Exercise WS20.1

1. Solve the following first-order difference equations and describe the time path of y.

   (a) $3y_t + y_{t-1} - 3 = 0$; $y_0 = 0$

From chapter 20 of the book we know that the difference equation $y_t = by_{t-1} + c$ has the solution $y_t = A(b)^t + \frac{c}{1-b}$. Here we have $b = -\frac{1}{3}$, $c = -1$, so the solution is

$$y_t = A(-\tfrac{1}{3})^t - \tfrac{3}{4}$$

Given $y_0 = 0$, this implies $0 = A(-\frac{1}{3})^0 - \frac{3}{4}$, from which $A = \frac{3}{4}$. So the complete solution is

$$y_t = \tfrac{3}{4}(-\tfrac{1}{3})^t - \tfrac{3}{4}$$

Time path: because $-\frac{1}{3}$ is a fraction, it gets smaller and smaller in absolute value as t increases and it is therefore raised to higher and higher powers. And because $-\frac{1}{3}$ is negative, its sign oscillates between positive and negative according to whether it is raised to an even or odd power. So $\frac{3}{4}(-\frac{1}{3})^t$ approaches zero but is alternately positive or negative at t is even or odd. Thus we have oscillatory convergence in which y gets closer and closer to $-\frac{3}{4}$ but is always slightly above or below it. (For example, when $t = 5$, $y_t = \frac{3}{4}(-\frac{1}{3})^5 - \frac{3}{4}$ = –0.75309 (slightly less than $-\frac{3}{4}$ = –0.75). When $t = 6$, $y_t = \frac{3}{4}(-\frac{1}{3})^6 - \frac{3}{4}$ = –0.74897 (slightly more than $-\frac{3}{4}$).

(b) $3y_t + y_{t-1} - 3 = 0$; $y_0 = 1$

Answer: This is the same difference equation as in (a) above, so the solution is again:

$$y_t = A(-\tfrac{1}{3})^t - \tfrac{3}{4}$$

However in this case we have $y_0 = 1$, this implies $1 = A(-\frac{1}{3})^0 - \frac{3}{4}$, from which $A = \frac{7}{4}$. So the complete solution is

$$y_t = \tfrac{7}{4}(-\tfrac{1}{3})^t - \tfrac{3}{4}$$

The time path is the same as in (a) except that the oscillations in y as t is even or odd are now a little larger. (For example, when $t = 5$, $y_t = \frac{7}{4}(-\frac{1}{3})^5 - \frac{3}{4}$ = –0.7476

(c) $y_t + y_{t-1} + 2 = 0$; $y_0 = 0$

Answer: Here $b = -1$, $c = -2$. With the same method as in (a) above, we get $y_t = A(-1)^t - 1$.

Given $y_0 = 0$, this implies $0 = A(-1)^0 - 1$, from which $A = 1$. So the complete solution is $y_t = (-1)^t - 1$.

Time path: Because $(-1)^t$ = +1 or -1 as t is even or odd, $y_t = (-1)^t - 1$ alternates between 0 and –2 as t is even or odd. Thus we have stable oscillation in y (that is, neither convergent nor divergent).

(d) $y_t - 2y_{t-1} + 6 = 0$; $y_2 = 4$

Answer: Here $b = 2$, $c = -6$. With the same method as in (a) above, we get $y_t = A(2)^t - 6$.

Given $y_2 = 4$, this implies $4 = A(2)^4 - 6$, from which $A = \frac{5}{8}$. So the complete solution is $y_t = \frac{5}{8}(2)^t - 6$.

Time path: 2^t increases without limit as t increases, so y_t also increases without limit (divergent or explosive growth).

(e) $y_t + 3y_{t-1} = 0$; $y_0 = -1$

Answer: Here $b = -3$, $c = 0$. Therefore $y_t = A(-3)^t$. The initial condition gives us $-1 = A(-3)^t$ from which $A = -1$. So the complete solution is $y_t = -1(-3)^t$.

