

Exercise WS19.1

1. Given: $\mathbf{A} = \begin{bmatrix} 4 & 3 \\ 1 & 2 \end{bmatrix}$ and $\mathbf{B} = \begin{bmatrix} 1 & 4 \\ 3 & 2 \end{bmatrix}$

Find:

(a) $\mathbf{A} + \mathbf{B}$ Answer: $\mathbf{A} + \mathbf{B} = \begin{bmatrix} 5 & 7 \\ 4 & 4 \end{bmatrix}$

(b) $2\mathbf{A} - 3\mathbf{B}$ Answer: $2\mathbf{A} - 3\mathbf{B} = \begin{bmatrix} 5 & -6 \\ -7 & -2 \end{bmatrix}$

(c) \mathbf{AB} Answer: $\mathbf{AB} = \begin{bmatrix} 13 & 22 \\ 7 & 8 \end{bmatrix}$

(d) $\mathbf{AB} + \mathbf{BA}$ Answer: $\mathbf{AB} + \mathbf{BA} = \begin{bmatrix} 21 & 33 \\ 21 & 21 \end{bmatrix}$

(e) $\mathbf{A}^T\mathbf{B}$ (where \mathbf{A}^T denotes the transpose of \mathbf{A})

Answer: $\mathbf{A}^T\mathbf{B} = \begin{bmatrix} 4 & 1 \\ 3 & 2 \end{bmatrix}$

(f) $\mathbf{B}^T\mathbf{A}$ Answer: $\mathbf{B}^T\mathbf{A} = \begin{bmatrix} 7 & 9 \\ 18 & 16 \end{bmatrix}$

(g) $\mathbf{A}^2\mathbf{B}$ Answer: $\mathbf{A}^2\mathbf{B} = \begin{bmatrix} 7 & 9 \\ 18 & 16 \end{bmatrix}$

2. Given: $\mathbf{A} = [a_1 \ a_2 \ a_3]$ and $\mathbf{B} = \begin{bmatrix} b_1 \\ b_2 \\ b_3 \end{bmatrix}$

Find \mathbf{AB} and \mathbf{BA}

Answer: $\mathbf{AB} = a_1b_1 + a_2b_2 + a_3b_3$ (a scalar)

$$\mathbf{BA} = \begin{bmatrix} b_1a_1 & b_1a_2 & b_1a_3 \\ b_2a_1 & b_2a_2 & b_2a_3 \\ b_3a_1 & b_3a_2 & b_3a_3 \end{bmatrix}$$

3. Given: $\mathbf{A} = \begin{bmatrix} a_1 \\ a_2 \\ a_3 \end{bmatrix}$ and $\mathbf{B} = \begin{bmatrix} b_1 \\ b_2 \\ b_3 \end{bmatrix}$

Find (if they exist) \mathbf{AB} , $\mathbf{A}^T\mathbf{B}$, \mathbf{BA} , $\mathbf{B}^T\mathbf{A}$, $\mathbf{A}^T\mathbf{B}^T$, and $\mathbf{B}^T\mathbf{A}^T$.

In each case, explain why the matrix product does (or does not) exist.

Answer: \mathbf{AB} does not exist, because the number of elements in the first row of \mathbf{A} (1), does not equal the number of elements in the first column of \mathbf{B} (3).

$$\mathbf{A}^T\mathbf{B} = a_1b_1 + a_2b_2 + a_3b_3 \quad (\text{a scalar})$$

\mathbf{BA} does not exist, because the number of elements in the first row of \mathbf{B} (1), does not equal the number of elements in the first column of \mathbf{A} (3).

$$\mathbf{B}^T\mathbf{A} = b_1a_1 + b_2a_2 + b_3a_3 \quad (\text{a scalar})$$

$\mathbf{B}^T\mathbf{A}^T$ does not exist, because the number of elements in the first row of \mathbf{B}^T (3), does not equal the number of elements in the first column of \mathbf{A}^T (1).

$\mathbf{A}^T\mathbf{B}^T$ does not exist, because the number of elements in the first row of \mathbf{A}^T (3), does not equal the number of elements in the first column of \mathbf{B}^T (1).

