Exercise WS21.1

1. Expand the following functions around $x = 0$, up to the second (second derivative) term. Calculate the error from using your expansion to approximate the value of y when $x = 1$.

(a) $y = \frac{x+2}{x+4}$

Answer: The first 3 terms of the Taylor expansion of *f(x)* are
$f(x) = f(a) + f'(a)(x-a) + \frac{1}{2!}f''(a)(x-a)^2$.
Here we have $f(x) = y = \frac{x+2}{x+4}$; $f'(x) = \frac{2}{(x+4)^2}$; $f''(x) = \frac{-4}{(x+4)^3}$. With $a = 0$, the Taylor expansion is therefore:
$f(x) = \frac{0+2}{0+4} + \frac{2}{(0+4)^2}(x-0) + \frac{1}{2!}\frac{-4}{(0+4)^3}(x-0)^2$
$= \frac{1}{2} + \frac{1}{8}x - \frac{1}{2}\frac{1}{16}x^2$. When $x = 1$, this expression becomes:
$f(1) = \frac{1}{2} + \frac{1}{8} - \frac{1}{32} = \frac{19}{32} = 0.594$. The true value if *f(x)* at $x = 1$ is
$f(x) = \frac{1+2}{1+4} = \frac{3}{5} = 0.6$. So the error in using the Taylor series is 0.006

(b) $y = e^{2x}$

Answer: Following the steps of (a) above, we have $f'(x) = 2e^{2x}$;
$f''(x) = 4e^{2x}$. With $a = 0$, the Taylor expansion is: $f(x) = 1 + 2x + \frac{4}{2}x^2$. When $x = 1$, this becomes $f(x) = 1 + 2 + 2 = 5$. The true value is $f(1) = e^2 = 7.389$.
So error is –2.389.

(c) $y = \ln(x^3 + 1)$

Answer: Again following the steps of (a) above, we have $f'(x) = \frac{3x^2}{x^3+1}$;
$f''(x) = \frac{3[2x(x^2+1) - 3x^2x^2]}{(x^3+1)^2} = \frac{3(2x - x^4)}{(x^3+1)^2}$. With $a = 0$, the Taylor expansion is:
$f(x) = \ln 1 + 0(x) + 0\left(\frac{x^2}{2}\right)$. When $x = 1$, this becomes $f(x) = \ln 1 = 0$. The true value is $f(1) = \ln 2 = 0.69$. So error is –0.69.

2. Expand the following functions around $x = 0$, up to the third (third derivative) term. Calculate the error from using your expansion to approximate the value of y when $x = 1$.

(a) $y = \frac{x+2}{x+4}$

Answer: This is the same question as (1)(a) above, except that we now have to include the 4th term (containing the third derivative) of the Taylor series. The first 4 terms of the Taylor expansion of *f(x)* are

$f(x) = f(a) + f'(a)(x-a) + \frac{1}{2!}f''(a)(x-a)^2 + \frac{1}{3!}f'''(a)(x-a)^3$. In this example the 3rd derivative is $f'''(x) = \frac{12}{(x+4)^4}$. With $a = 0$, the Taylor expansion is therefore: $f(x) = \frac{0+2}{0+4} + \frac{2}{\left(0+4\right)^2}(x-0) + \frac{1}{2!}\frac{-4}{(0+4)^3}(x-0)^2 + \frac{1}{3!}\frac{12}{(0+4)^4}(x-0)^3$

$= \frac{1}{2} + \frac{1}{8}x - \frac{1}{2}\frac{1}{16}x^2 + \frac{1}{6}\frac{3}{64}x^3$. When $x = 1$, this expression becomes:

$f(1) = \frac{1}{2} + \frac{1}{8} - \frac{1}{32} + \frac{1}{128} = \frac{19}{32} + \frac{1}{128} = 0.602$. The true value if *f(x)* at $x = 1$ is

$f(x) = \frac{1+2}{1+4} = \frac{3}{5} = 0.6$. So the error in using the Taylor series is 0.002.