Time path: As t increases, $(-3)^t$ increases without limit in absolute value and is alternately positive or negative as t is even or odd. Therefore y_t increases without limit in absolute value and is alternately positive or negative as t is odd or even. Thus we have explosive or divergent oscillations.

Exercise WS20.2

1. The demand and supply functions for a product are:

$$q_t^{\mathrm{D}} = 110 - p_t$$

$$q_t^{\mathrm{S}} = 5p_t + 5p_{t-1}$$

(a) Find the time path of price if $p_0 = 4$.

Answer: Assuming supply and demand are equal in any time period, in any period we will have $q_t^{\mathrm{S}} = q_t^{\mathrm{D}}$ and therefore

$110 - p_t = 5p_t + 5p_{t-1}$ which re-arranges as

$p_t = (-\frac{5}{6})p_{t-1} + \frac{110}{6}$. Solving this with the method of Ex WS20.1 we get

$p_t = A(-\frac{5}{6})^t + 10$

Given $p_0 = 4$, this implies $4 = A(4)^0 + 10$ from which $A = -6$. So the full solution is $p_t = -6(-\frac{5}{6})^t + 10$.

(b) Is this market stable or unstable? How is your answer related to the parameter values?

Answer: We see that $(-\frac{5}{6})^t$ decreases in absolute value as t increases, and alternates between positive and negative as t is even or odd. So $-6(-\frac{5}{6})^t$ also decreases in absolute value as t increases, and alternates between positive and negative as t is odd or even. Therefore p_t oscillates and converges towards a limiting value of 10. (The price $p_t = 10$ is often referred to as the long run equilibrium price. If $p_t = p_{t-1} = 10$, then we have $q_t = q_{t-1} = 100$. However $p_t = 10$ is approached but never reached.)

2. Consider the following reduced-form macroeconomic model:

$Y_t = C_t + I_t$ (equilibrium between output and aggregate demand)

$C_t = 50 + 0.8Y_{t-1}$ (consumption behavioural equation, where *C* denotes both planned and actual consumption)

$I_t = 0.1(Y_t - Y_{t-1})$ (investment behavioural equation)

Find the time path of *Y*. If Y_0 = 300, will *Y* increase or decrease through time?

Answer: It is useful to first find the general solution to economic models of this form. The general form may be written as:

$$Y_t = C_t + I_t \quad ; \quad C_t = b + aY_{t-1} \quad ; \quad I_t = v(Y_t - Y_{t-1})$$

where *a, b* and *v* are constants.

Combining these 3 equations we get

$Y_t = b + aY_{t-1} + vY_t + (a-v)Y_{t-1}$ This re-arranges as

$$Y_t = \frac{b}{1-v} + \left(\frac{a-v}{1-v}\right)Y_{t-1}$$

This is a standard first-order linear difference equation. Solving it using the methods of Ex WS20.1 we get

$Y_t = A\left(\frac{a-v}{1-v}\right)^t + \frac{b}{1-a}$. In the given equation we have *a* = 0.8, *b* =50, *v* = 0.1.

Substituting these values into the general solution gives

$$Y_t = A\left(\frac{0.8-0.1}{1-0.1}\right)^t + \frac{50}{1-0.8} = A\left(\tfrac{7}{9}\right)^t + 250.$$

Given Y_0 = 300, we have $300 = A\left(\tfrac{7}{9}\right)^0 + 250$, from which *A* = 50. So the complete solution is $Y_t = 50\left(\tfrac{7}{9}\right)^t + 250$

Time path of *Y*. Because $\left(\tfrac{7}{9}\right)^t$ approaches zero as *t* increases, Y_t approaches a limiting value of 250 as *t* increases. The approach is smooth (that is, without oscillations), and the approach is from above (that is, Y_t is always greater than 250).

