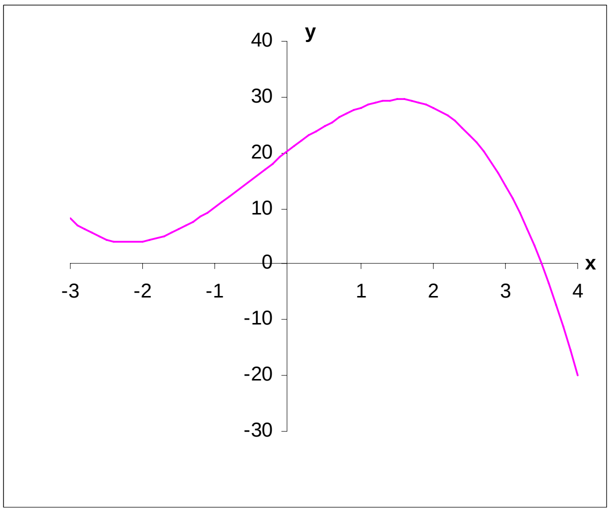
(a) $y = x^3 - 2x^2 - 5x + 6$. Take integer values of *x* between –3 and +4.

From the sketch, we see that the roots of this cubic equation are $x = 3, 1, -2$. Therefore it follows that if we expand (multiply out) $(x-3)(x-1)(x+2)$, the result will be $x^3 - 2x^2 - 5x + 6$.



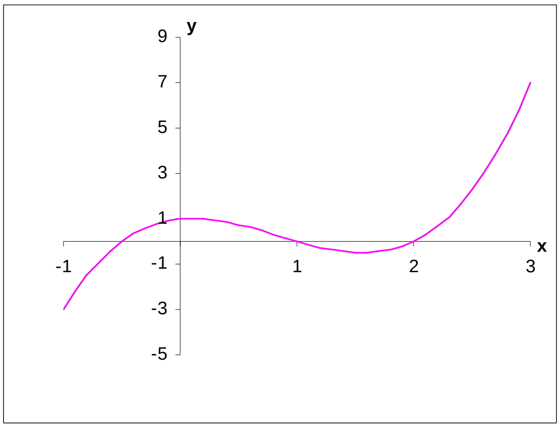
(b) $y = -x^3 - x^2 + 10x + 20.125$. Take values of *x* between –3 and +4, at intervals of 0.5.

From the sketch, we see that there is only one root, at *x* = approx 3.5. If we substitute *x* = 3.5 into the equation, the result is *y* = 0. This confirms that the root is exactly 3.5.



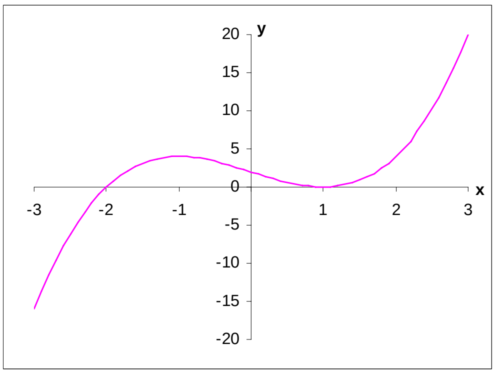
(c) $y = x^3 - 2.5x^2 + 0.5x + 1$. Take values of x between –1 and +3, at intervals of 0.5.

From the sketch, we see that there are 3 roots, $x = -0.5, 1, 2$. As in (a) above, it follows that if we expand (multiply out) $(x+0.5)(x-1)(x-2)$, the result will be $x^3 - 2.5x^2 + 0.5x + 1$.