(b) $y = 2^x$

Answer: Following the steps of (a) above, we have $f(x) = 2^x$, so

$f(x) = 2^x$, so $f'(x) = (\ln 2)2^x$; $f''(x) = (\ln 2)^2 2^x$; $f'''(x) = (\ln 2)^3 2^x$ (For the technique for finding these derivatives, see Generalizing the Exponential and Logarithmic Functions, on the Online Resource Centre). With $a = 0$, the Taylor expansion is therefore:

$f(x) = 2^0 + (\ln 2)x + \tfrac{1}{2}(\ln 2)^2 x^2 + \tfrac{1}{6}(\ln 2)^3 x^3$. When $x = 0$, this becomes

$f(1) = 1 + 0.693 + 0.240 + 0.056 = 1.989$. The true value is $f(1) = 2^1 = 2$, so the error is: –0.011.

(c) $y = x \ln (x + 2)$

Answer: Following the steps of (a) above, we have $f(x) = x\ln(x+2)$, so

$$f'(x) = \ln(x+2) + \frac{x}{x+2} \text{ ; } f''(x) = \frac{1}{x+2} + \frac{2}{(x+2)^2} \text{ ; } f'''(x) = -\frac{1}{(x+2)^2} - \frac{4}{(x+2)^3}.$$

The Taylor series is $f(x) = 0 + 0.693x + \left(\tfrac{1}{2} + \tfrac{1}{2}\right)\frac{x^2}{2} - \left(\tfrac{1}{4} + \tfrac{1}{2}\right)\frac{x^3}{6}$. When x = 0, this becomes $f(1) = 0.693 + 0.5 - 0.125 = 1.068$. The true value is ln 3 = 1.099, so error is: –0.031.

3. Find and classify the stationary points for the functions given below.

(a) $z = x^2 + 2y^2 - 10x - 12y$

Answer: We set the first order partial derivatives, z_x and z_y, equal to zero and solve as simultaneous equations. Thus $z_x = 2x - 10 = 0$; $z_y = 4y - 12 = 0$. Solution is x = 5, y = 3. We also have the 2nd derivatives $z_{xx} = 2$, $z_{yy} = 4$, $z_{xy} = z_{yx} = 0$. So because $z_{xx} > 0$, $z_{yy} > 0$, and $z_{xx}z_{yy} - (z_{xy})^2 > 0$, this is a minimum.

(b) $z = 10x - x^2 - 2y^2 + 8y$

Answer: Following method of (a) above: $z_x = 10 - 2x = 0$; $z_y = -4y + 8 = 0$. Solution is x = 5, y = 2. Second derivatives are $z_{xx} = -2$; $z_{yy} = -4$; $z_{xy} = z_{yx} = 0$. So because $z_{xx} < 0$, $z_{yy} < 0$, and $z_{xx}z_{yy} - (z_{xy})^2 > 0$, this is a maximum.

(c) $y = (1 - x)^3$

Answer: We set $y_x = 0$, giving $y_x = -3(1 - x)^2 = 0$, with solution $x = 1$. We also have $y_{xx} = 6(1 - x)$ and $y_{xxx} = -6$. Since when $x = 1$ we have $y_{xx} = 0$ and $y_{xxx} < 0$, this is a point of inflection.

(d) $y = (1 - x)^4$

Answer: We set $y_x = 0$, giving $y_x = -4(1 - x)^3 = 0$, with solution $x = 1$. We also have $y_{xx} = 12(1 - x)^2$ and $y_{xxx} = -24(1 - x)$. Since when $x = 1$ we have $y_{xx} = 0$ and $y_{xxx} = 0$, we must take further derivatives to determine the behaviour of the function at $x = 1$. The fourth derivative is $y_{xxxx} = 24$. As this is the first derivative that is non-zero when $x = 1$, and as this is the 4th derivative and 4 is an even-number, and the derivative has a positive value, this point is a minimum.

4. Find and classify the stationary points of the function

$$z = 4y^2 - 4y - 2x^2y - x^3$$

Answer: We have $z_x = -4xy - 3x^2 = 0$ (1)

and $z_y = 8y - 4 - 2x^2 = 0$ (2)

From equation (1), $-x(4y + 3x) = 0$, hence x = 0 or $x = -\frac{4}{3}y$. In equation (2), when x = 0, $y = \frac{1}{2}$. In (2), when $x = -\frac{4}{3}y$, $8y - 4 - 2\left(\frac{-4}{3}y^2\right) = 0$. The roots of this quadratic equation are $y = \frac{3}{4}$ or $y = \frac{3}{2}$. When $y = \frac{3}{4}$, $x = -\frac{4}{3}\frac{3}{4} = -1$. When $y = \frac{3}{2}$, $x = -\frac{4}{3}\frac{3}{2} = -2$. So the solutions are $\left(0,\ \frac{1}{2}\right)$; $\left(-1,\ \frac{3}{4}\right)$; $\left(-2,\ \frac{3}{2}\right)$.