Exercise WS20.3

1. Solve the following first-order differential equations with the given initial conditions.

(a) $\frac{\mathrm{d}y}{\mathrm{d}t} + 3y = 12$; $y(0) = 1$

Answer: From chapter 20 of the book we know that a first-order differential equation of the form $\frac{\mathrm{d}y}{\mathrm{d}t} = by + c$ has the solution $y = Ae^{bt} - \frac{c}{b}$. In this example we have $b = -3,\ c = 12$ so the solution is

$$y = Ae^{-3t} + 4$$

Given $y(0) = 1$, we can find A as $1 = Ae^{-3(0)} + 4$, from which $A = -3$. So the complete solution is

$$y = -3e^{-3t} + 4$$

We can check this. The derivative of our answer is $\frac{dy}{dt} = 9e^{-3t}$. Substituting this and our answer into the given equation we get $9e^{-3t} + 3\left(-3e^{-3t} + 4\right) = 12$. The left hand side simplifies to 12, so our solution satisfies the given equation and is therefore correct. Also, we can check that our solution satisfies the initial condition, that $y(0) = 1$, by substituting $y = 1$, $t = 0$ into our solution. It then becomes

$1 = -3e^{-3(0)} + 4$ which is an identity since $e^{-3(0)} = 1$.

(b) $\frac{\mathrm{d}y}{\mathrm{d}t} - y = 2$; $y(0) = 0$

Answer: Using the method of solution from (a) above, in this example we have $b = 1,\ c = 2$ so the solution is

$$y = Ae^{t} - 2$$

Given $y(0) = 0$ we can find A as $0 = Ae^{(0)} - 2$, from which $A = 2$. So the complete solution is

$$y = 2e^{t} - 2$$

(c) $3\frac{dy}{dt}+4y=4$; $y(1)=0$

Answer: Using the method of solution from (a) above, in this example we have $b=-\frac{4}{3}$, $c=\frac{4}{3}$ so the solution is

$$y=Ae^{-\frac{4}{3}t}+1$$

Given $y(1)=0$ we can find A as $0=Ae^{-\frac{4}{3}(1)}+1$, from which $Ae^{-\frac{4}{3}(1)}+1=\frac{-1}{e^{-\frac{4}{3}}}=-e^{\frac{4}{3}}=-3.79$. So the complete solution is

$$y=-3.79e^{-\frac{4}{3}t}+1$$

2. In the economy of Utopia, planned investment is given by

$$\hat{I}_t=2\frac{dY}{dt}$$

(The idea underlying this is that the level of investment will be higher when aggregate demand is growing.)

Planned consumption is given by

$$\hat{C}_t=0.8Y_t.$$

(a) Given the equilibrium condition $Y_t=\hat{C}_t+\hat{I}_t$, and assuming the economy is always in equilibrium (that is, $C_t=\hat{C}_t$ and $I_t=\hat{I}_t$), find the time path of Y.

Answer: given the assumption that planned consumption and investment are always equal to the actual values, the model becomes

$$I_t=2\frac{dY}{dt} \quad ; \quad C_t=0.8Y_t \quad ; \quad Y_t=C_t+I_t$$

Substituting the first two equations into the third gives $Y_t=0.8Y_t+2\frac{dY}{dt}$. This rearranges as $\frac{dY}{dt}=0.1Y_t$. This is a first-order differential equation which we can solve using the method of the previous question. With b = 0.1, c = 0, the solution is $Y_t=Ae^{0.1t}$. (Note that the t subscripts on the variables are not strictly necessary but were put in to emphasise that the values of the variables are changing through time.)

(b) What is the rate of growth of *Y* along the equilibrium path?

Answer: from the solution, $Y_t = Ae^{0.1t}$, we see that $e^{0.1t}$ increases without limit as *t* increases, so therefore does $Y_t = Ae^{0.1t}$ (provided *A* is positive, which we can reasonably assume to be the case). The rate of growth at any moment in time is given by $\frac{1}{Y_t}\frac{dY_t}{dt}$ (see rule 13.7 in the book). Here $\frac{1}{Y_t}\frac{dY_t}{dt} = 0.1$, or 10% (see rule 13.8).