4. Given: $\mathbf{A} = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \end{bmatrix}$ and $\mathbf{B} = \begin{bmatrix} b_{11} & b_{12} \\ b_{21} & b_{22} \\ b_{31} & b_{32} \end{bmatrix}$

Find (if they exist) \mathbf{AB} and \mathbf{BA}

Answer: \mathbf{AB} exists, because the number of elements in the first row of \mathbf{A} (3) equals the number of elements in the first column of \mathbf{B} . And \mathbf{BA} exists, because the number of elements in the first row of \mathbf{B} (2) equals the number of elements in the first column of \mathbf{A} .

$$\mathbf{AB} = \begin{bmatrix} a_{11}b_{11} + a_{12}b_{21} + a_{13}b_{31} & a_{11}b_{12} + a_{12}b_{22} + a_{13}b_{32} \\ a_{21}b_{11} + a_{22}b_{21} + a_{23}b_{31} & a_{21}b_{12} + a_{22}b_{22} + a_{23}b_{32} \end{bmatrix}$$

$$\mathbf{BA} = \begin{bmatrix} a_{11}b_{11} + a_{21}b_{12} & a_{12}b_{11} + a_{22}b_{12} & a_{13}b_{11} + a_{23}b_{12} \\ a_{11}b_{21} + a_{21}b_{22} & a_{12}b_{21} + a_{22}b_{22} & a_{13}b_{21} + a_{23}b_{22} \\ a_{11}b_{31} + a_{21}b_{32} & a_{12}b_{31} + a_{22}b_{32} & a_{13}b_{31} + a_{23}b_{32} \end{bmatrix}$$

5. An investor buys 1000 shares in company x, 500 shares in company y and 100 shares in company z. The respective prices (in euros) are 150, 60, and 50 per share.

- (a) Define two vectors, one for quantities of shares purchased and the other for their prices, such that the product of the two vectors will give the total cost of the portfolio of shares.

Answer: we define the **row** vector of prices, $\mathbf{p} = [p_x, p_y, p_z]$. We define the

column vector of quantities purchased (owned), $\mathbf{q} = \begin{bmatrix} q_x \\ q_y \\ q_z \end{bmatrix}$. Then the value

of the portfolio (shares purchased or owned) is given by

$$V = \mathbf{p}\mathbf{q} = p_x q_x + p_y q_y + p_z q_z \text{ (a scalar) (It may help you if you think of this as a } 1 \times 1 \text{ matrix.)}$$

- (b) Calculate the total cost of the portfolio, using matrix multiplication.

The value of the portfolio (shares purchased or owned) is given by

$$V = \mathbf{p}\mathbf{q} = p_x q_x + p_y q_y + p_z q_z \text{ (a scalar)}$$

- (c) If the price of shares in company z doubles, and the prices of shares in companies x and y fall by 20%, calculate by matrix multiplication the change in the value of the portfolio.

The new price vector is $\mathbf{p}^* = [0.8p_x, 0.8p_y, 2p_z]$. Thus the new value of the portfolio is

$$V^* = \mathbf{p}^* \mathbf{q} = 0.8p_x q_x + 0.8p_y q_y + 2p_z q_z.$$

Therefore the change in the value of the portfolio is

$$\begin{aligned} V^* - V &= (\mathbf{p}^* - \mathbf{p})\mathbf{q} = (0.8 - 1)p_x q_x + (0.8 - 1)p_y q_y + (2 - 1)p_z q_z \\ &= 0.2p_x q_x + 0.2p_y q_y + p_z q_z \end{aligned}$$

Exercise WS19.2

1. Find the inverses (if they exist) of:

$$\mathbf{A} = \begin{bmatrix} 4 & 3 \\ 1 & 2 \end{bmatrix} ; \quad \mathbf{B} = \begin{bmatrix} 1 & 4 \\ 3 & 2 \end{bmatrix} ; \quad \mathbf{C} = \begin{bmatrix} 1 & 4 \\ -2 & -8 \end{bmatrix} ; \quad \mathbf{D} = \begin{bmatrix} 4 & 3 \\ 1 & 0.5 \end{bmatrix}$$