(d) $y = x^3 - 3x + 2$. Take integer values of x between –3 and +3. soln roots 1, 1, –2

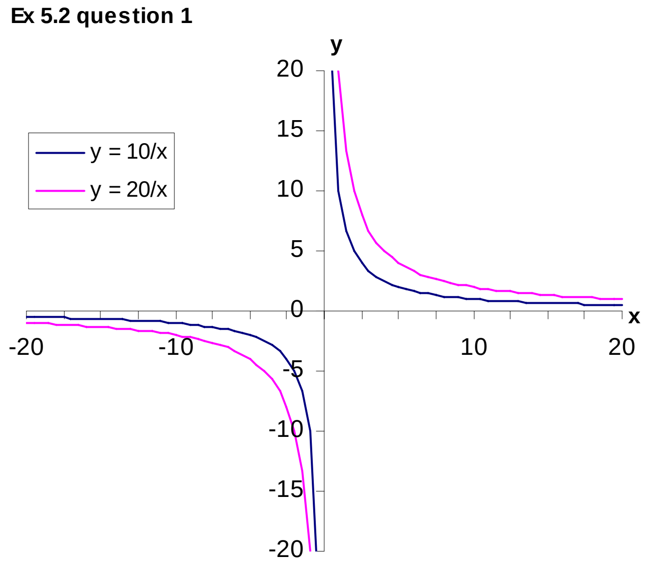
From the sketch, we see that there are 3 roots, $x = -2, 1, 1$. Thus 1 is a repeated root. As in (a) above, it follows that if we expand (multiply out) $(x+2)(x-1)(x-1)$, the result will be $x^3 - 3x + 2$. The curve is tangent to the x axis at $x = 1$.



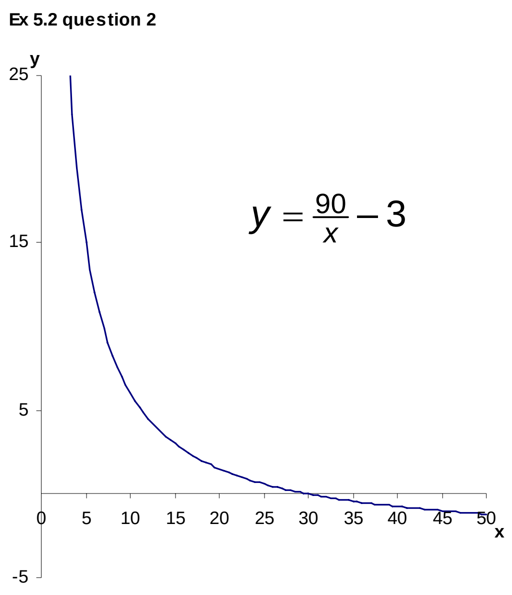
Progress exercise 5.2

In answering these questions, do not try to produce highly accurate graphs, but try to show the key features of the shape of each function.

1. On the same axes, plot the graphs of $y = \frac{10}{x}$ and $y = \frac{20}{x}$. Take values of x from –20 to –0.1 and from 0.1 to 20. For each function, identify any limiting values and discontinuities.
   Answer: For both functions, $y \to 0$ as $x \to \pm\infty$. For both functions there is a discontinuity at $x = 0$ (that is, at $x = 0$, y is undefined).



2. Plot the graph of $y = \frac{90}{x} - 3$ Take values of x from 0 to 45. Identify limiting values and any discontinuity.