The second derivatives are $z_{xx} = -4y - 6x$; $z_{yy} = 8$; and $z_{xy} = z_{yx} = -4x$. Therefore the Hessian matrix is $H = \begin{bmatrix} -4y - 6x & -4x \\ -4x & 8 \end{bmatrix}$

For the solution $\left(0,\ \frac{1}{2}\right)$ we have $H = \begin{bmatrix} -2 & 0 \\ 0 & 8 \end{bmatrix}$. So det. H = -16. As this is negative, this is a saddle point. (As $z_{xx} < 0$ and $z_{yy} > 0$, this is a maximum for variation in x alone and a minimum for variation in y alone.)

For the solution $\left(-1,\ \frac{3}{4}\right)$ we have $H = \begin{bmatrix} 3 & -4 \\ -4 & 8 \end{bmatrix}$. So det. H = 8. As $z_{xx} > 0$ and $z_{yy} > 0$, and H is positive, this is a minimum.

For the solution $\left(-2,\ \frac{3}{2}\right)$ we have $H = \begin{bmatrix} 6 & 8 \\ 8 & 8 \end{bmatrix}$. So det. H = -16. As this is negative, this is a saddle point. (As $z_{xx} > 0$ and $z_{yy} > 0$, this is a minimum for variation in x or y alone.)

5. Which of the following functions are concave for x>0?

Answers: a function $y = f(x)$ is concave if its 2nd derivative, y'', is negative.

(a) $y = -x^{-2}$ Here $y'' = -6x^{-4} < 0$ when $x > 0$, so concave.

(b) $y = x^{-1/2}$ Here $y'' = \frac{3}{4}x^{-\frac{5}{2}} > 0$ when $x > 0$, so not concave.

(c) $y = -x(1-2x)$ Here $y'' = 4 > 0$ when $x > 0$, so not concave.

(d) $y = -(x^{-2} + x^{-1/2})$ Here $y'' = -6x^{-4} - \frac{3}{4}x^{-\frac{5}{2}} < 0$ when $x > 0$, so concave.

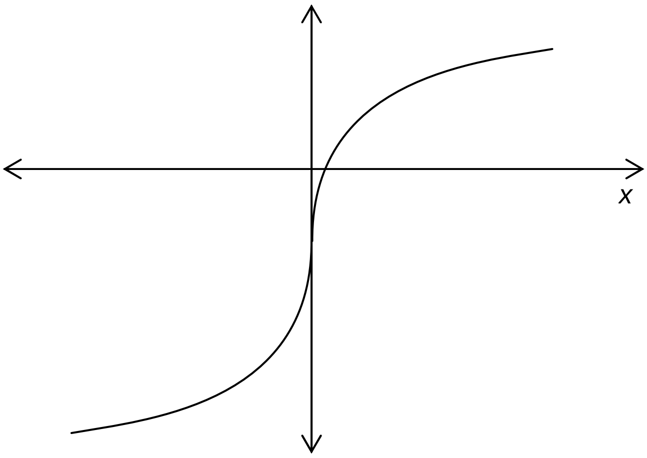
(e) $y = -x^{-1/2} + x(1-2x)$ Here $y'' = -\frac{3}{4}x^{-\frac{5}{2}} - 4 < 0$ when $x > 0$, so concave.

(f) $y = x + (x-a)^2$ Here $y'' = 2 > 0$ when $x > 0$, so concave.

6. "A linear function is neither strictly concave nor strictly convex." True or false?

Answer: True

7. Draw a function which is strictly concave for $x > 0$ and strictly convex for $x < 0$.



8. (a) What is the maximum of $x - 2(x-a)^2$ (where x is constrained to be non-negative) for any value of a ?