Answer: $\mathbf{A}^{-1} = \begin{bmatrix} \frac{2}{5} & \frac{-3}{5} \\ \frac{-1}{5} & \frac{4}{5} \end{bmatrix} ; \quad \mathbf{B}^{-1} = \begin{bmatrix} \frac{-1}{5} & \frac{2}{5} \\ \frac{3}{10} & \frac{-1}{10} \end{bmatrix}$

The inverse of \mathbf{C} does not exist, because the determinant of \mathbf{C} , $\det \mathbf{C} = (1 \times -8) - (-2 \times 4) = 0$.

$$\mathbf{D}^{-1} = \begin{bmatrix} \frac{2}{5} & \frac{-3}{5} \\ \frac{-1}{5} & \frac{4}{5} \end{bmatrix}$$

2. Using your answers from (1) above, solve where possible the following sets of simultaneous equations. If you find that no solution is possible, explain why.

(a) $4x + 3y = 5$

$x + 2y = 2$

Answer: we can write this set of equations in matrix notation as $\mathbf{Ax} = \mathbf{b}$, where

$$\mathbf{A} = \begin{bmatrix} 4 & 3 \\ 1 & 2 \end{bmatrix} ; \quad \mathbf{x} = \begin{bmatrix} x \\ y \end{bmatrix} ; \quad \mathbf{b} = \begin{bmatrix} 5 \\ 2 \end{bmatrix}$$

Therefore $\mathbf{x} = \mathbf{A}^{-1}\mathbf{b}$. From question 1 above, we have $\mathbf{A}^{-1} = \begin{bmatrix} \frac{2}{5} & \frac{-3}{5} \\ \frac{-1}{5} & \frac{4}{5} \end{bmatrix}$, so

$$\mathbf{x} = \mathbf{A}^{-1}\mathbf{b} = \begin{bmatrix} \frac{2}{5} & \frac{-3}{5} \\ \frac{-1}{5} & \frac{4}{5} \end{bmatrix} \begin{bmatrix} 5 \\ 2 \end{bmatrix} = \begin{bmatrix} \frac{4}{5} \\ \frac{3}{5} \end{bmatrix}. \text{ That is, the solution is } x = \frac{4}{5}, y = \frac{3}{5}. \text{ (Obviously,}$$

these values for x and y may be checked by the ordinary algebraic method of solution.)

(b) $x + 4y = -7$

$3x + 2y = -1$

Answer: following the method of (a) above, we have $\mathbf{Bx} = \mathbf{b}$, where

$$\mathbf{B} = \begin{bmatrix} 1 & 4 \\ 3 & 2 \end{bmatrix} \quad \mathbf{x} = \begin{bmatrix} x \\ y \end{bmatrix} ; \quad \mathbf{b} = \begin{bmatrix} -7 \\ -1 \end{bmatrix} \text{ and therefore, using question 1:}$$

$$\mathbf{x} = \mathbf{B}^{-1}\mathbf{b} = \begin{bmatrix} \frac{-1}{5} & \frac{2}{5} \\ \frac{3}{10} & \frac{-1}{10} \end{bmatrix} \begin{bmatrix} -7 \\ -1 \end{bmatrix} = \begin{bmatrix} \frac{9}{5} \\ \frac{-11}{5} \end{bmatrix}. \text{ That is, the solution is } x = \frac{9}{5}, y = \frac{-11}{5}.$$

(c) $x + 4y = 1$
 $-2x - 8y = 1$

Answer: following the method of (a) above, we have $\mathbf{C}\mathbf{x} = \mathbf{b}$, where

$$\mathbf{C} = \begin{bmatrix} 1 & 4 \\ -2 & -8 \end{bmatrix} \quad \mathbf{x} = \begin{bmatrix} x \\ y \end{bmatrix} \quad ; \quad \mathbf{b} = \begin{bmatrix} 1 \\ 1 \end{bmatrix} \text{ and therefore, using question 1:}$$

$$\mathbf{x} = \mathbf{C}^{-1}\mathbf{b}.$$

But from question 1, we know that \mathbf{C}^{-1} does not exist, therefore $\mathbf{x} = \mathbf{C}^{-1}\mathbf{b}$ cannot be evaluated. This means that no unique solution exists to this pair of simultaneous equations. This may be seen if we try to solve them by ordinary algebra. Equating the two right hand sides, we get $x = -4y$. Substituting this into the first equation, we get $0 = 0$. (The graphs of $x + 4y = 1$ and $-2x - 8y = 1$ are parallel lines.)