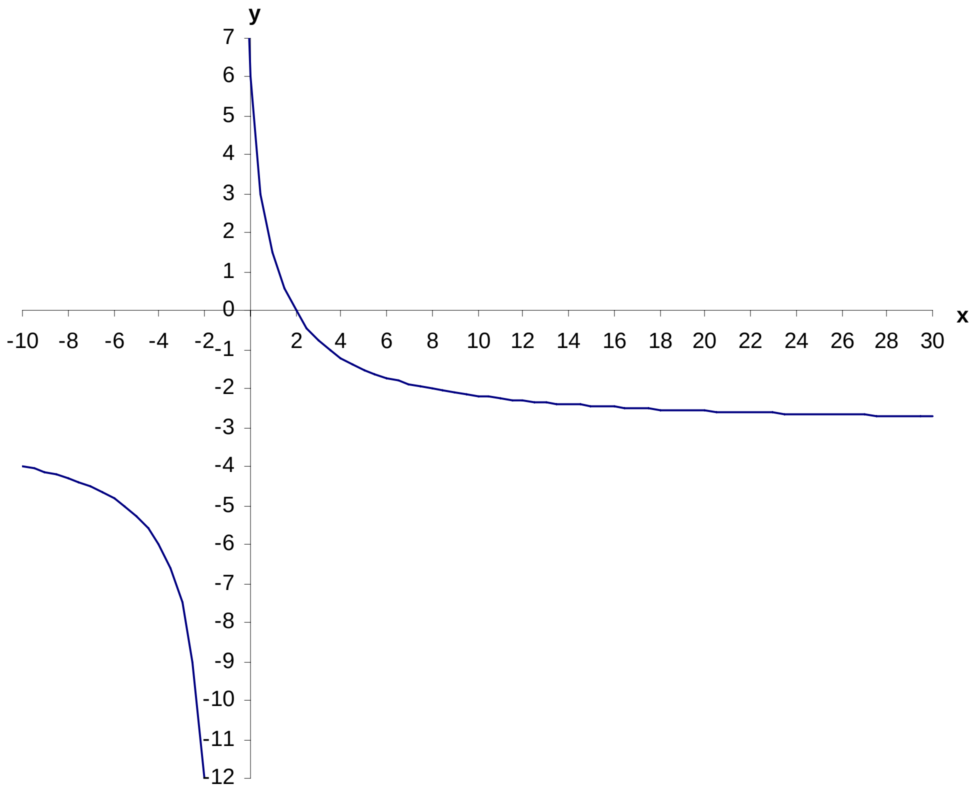


2. (cont'd) From inspection of $y = \frac{90}{x} - 3$, we see that $\frac{90}{x} \to 0$ as x approaches $+\infty$, and therefore y approaches a limiting value of –3. There is a discontinuity at $x = 0$. The graph cuts the x axis at $x = 30$. (Note that to make the graph above clearer, y has been restricted to a maximum value of 25.)

3. Plot the graph of $y = \frac{9}{x+1} - 3$. Take values of x from –10 to +30. Show that the horizontal asymptote is at $y = -3$, and the vertical asymptote at $x = -1$. Identify the discontinuity. Could this function be a plausible demand function with y as quantity demanded, with y positive? Give reasons for your answer.

Answer: From inspection of $y = \frac{9}{x+1} - 3$, and also from the graph, we can see that as x approaches $+\infty$, $\frac{9}{x+1}$ approaches zero, hence y approaches a limiting value of –3. This is therefore the horizontal asymptote. There is a discontinuity at $x = -1$, for then $\frac{9}{x+1}$ is undefined, and so therefore is y. This is therefore the vertical asymptote. The graph cuts the x axis at $x = 2$, for then $y = 0$.

Ex 5.2 question 2

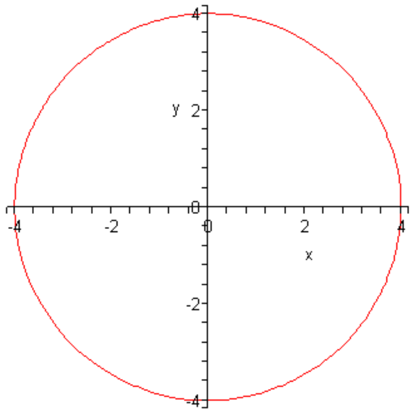


Progress exercise 5.3

On the same axes, sketch the graphs of

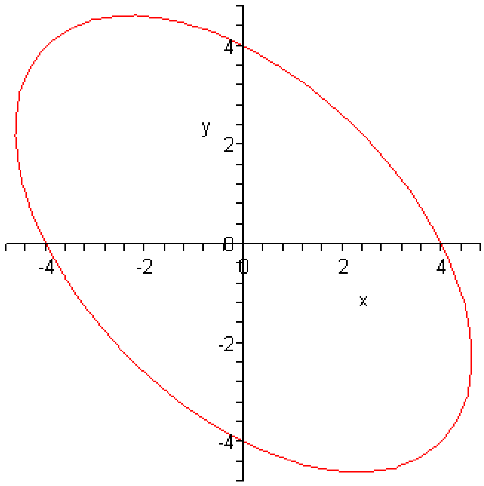
(a) $x^2 + y^2 = 16$.