Answer: given $y = x - 2(x-a)^2$, we have $y' = 1 - 4(x-a) = 1 - 4x - 4a = 0$, with solution $x = \frac{1}{4} + a$. Since $y'' = -4 < 0$, this is a maximum. Given the constraint $x \geq 0$, the solution is $x = \frac{1}{4} + a$ if $a > \frac{1}{4}$, and $x = 0$ if $a \leq \frac{1}{4}$.

(b) What is the maximum of $x - 2(x-a)^2$ (where x is constrained by $x \geq 1$) for any value of a?

Answer: as in (a) above, the solution is $x = \frac{1}{4} + a$. Since $y'' = -4 < 0$, this is a maximum. Given the constraint $x \geq 1$, the solution is $x = \frac{1}{4} + a$ if $a \geq \frac{3}{4}$, and $x = 0$ if $a < \frac{3}{4}$.

Exercise WS21.2

1. If $z = bx - xy - x^2 + cy - 2y^2$, find the maximum value of z and determine how the optimising values of *x* and *y* change (a) as *b* changes and (b) as *c* changes.

Answer: To find maximum value of *z*, e have $z_x = b - y - 2x = 0$ (1)

and $z_y = -x + c - 4y = 0$ (2)

From (1), $y = b - 2x$. Substituting this into (2) gives $-x + c - 4(b - 2x) = 0$, from which $x = \frac{4b - c}{7}$. Therefore $y = b - 2x = \frac{-b + 2c}{7}$. Because $z_{xx} = -2$, $z_{yy} = -4$, and $z_{xy} = -1$, this is a maximum. The solution for *z* is

$$z = bx - xy - x^2 + cy - 2y^2 = b\left(\frac{4b - c}{7}\right) - \frac{(4b - c)(-b + 2c)}{7^2} - \frac{(4b - c)^2}{7^2} + \frac{c(-b + 2c)}{7} - \frac{2(-b + 2c)^2}{7^2}.$$

To find effects of change in *b* and *c* on the optimising value of x (denoted by *x**), we have, from above, $x^* = \frac{4b - c}{7}$, so $\frac{dx^*}{db} = \frac{4}{7}$ and $\frac{dx^*}{dc} = -\frac{1}{7}$. Similarly we have $y^* = \frac{-b + 2c}{7}$, so $\frac{dy^*}{db} = -\frac{1}{7}$ and $\frac{dy^*}{dc} = \frac{2}{7}$.

2. If the number of cigarettes sold (in millions, and denoted by *q*) depends on the price $p + t$, where *t* is a tax, according to the demand function $q = 100 - 2(p+t)$, and if *p* is fixed by the international price of cigarettes, find the tax *t* which maximises the government's tax revenue. Find how *t* responds to a change in *p.*

Answer: Each cigarette sold yields tax revenue of *t*. So if *q* cigarettes are sold, tax revenue, *R*, is given by $R = tq$. Using the demand function, we have $R = tq = 100t - 2t(p + t) = 100t - 2tp - 2t^2$. With *p* given, $\frac{dR}{dt} = 100 - 2p - 4t = 0$, with solution $t = 25 - \frac{1}{2}p$. (This is a maximum, because $\frac{d^2R}{dt^2} = -4 < 0$.) If we write *t** to denote the optimum (revenue maximising) tax rate, we have $t^* = 25 - \frac{1}{2}p$ and therefore $\frac{dt^*}{dp} = -\frac{1}{2}$. (Thus if the international price of cigarettes rises by 10 pence, the tax should be reduced by 5 pence to maintain maximum revenue.)

3. A monopolistic producer of two goods has a joint total cost function

$$TC = 10Q_1 + Q_1Q_2 + 10Q_2$$

where Q_1 and Q_2 denote the quantities of good 1 and good 2 respectively. If P_1 and P_2 denote the corresponding prices, then the demand equations are

$$P_1 = 50 - 2Q_1 + Q_2 \qquad \text{and} \qquad P_2 = 30 + 2Q_1 - 4Q_2$$

(a) Determine the total revenue function $TR = P_1Q_1 + P_2Q_2$ in terms of Q_1 and Q_2 only.

Answer: using the (inverse) demand functions, we have

$TR = P_1Q_1 + P_2Q_2 = (50 - 2Q_1 + Q_2)Q_1 + (30 + 2Q_1 - 4Q_2)Q_2$. This rearranges as

$TR = 50Q_1 + 30Q_2 + 3Q_1Q_2 - 2Q_1^2 - 4Q_2^2$

(b) Find the profit function in terms of Q_1 and Q_2.