(d) $4x + 3y = 1$
 $x + 0.5y = -2$

$$\mathbf{D}^{-1} = \begin{bmatrix} \frac{2}{5} & \frac{-3}{5} \\ \frac{-1}{5} & \frac{4}{5} \end{bmatrix}$$

Answer: following the method of (a) above, we have $\mathbf{D}\mathbf{x} = \mathbf{b}$, where

$$\mathbf{D} = \begin{bmatrix} 4 & 3 \\ 1 & 0.5 \end{bmatrix} \quad \mathbf{x} = \begin{bmatrix} x \\ y \end{bmatrix} \quad ; \quad \mathbf{b} = \begin{bmatrix} 1 \\ -2 \end{bmatrix} \text{ and therefore, using question 1:}$$

$$\mathbf{x} = \mathbf{D}^{-1}\mathbf{b} = \begin{bmatrix} \frac{2}{5} & \frac{-3}{5} \\ \frac{-1}{5} & \frac{4}{5} \end{bmatrix} \begin{bmatrix} 1 \\ -2 \end{bmatrix} = \begin{bmatrix} \frac{8}{5} \\ \frac{-9}{5} \end{bmatrix}. \text{ That is, the solution is } x = \frac{8}{5}, y = \frac{-9}{5}.$$

3. Find the determinant, all minors and cofactors, and the inverse of each of the following matrices:

$$(a) \quad \mathbf{A} = \begin{bmatrix} 3 & 0 & -1 \\ 2 & 1 & 2 \\ 4 & 0 & 0 \end{bmatrix}$$

Answer: the minor $|M_{11}|$ is found by deleting the first row and first column of the given matrix, then evaluating the determinant of the resulting matrix. Thus $|M_{11}| = (1 \times 0) - (0 \times 2) = 0$. Similarly $|M_{12}| = (2 \times 0) - (4 \times 2) = -8$, and $|M_{13}| = (2 \times 0) - (4 \times 1) = -4$. Other minors may be found in the same way.

The cofactors are the signed minors, the sign being positive if the sum of the subscripts is even, negative if odd. So the cofactors are

$$+|M_{11}| = 0 \quad ; \quad -|M_{12}| = 8 \quad ; \quad +|M_{13}| = -4$$

The determinant $|\mathbf{A}|$ is $+a_{11}|M_{11}| - a_{12}|M_{12}| + a_{13}|M_{13}| = (3 \times 0) - (0 \times 8) + (-1 \times -4) = 4$

$$(b) \quad \mathbf{B} = \begin{bmatrix} 1 & 1 & 1 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{bmatrix}$$

Answer: $|M_{11}| = (1 \times 1) - (0 \times 1) = 1$; $|M_{12}| = (0 \times 1) - (0 \times 1) = 0$;
 $|M_{13}| = (0 \times 0) - (0 \times 1) = 0$. Other minors may be found in the same way.

The cofactors are: $+|M_{11}| = 1$; $-|M_{12}| = 0$; $+|M_{13}| = 0$

The determinant $|\mathbf{B}|$ is $+a_{11}|M_{11}| - a_{12}|M_{12}| + a_{13}|M_{13}| = (1 \times 1) - (1 \times 0) + (1 \times 0) = 1$

$$(c) \quad \mathbf{C} = \begin{bmatrix} -1 & -1 & -1 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{bmatrix}$$

Answer: $|M_{11}| = (1 \times 1) - (0 \times 1) = 1$; $|M_{12}| = (0 \times 1) - (0 \times 1) = 0$;
 $|M_{13}| = (0 \times 0) - (0 \times 1) = 0$. Other minors may be found in the same way.