As we see in the graph, this is a circle with centre at the origin and radius 4 (=√16)

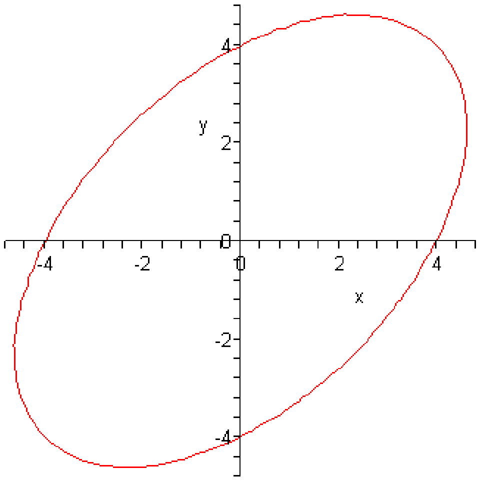


(b) $x^2 + xy + y^2 = 16$

From the graph we see that this is an ellipse centred at origin. Compared with the circle in (a) above, the addition of the $+xy$ term causes the curve to lie closer to the origin when x and y have the same sign and xy is therefore positive. The curve lies further from the origin when x and y have opposite signs and xy is therefore negative.



(c) $x^2 - xy + y^2 = 16$



This is the reverse of (b) because we have $-xy$ instead of $+xy$ as the additional term. Compared with the circle in (a), this ellipse is “stretched” away from the origin when x and y are both positive, and “squeezed” towards the origin when they have opposite signs.

Progress exercise 5.4

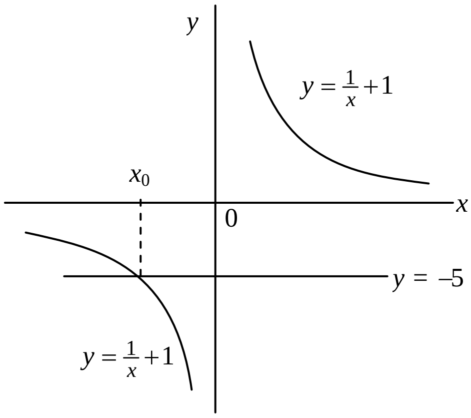
(1) Given $0 < x < 2$. After multiplying by 2 and adding 1, this becomes $1 < 2x + 1 < 5$. Therefore $0 < x < 2$ implies that $y = 2x + 1$ must be greater than 1 but less than 5.

(2) Given $\frac{1}{x} + 1 > -5$. Subtracting 1 gives $\frac{1}{x} > -6$

(i) Assume $x > 0$. Multiplying both sides of $\frac{1}{x} > -6$ by x gives: $1 > -6x$ (inequality not reversed, since $x > 0$). Dividing this by –6, we get: $-\frac{1}{6} < x$ (inequality reversed because we divided by a negative number). This is the same as $x > -\frac{1}{6}$. This condition is satisfied when $x > 0$ or $-\frac{1}{6} < x \le 0$. But the latter case is ruled out due to our assumption $x > 0$. Therefore the given condition, $\frac{1}{x} + 1 > -5$, is satisfied when x is positive.

(ii) Assume $x < 0$. Then multiplying both sides of $\frac{1}{x} > -6$ by x gives: $1 < -6x$ (inequality reversed, since $x < 0$). Dividing this by –6, we get: $-\frac{1}{6} > x$ (inequality again reversed because we divided by a negative number). This is the same as $x < -\frac{1}{6}$. So when x is negative, the condition $\frac{1}{x} + 1 > -5$ is satisfied when $x < -\frac{1}{6}$. Combining (i) and (ii), the given inequality is satisfied when $x > 0$ or $x < -\frac{1}{6}$.

The solution above is confirmed by the figure below, where we see that $y = \frac{1}{x} + 1$ is a rectangular hyperbola, while $y = -5$ is a horizontal line 5 units below the x axis. From the graph, we see that the graph of $y = \frac{1}{x} + 1$ lies above the graph of $y = -5$ in two cases: (a) when $x > 0$; (b) when x is negative and less than x_0. We have found that $x_0 = -\frac{1}{6}$.