Answer: From above, we have $TR = 50Q_1 + 30Q_2 + 3Q_1Q_2 - 2Q_1^2 - 4Q_2^2$ and $TC = 10Q_1 + Q_1Q_2 + 10Q_2$. By definition, profit, Π, is given by $\Pi = TR - TC$. Combining these three equations gives
$\Pi = 50Q_1 + 30Q_2 + 3Q_1Q_2 - 2Q_1^2 - 4Q_2^2 - \left(10Q_1 + Q_1Q_2 + 10Q_2\right)$. This simplifies to

$\Pi = 40Q_1 + 20Q_2 + 2Q_1Q_2 - 2Q_1^2 - 4Q_2^2$

(c) Find the profit-maximising levels of Q_1 and Q_2.

We will write Π_1 and Π_2 as the partial derivatives of the profit function with respect to Q_1 and Q_2 respectively. Taking these partial derivatives and setting them equal to zero, we have

$\Pi_1 = 40 - 4Q_1 + 2Q_2 = 0$ and $\Pi_2 = 20 + 2Q_1 - 8Q_2 = 0$. Solving these simultaneously gives $Q_1 = \frac{90}{7}$ and $Q_2 = \frac{40}{7}$.

(d) Find the maximum profit.

Answer: Substituting $Q_1 = \frac{90}{7}$ and $Q_2 = \frac{40}{7}$ into the profit function, we get

$\Pi = 40Q_1 + 20Q_2 + 2Q_1Q_2 - 2Q_1^2 - 4Q_2^2 = 40\left(\frac{90}{7}\right) + 20\left(\frac{40}{7}\right) + 2\left(\frac{90}{7}\right)\left(\frac{40}{7}\right) - 2\left(\frac{90}{7}\right)^2 - 4\left(\frac{40}{7}\right)^2$
$= \frac{2200}{7} = 314.2857$

(e) Show that second-order conditions hold.

Answer: This is a maximum because $\Pi_{11} < 0$; $\Pi_{22} < 0$; and $\Pi_{11}\Pi_{22} - \Pi_{12}^2 = 12 > 0$.

4. Repeat question 3 with the following difference: an amount *t* of the price received for each unit of good 1 is paid in tax to the government, and the firm maximises profit paid net of tax. Suppose *t* changes by a small amount d*t*. Find the changes in optimal Q_1 and Q_2. Find the change in maximum profit. Show that this change is equal to the optimal level of Q_1.

Answer to (a): The firm's total revenue function is now
$TR = (P_1 - t)Q_1 + P_2Q_2 = P_1Q_1 + P_2Q_2 - tQ_1$

Answer to (b): the (inverse) demand functions $P_1 = 50 - 2Q_1 + Q_2$ and $P_2 = 30 + 2Q_1 - 4Q_2$ are the same as in question 3. Substituting these into the total revenue function from 4(a) above gives:

$$TR = 50Q_1 + 30Q_2 + 3Q_1Q_2 - 2Q_1^2 - 4Q_2^2 - tQ_1$$

The total cost function $TC = 10Q_1 + Q_1Q_2 + 10Q_2$ is the same as in question 3. So the profit function is $\Pi = TR - TC$
$= 50Q_1 + 30Q_2 + 3Q_1Q_2 - 2Q_1^2 - 4Q_2^2 - tQ_1 - \left(10Q_1 + Q_1Q_2 + 10Q_2\right)$. This simplifies to

$$\Pi = 40Q_1 + 20Q_2 + 2Q_1Q_2 - 2Q_1^2 - 4Q_2^2 - tQ_1$$

Answer to (c): We take the partial derivatives of the profit function in 4(b) above and set them equal to zero, giving
$\Pi_1 = 40 - 4Q_1 + 2Q_2 - t = 0$ and $\Pi_2 = 20 + 2Q_1 - 8Q_2 = 0$. Solving these simultaneously gives $Q_1 = \dfrac{90-2t}{7}$ and $Q_2 = \dfrac{80-t}{14}$.