The cofactors are: $+|M_{11}| = 1$; $-|M_{12}| = 0$; $+|M_{13}| = 0$

The determinant $|\mathbf{C}|$ is

$$+a_{11}|M_{11}| - a_{12}|M_{12}| + a_{13}|M_{13}| = (-1 \times 1) - (-1 \times 0) + (-1 \times 0) = -1$$

$$(d) \quad \mathbf{D} = \begin{bmatrix} 2 & 1 & 1 \\ 1 & 1 & 2 \\ 4 & 3 & 5 \end{bmatrix}$$

Answer: $|M_{11}| = (1 \times 5) - (3 \times 2) = -1$; $|M_{12}| = (1 \times 5) - (4 \times 2) = -3$;
 $|M_{13}| = (1 \times 3) - (4 \times 1) = -1$. Other minors may be found in the same way.

The cofactors are: $+|M_{11}| = -1$; $-|M_{12}| = 3$; $+|M_{13}| = -1$

The determinant $|\mathbf{C}|$ is

$$+a_{11}|M_{11}| - a_{12}|M_{12}| + a_{13}|M_{13}| = (2 \times -1) - (1 \times -3) + (1 \times -1) = 0$$

Exercise WS19.3

1. Use Cramer's rule to find x in the equation systems below.

$$(a) \quad 3x \quad -z = -2$$

$$2x + y + 2z = 0$$

$$4x \quad = 1$$

Answer: in matrix notation we can write the above as $\mathbf{Ax} = \mathbf{b}$, where

$$\mathbf{A} = \begin{bmatrix} 3 & 0 & -1 \\ 2 & 1 & 2 \\ 4 & 0 & 0 \end{bmatrix} ; \quad \mathbf{x} = \begin{bmatrix} x \\ y \\ z \end{bmatrix} ; \quad \mathbf{b} = \begin{bmatrix} -2 \\ 0 \\ 1 \end{bmatrix}$$

(Note that \mathbf{A} is the same as the matrix \mathbf{A} in Ex WS19.2 question 3, above; so we already know that its determinant is 4.)

To find x we form the matrix \mathbf{A}_1 in which the first column of \mathbf{A} is replaced by \mathbf{b} . So

$$\mathbf{A}_1 = \begin{bmatrix} -2 & 0 & -1 \\ 0 & 1 & 2 \\ 1 & 0 & 0 \end{bmatrix} \quad \text{Then } x = \frac{|\mathbf{A}_1|}{|\mathbf{A}|} = \frac{1}{4}$$

$$\text{Similarly } \mathbf{A}_2 = \begin{bmatrix} 3 & -2 & -1 \\ 2 & 0 & 2 \\ 4 & 1 & 0 \end{bmatrix} \quad \text{and } y = \frac{|\mathbf{A}_2|}{|\mathbf{A}|} = \frac{-24}{4} = -6$$

$$\text{Similarly } \mathbf{A}_3 = \begin{bmatrix} 3 & 0 & -2 \\ 2 & 1 & 0 \\ 4 & 0 & 1 \end{bmatrix} \quad \text{and } z = \frac{|\mathbf{A}_3|}{|\mathbf{A}|} = \frac{11}{4}$$

(Of course, these solution values may be checked by direct substitution into the given equations).

$$(b) \quad x + y + z = 1$$

$$y + z = 0$$

$$z = -1$$

Answer: in matrix notation we can write the above as $\mathbf{B}\mathbf{x} = \mathbf{b}$, where

$$\mathbf{B} = \begin{bmatrix} 1 & 1 & 1 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{bmatrix} ; \quad \mathbf{x} = \begin{bmatrix} x \\ y \\ z \end{bmatrix} ; \quad \mathbf{b} = \begin{bmatrix} 1 \\ 0 \\ -1 \end{bmatrix}$$

(Note that \mathbf{B} is the same as the matrix \mathbf{B} in Ex W19.2 question 3, above; so we already know that its determinant is 1.)