(3) This could be solved in a way similar to (2) above, but we'll do it slightly differently for the sake of variety. Given $\frac{1}{x}+1>2x$

(i) Assume $x > 0$, then multiply both sides of $\frac{1}{x}+1>2x$ by x, giving $1 + x > 2x^2$ (since $x > 0$, inequality is not reversed). This rearranges as $0 > 2x^2 - x - 1$, or $2x^2 - x - 1 < 0$.

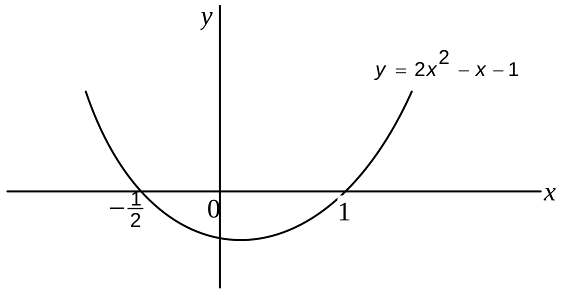
We now define $y = 2x^2 - x - 1$ and sketch its graph (below). (Note that to sketch the graph, we factorised $2x^2 - x - 1 = 0$. The factors are $(2x + 1)$ and $(x - 1)$.)

The graph tells us that $y < 0$ when $-\frac{1}{2}<x<1$. And when $y < 0$, we have by definition $2x^2 - x - 1 < 0$. So when $x > 0$, the given condition is satisfied when $-\frac{1}{2}<x<1$. Combining the two conditions ($x > 0$ and $-\frac{1}{2}<x<1$) we conclude that the given condition is satisfied when $x > 0$ and $0<x<1$.

(ii) Assume $x < 0$, then multiply both sides of $\frac{1}{x}+1>2x$ by x, giving $1 + x < 6x^2$ (since $x < 0$, inequality is reversed). This rearranges as $0 < 2x^2 - x - 1$, or $2x^2 - x - 1 > 0$. From the graph below we see that this is true when $x > 1$ or $x < -\frac{1}{2}$. Since we have assumed $x < 0$, the first of these must be rejected, so we conclude that when $x < 0$ the given condition is satisfied when $x < -\frac{1}{2}$.

(ii) Finally (I nearly forgot this!) we have to consider the case of $x = 0$. In that case, $\frac{1}{x}$ and therefore $\frac{1}{x}+1$ are undefined, and the question whether the given inequality is satisfied becomes literally meaningless.

Combining (i), (ii) and (iii), we conclude that the given condition is satisfied when $x > 0$ or $x < -\frac{1}{2}$.



(4) (Here $\Rightarrow$ means 'implies')

Given $(x-1)^2 < 2-2x$. Multiply out the left hand side

$\Rightarrow$ $x^2 - 2x + 1 < 2 - 2x$

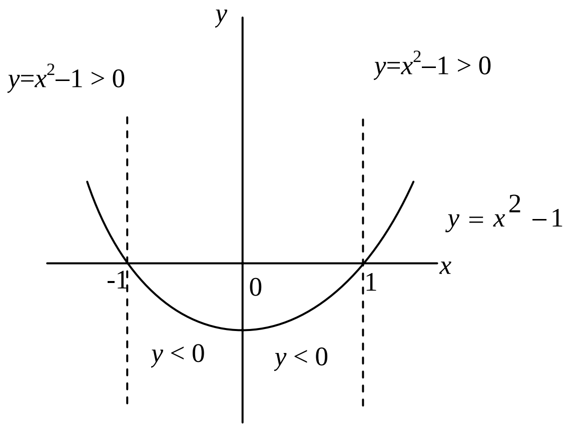
$\Rightarrow$ $x^2 - 1 < 0$. Factorise left hand side