Answer to (d): as in question 3, we substitute the solution values of Q_1 and Q_2 from 4(c) into the profit function found in 4(b). This gives:

$$\Pi = 40Q_1 + 20Q_2 + 2Q_1Q_2 - 2Q_1^2 - 4Q_2^2 - tQ_1$$

$$= 40\left(\frac{90-2t}{7}\right) + 20\left(\frac{80-t}{14}\right) + 2\left(\frac{90-2t}{7}\right)\left(\frac{80-t}{14}\right) - 2\left(\frac{90-2t}{7}\right)^2 - 4\left(\frac{80-t}{14}\right)^2 - t\left(\frac{90-2t}{7}\right)$$

This simplifies to:

$$\Pi = \tfrac{2200}{7} - \tfrac{90}{7}t + \tfrac{1}{7}t^2$$

This says that profit is a quadratic function of *t*. By differentiation we can easily show that profit reaches a minimum when *t* is slightly greater than 45. (Check for yourself).

Beyond this point, further increases in t increase profit. This is a little disturbing, as it is hard to see how an increase in the tax rate could increase profit. However, on looking again at the model we see that t cannot exceed 45, as then our solution for Q_1 becomes negative, which we implicitly rule out.

Answer to (e): As in question 3, this is a maximum because $\Pi_{11} < 0$; $\Pi_{22} < 0$; and $\Pi_{11}\,\Pi_{22} - \Pi_{12}^2 = 12 > 0$.

Further answer: effect of small change in t. From 4(b) above, we have
$\Pi = 40Q_1 + 20Q_2 + 2Q_1Q_2 - 2Q_1^2 - 4Q_2^2 - tQ_1$

Let Q_1^* and Q_2^* denote the profit maximising values of Q_1 and Q_2. Then, when profits are maximised we will have $\Pi = 40Q_1^* + 20Q_2^* + 2Q_1^*Q_2^* - 2\left(Q_1^*\right)^2 - 4\left(Q_2^*\right)^2 - tQ_1^*$
The effect on maximised profits of a small change in t is:
$\dfrac{d\Pi}{dt} = -Q_1^*$. Moreover, since from 4(c) above we have $Q_1^* = \dfrac{90 - 2t}{7}$, we have that $\dfrac{d\Pi}{dt} = -\dfrac{90 - 2t}{7}$

Exercise WS21.3

1. Solve each of the following second-order homogeneous difference equations. In each case, determine whether the solution is convergent or divergent, oscillatory or otherwise.

   (a) $Y_t - 11Y_{t-1} + 10Y_{t-2} = 0$

Answer: the characteristic equation is: $m^2 - 11m + 10 = 0$. This quadratic factorises as $(m - 10)(m - 1) = 0$, so the roots are $m_1 = 10$ and $m_2 = 1$. So the general or complete solution is $Y_t = A_1 10^t + A_2 1^t$. (where A_1 and A_2 are arbitrary unless we are given initial conditions that permit us to determine their values).

In the solution, the behaviour of Y_t is determined solely by the10^t term, since $A_2 1^t$ is constant. As 10^t increases as t increases, the path of Y_t is divergent.

   (b) $9Y_t - Y_{t-2} = 0$

Answer: the characteristic equation is: $9m^2 - 1 = 0$, with single (repeated) root $m = \pm\frac{1}{3}$. The complete solution is therefore

$$Y_t = A_1\left(\tfrac{1}{3}\right)^t + A_2\left(-\tfrac{1}{3}\right)^t$$

As $\left(\frac{1}{3}\right)^t$ and $\left(-\frac{1}{3}\right)^t$ both become smaller and smaller in absolute value as *t* increases, the behaviour of Y_t is convergent. But since $\left(-\frac{1}{3}\right)^t$ is alternately positive or negative as *t* is even or odd, the behaviour of Y_t is oscillatory.

(c) $Y_t - 10Y_{t-1} + 21Y_{t-2} = 0$

Answer: the characteristic equation is: $m^2 - 10m + 21 = 0$. This quadratic factorises as $(m - 7)(m - 3) = 0$, so the roots are $m_1 = 7$ and $m_2 = 3$. The complete solution is therefore

$Y_t = A_1 7^t + A_2 3^t$. The path of Y_t is divergent.

2 Find the general solution of each of the following second order non-homogeneous difference equations, and comment on their time paths.