To find x we form the matrix \mathbf{B}_1 in which the first column of \mathbf{B} is replaced by \mathbf{b} . So

$$\mathbf{B}_1 = \begin{bmatrix} 1 & 1 & 1 \\ 0 & 1 & 1 \\ -1 & 0 & 1 \end{bmatrix} \quad \text{Then } x = \frac{|\mathbf{B}_1|}{|\mathbf{B}|} = \frac{1}{1} = 1$$

$$\text{Similarly } y = \frac{|\mathbf{B}_2|}{|\mathbf{B}|} = \frac{1}{1} = 1 \quad ; \quad z = \frac{|\mathbf{B}_3|}{|\mathbf{B}|} = \frac{-1}{1} = -1$$

2. Solve the following system of linear equations using (a) matrix inversion, and (b) Cramer's rule.

$$\begin{bmatrix} 2 & 1 & 1 \\ 1 & 1 & 2 \\ 4 & 3 & 4 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}$$

Answer: (a) In matrix notation we can write the above as $\mathbf{A}\mathbf{x} = \mathbf{b}$, where

$$\mathbf{A} = \begin{bmatrix} 2 & 1 & 1 \\ 1 & 1 & 2 \\ 4 & 3 & 4 \end{bmatrix} ; \quad \mathbf{x} = \begin{bmatrix} x \\ y \\ z \end{bmatrix} ; \quad \mathbf{b} = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}$$

$$\text{Then } \mathbf{x} = \mathbf{A}^{-1}\mathbf{b} = \begin{bmatrix} 2 & 1 & -1 \\ -4 & -4 & 3 \\ 1 & 2 & -1 \end{bmatrix} \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} = \begin{bmatrix} 1 \\ -4 \\ 2 \end{bmatrix}$$

(b) Cramer's rule: using the method of question 1, we have

$$|\mathbf{A}| = -1 ; |\mathbf{A}_1| = -1 ; |\mathbf{A}_2| = 4 ; |\mathbf{A}_3| = -2$$

$$\text{So } x = \frac{|\mathbf{A}_1|}{|\mathbf{A}|} = \frac{-1}{-1} = 1 ; y = \frac{|\mathbf{A}_2|}{|\mathbf{A}|} = \frac{4}{-1} = -4 ; z = \frac{|\mathbf{A}_3|}{|\mathbf{A}|} = \frac{-2}{-1} = 2$$

(Of course, (a) and (b) give the same solution.)

3. The demand functions for apples (subscripted a) and bananas (subscripted b) are inter-connected by the following demand functions:

$$Q_a^D = 100 - 15P_a + 2P_b \quad \text{and} \quad Q_b^D = 67 + 3P_a - 3P_b$$

The supply functions for apples and bananas (similarly subscripted) are:

$$Q_a^S = -4 + 25P_a \quad \text{and} \quad Q_b^S = -4 + 7P_b$$

(a) By imposing the equilibrium condition that quantity supplied equals quantity demanded in both markets, find a pair of simultaneous equations in the variables P_a and P_b .

Answer: the equilibrium condition in the market for apples, $Q_a^D = Q_a^S$, implies $100 - 15P_a + 2P_b = -4 + 25P_a$, which rearranges as $40P_a - 2P_b = 100$.

Similarly the equilibrium condition in the market for bananas, $Q_b^D = Q_b^S$, implies $67 + 3P_a - 3P_b = -4 + 7P_b$, which rearranges as $-3P_a + 10P_b = 71$.

Thus we have the simultaneous equations

$$\begin{aligned} 40P_a - 2P_b &= 100 & \text{and} \\ -3P_a + 10P_b &= 71 \end{aligned}$$

(Note, we have eliminated 2 of the 4 variables in the original equations.)