$\Rightarrow$ $(x+1)(x-1) < 0$

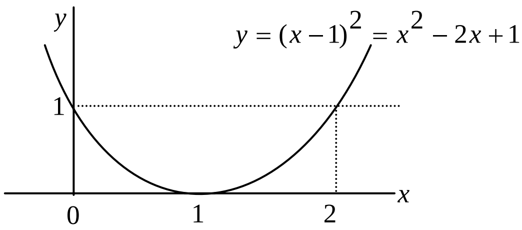
Left hand side is negative if $x + 1$ and $x - 1$ have opposite signs; that is, one positive and the other negative. Now, $x + 1$ and $x - 1$ are both positive if $x > 1$, so we require $x < 1$ for them to have opposite signs. However they are both negative if $x < -1$, so we also require $x > -1$ for them to have opposite signs. Combining these, the given inequality is satisfied if $-1 < x < 1$.

(This is not the only method of solution.)

This solution is confirmed by graph of $y = x^2 - 1$ below where we see that $y < 0$ if $-1 < x < 1$.



(5) There are two possible methods of solving this. Method (a): We expand $y=(x-1)^2$, giving $y=x^2-2x+1$. This is graphed below. We immediately see that when *x* lies between 0 and 1, y also lies between 0 and 1. And when *x* lies between 1 and 2, *y* again lies between 0 and 1. In symbols, we write this as: when $0 < x < 1$, $0 < y < 1$; and when $1 < x < 2$, $0 < y < 1$. Also, when $x = 1$, $y = 0$ (the point of tangency of the curve with the *x* axis).
Collecting results, we conclude: when $0 < x < 2$, $0 \le y < 1$.



Method (b): We can say immediately that $y=(x-1)^2$ is positive for all values of *x*. (This is true because any non-zero number becomes positive, when squared.) The only exception is when $x = 1$, when $(x-1)^2=0$.

When $x \ne 1$, and given $0 < x < 2$, there are two cases:

(i) When $0 < x < 1$. Subtracting 1 gives $-1 < x-1 < 0$. Multiply by $x-1$ (which reverses inequality, since $x-1 < 0$). This gives $-(x-1) > (x-1)^2 > 0$, which re-arranges as: $0<(x-1)^2<1-x$, where $1-x<1$ since *x* is positive. So therefore $0<(x-1)^2<1$ when $0 < x < 1$.

(ii) When $1 < x < 2$. Subtracting 1 gives $0<x-1<1$. Multiply by $x-1$ (which does not reverse inequality, since $x-1 > 0$). This gives $0<(x-1)^2<x-1$, where $x-1<1$ since *x* is less than 2. So again $0<(x-1)^2<1$.

Overall conclusion: when $0<x<2$, $0\le(x-1)^2<1$ (note solution includes $0=(x-1)^2$, when $x = 1$).

Note: in section 5.11 of the book, we warned of the dangers of raising an inequality to a power. We can illustrate this here. Given the condition $0 < x < 2$, if we subtract 1 from all sides we get $-1 < x-1 < 1$. If we then raise all sides to the power 2, we get $1<(x-1)^2<1$ (since $(-1)^2=1$). This is a contradiction, since it is impossible for $(x-1)^2$ to be both greater than 1 and less than 1.

(6) (a) (i) $3x + 4y \leq 120$; (ii) $y \leq 30 - \frac{3}{4}x$

(b) Opportunity cost of x in terms of y is $\frac{P_x}{P_y} = \frac{3}{4}$. This means that in order to increase her purchases of x by 1 unit, the consumer must reduce her purchases of y by $\frac{3}{4}$ of a unit.

(c) (i) 30, because this is the maximum quantity of y that her money income can buy. (This is found from the strong budget constraint $y = 30 - \frac{3}{4}x$, with $x = 0$.)

(ii) 40, because this is the maximum quantity of x that her money income can buy. (This is found from the strong budget constraint $y = 30 - \frac{3}{4}x$, with $y = 0$.) (Note that if the price of x increases, her real income measured in units of x falls, but her real income measured in units of y is unchanged.)

(d) No, because total expenditure is then 3(20) + 4(12) = 112(< 120).

(e) 40% of 120 is 48, so she buys 48/3 = 16 units of x. And 60% of 120 is 72, so she buys 72/4 = 18 units of y.