Note that parts (a) – (c) below involve the same equations as in question 1 (a) – (c) above, except for the addition of a constant term on the right hand side (which by definition makes them non-homogeneous). This means that we have to find a particular solution (*PS*) to the given equation, and add it to the solution we found in question 1 above.

(a) $Y_t - 11Y_{t-1} + 10Y_{t-2} = 4$

Answer: We normally hope to find the particular solution as *PS* = *Y**, where *Y** is any constant value of *Y* that satisfies the given equation. Since *Y** is independent of *t*, it follows that $Y_t = Y_{t-1} = Y_{t-2} = Y^*$. But in this example, this method fails because if we substitute $Y_t = Y_{t-1} = Y_{t-2} = Y^*$ into the given equation, it becomes $Y^* - 11Y^* + 10Y^* = 0$, which is an identity and therefore does not permit us to determine any unique value for *Y**. So instead we assume *PS* = *Y*t* (a linear function of *t,* since *Y** is assumed constant). Then we have $Y_t = Y^*t$, $Y_{t-1} = Y^*(t-1)$, and $Y_{t-2} = Y^*(t-2)$. Substituting these values into the given equation we get

$Y^*t - 11Y^*(t-1) + 10Y^*(t-2) = 4$. Multiplying out and simplifying gives

$$Y^*t - 11Y^*t + 11Y^* + 10Y^*t - 20Y^* = 4$$

$-9Y^* = 4$, so $Y^* = -\frac{4}{9}$

Therefore the complete solution is $Y_t = A_1 10^t + A_2 1^t - \frac{4}{9}t$. This is the same solution as the solution to 1(a) above, except for the addition of the term $-\frac{4}{9}t$. As in 1(a), the 10^t term continues to dominate the other terms, so the behaviour of Y_t is divergent.

(b) $9Y_t - Y_{t-2} = 1$

Answer: we find the particular solution by assuming *PS* = *Y**, which is independent of *t*, and therefore substitute $Y_t = Y_{t-2} = Y^*$ into the given equation, so it becomes $9Y^* - Y^* = 1$, from which $Y^* = \frac{1}{8}$. The complete solution is therefore

$Y_t = A_1\left(\frac{1}{3}\right)^t + A_2\left(-\frac{1}{3}\right)^t + \frac{1}{8}$. Comparing this with our solution to 1(b), we see that the path of Y_t is shifted up by $\frac{1}{8}$, but is otherwise unchanged. It therefore remains oscillatory.

(c) $Y_t - 10Y_{t-1} + 21Y_{t-2} = 8$

Answer: again we find the particular solution by assuming *PS* = *Y**, which is independent of *t*, and therefore substitute $Y_t = Y_{t-1} = Y_{t-2} = Y^*$ into the given equation, so it becomes

$Y^* - 10Y^* + 21Y^* = 8$, from which $Y^* = \frac{2}{3}$.

The complete solution is therefore $Y_t = A_1 7^t + A_2 3^t + \frac{2}{3}$. The addition of the constant term, 8, in the given equation causes the path of Y_t to be shifted up by $\frac{2}{3}$, but the path remains divergent.

(d) $2Y_t - 2Y_{t-1} + Y_{t-2} = 6$

Answer: again we find the particular solution by assuming *PS* = *Y**, which is independent of *t*, and therefore substitute $Y_t = Y_{t-1} = Y_{t-2} = Y^*$ into the given equation, so it becomes

$2Y^* - 2Y^* + Y^* = 6$, from which $Y^* = 6$.

Next, we have to solve the reduced form equation $2Y_t - 2Y_{t-1} + Y_{t-2} = 0$ (as in this case the solution was not found in question (1) above. The characteristic equation is: $2m^2 - 2m + 1 = 0$. Using the formula, the roots of this quadratic are

$$m = \frac{2 \pm \sqrt{4-8}}{4} = \frac{2 \pm \sqrt{-4}}{4} = \frac{2 \pm i\sqrt{4}}{4} = \tfrac{1}{2} \pm \tfrac{1}{2}i$$

Therefore the solution to the reduced form is

$$Y_t = \left(\tfrac{1}{2}\right)^{t/2}\left(A_1 \cos\tfrac{\pi}{4}t + A_2 \sin\tfrac{\pi}{4}t\right)$$

and the complete solution to the given equation is therefore:

$$Y_t = \left(\tfrac{1}{2}\right)^{t/2}\left(A_1 \cos\tfrac{\pi}{4}t + A_2 \sin\tfrac{\pi}{4}t\right) + 6$$

3. Find the solutions to parts (b), (c) and (d) of question 2, subject to the given initial conditions in each case:

   (b) $Y_0 = 0, \quad Y_1 = 1$

Answer: from 2(b), above, our solution was: $Y_t = A_1\left(\frac{1}{3}\right)^t + A_2\left(-\frac{1}{3}\right)^t + \frac{1}{8}$. When t = 0, the value of this expression is $Y_0 = A_1\left(\frac{1}{3}\right)^0 + A_2\left(-\frac{1}{3}\right)^0 + \frac{1}{8} = A_1 + A_2 + \frac{1}{8}$. But we also have the initial condition, or information, that $Y_0 = 0$. Combining these gives $0 = A_1 + A_2 + \frac{1}{8}$.
Proceeding in the same way for t = 1, we get $Y_1 = A_1\left(\frac{1}{3}\right)^1 + A_2\left(-\frac{1}{3}\right)^1 + \frac{1}{8} = \frac{1}{3}(A_1 - A_2) + \frac{1}{8}$, and $Y_1 = 1$. Combining these gives $1 = \frac{1}{3}(A_1 - A_2) + \frac{1}{8}$. We now have a pair of simultaneous equations ($0 = A_1 + A_2 + \frac{1}{8}$ and $1 = \frac{1}{3}(A_1 - A_2) + \frac{1}{8}$) in the unknowns A_1 and A_2. Solving them gives $A_1 = \frac{5}{4}$ and $A_2 = -\frac{11}{8}$. Our solution from 2(b) above therefore becomes $Y_t = \frac{5}{4}\left(\frac{1}{3}\right)^t - \frac{11}{8}\left(-\frac{1}{3}\right)^t + \frac{1}{8}$

   (c) $Y_0 = \frac{1}{2}, \quad Y_1 = 1$

Using the same method as in (b) above, we obtain the simultaneous equations $\frac{1}{2} = A_1 + A_2 + \frac{2}{3}$ and $1 = 7A_1 + 3A_2 + \frac{2}{3}$. The solution is $A_1 = \frac{5}{24}$ and $A_2 = -\frac{9}{24}$. Our solution from 2(c) above therefore becomes $Y_t = \left(\frac{5}{24}\right)7^t - \left(\frac{9}{24}\right)3^t + \frac{2}{3}$

   (d) $Y_0 = 2, \quad Y_1 = 3$

Answer: from 2(d), above, our solution was $Y_t = \left(\frac{1}{2}\right)^{t/2}\left(A_1 \cos\frac{\pi}{4}t + A_2 \sin\frac{\pi}{4}t\right) + 6$.
When t =0, the value of this expression is $Y_0 = A_1 + 6$ (because cos 0 = 1 and sin 0 = 0). Since we are given the initial condition $Y_0 = 2,$, we have $2 = A_1 + 6$, hence $A_1 = -4$. Similarly, when t = 1, we have

$$3 = Y_1 = \left(\frac{1}{2}\right)^{1/2}\left(A_1 \cos\frac{\pi}{4} + A_2 \sin\frac{\pi}{4}\right) + 6 = \left(\frac{1}{2}\right)^{1/2}\left((-4)\cos\frac{\pi}{4} + A_2 \sin\frac{\pi}{4}\right) + 6$$ (since $A_1 = -4$).

Since $\cos\frac{\pi}{4} = \sin\frac{\pi}{4} = \left(\frac{1}{2}\right)^{1/2}$, this becomes

$$-3 = \left(\frac{1}{2}\right)^{1/2}\left((-4)\left(\frac{1}{2}\right)^{1/2} + A_2\left(\frac{1}{2}\right)^{1/2}\right) = (-4)\frac{1}{2} + A_2\frac{1}{2}$$, from which A_2 = –2.

So the complete solution from 2(d) above becomes $Y_t = \left(\frac{1}{2}\right)^{t/2}\left(-4\cos\frac{\pi}{4}t - 2\sin\frac{\pi}{4}t\right) + 6$