- (b) Show that this pair of simultaneous equations can be written in matrix form as

$$\mathbf{Ap} = \mathbf{b} \quad \text{where } \mathbf{A} = \begin{bmatrix} 40 & -2 \\ -3 & 10 \end{bmatrix} ; \mathbf{p} = \begin{bmatrix} P_a \\ P_b \end{bmatrix} ; \mathbf{b} = \begin{bmatrix} 104 \\ 71 \end{bmatrix}$$

Answer: $\mathbf{Ap} = \mathbf{b}$ implies $\begin{bmatrix} 40 & -2 \\ -3 & 10 \end{bmatrix} \begin{bmatrix} P_a \\ P_b \end{bmatrix} = \begin{bmatrix} 104 \\ 71 \end{bmatrix}$. Using matrix multiplication to

multiply out the left hand side, we get $\begin{bmatrix} 40P_a - 2P_b \\ -3P_a + 10P_b \end{bmatrix} = \begin{bmatrix} 104 \\ 71 \end{bmatrix}$, which means that $40P_a - 2P_b = 100$ and $-3P_a + 10P_b = 71$.

- (c) By inverting the matrix \mathbf{A} , find the equilibrium values of P_a and P_b . Hence find the equilibrium quantities of apples and bananas.

Answer: $\mathbf{Ap} = \mathbf{b}$ implies $\mathbf{A}^{-1}\mathbf{b} = \mathbf{p}$. We calculate \mathbf{A}^{-1} as $\mathbf{A}^{-1} = \begin{bmatrix} \frac{5}{197} & \frac{1}{197} \\ \frac{3}{394} & \frac{20}{197} \end{bmatrix}$.

Therefore $\mathbf{A}^{-1}\mathbf{b} = \begin{bmatrix} \frac{5}{197} & \frac{1}{197} \\ \frac{3}{394} & \frac{20}{197} \end{bmatrix} \begin{bmatrix} 104 \\ 71 \end{bmatrix} = \begin{bmatrix} 3 \\ 8 \end{bmatrix}$. Thus $\mathbf{p} = \mathbf{A}^{-1}\mathbf{b} = \begin{bmatrix} 3 \\ 8 \end{bmatrix}$. In ordinary algebra, $P_a = 3$, $P_b = 8$. Substituting these values into any of the given equations gives $Q_a^S = Q_a^D = 71$, $Q_b^S = Q_b^D = 52$

(Of course, we can check this is correct by substitution into the given equations.)

4. A businessperson owns 3 shops. Shop *S* is in the southern part of town; shop *E* is in the east; and shop *W* is in the west. Each sells 3 products: cigarettes, newspapers and ice-cream. Last month shop *S* sold 1000 packets of cigarettes, 2000 newspapers and 3000 ice creams. Shop *E* sold 2000 packets of cigarettes, 500 newspapers and 1000 ice creams. Shop *W* sold 4000 packets of cigarettes, 1000 newspapers and 500 ice creams. The cigarettes were sold at a price of 10 euros per packet; the newspapers at 3 euros each; and the ice creams at 2 euros each.

(a) Set up a matrix **A** showing the sales of each product by each shop.

Answer: To help us think clearly, we first construct the following table of quantities sold:

	south shop	east shop	west shop
cigarettes	1000	2000	4000
newspapers	2000	500	1000
ice creams	3000	1000	500

Then we define a matrix **A** such that element a_{ij} gives quantity sold of product i in shop j . (That is, each row gives the quantity sold of a product, and each column gives the quantities sold of a shop, as in the table above.) Therefore

$$\mathbf{A} = \begin{bmatrix} 1000 & 2000 & 4000 \\ 2000 & 500 & 1000 \\ 3000 & 1000 & 500 \end{bmatrix}$$

(b) Show that the total revenue of each shop can be obtained, in vector form, by appropriate multiplication of the matrix of quantities **A** by a vector of prices, **p**.

We define a **row** vector of prices, $\mathbf{p} = [p_c \quad p_n \quad p_i] = [10 \quad 3 \quad 2]$ where p_c = price of cigarettes, and so on. Then we can **pre-multiply** **A** by **p** and get

$$\mathbf{pA} = [10 \quad 3 \quad 2] \begin{bmatrix} 1000 & 2000 & 4000 \\ 2000 & 500 & 1000 \\ 3000 & 1000 & 500 \end{bmatrix} = [22000 \quad 23500 \quad 44000]$$

Thus the first element of this row vector gives total sales revenue of the south shop (22000), the second element gives total sales revenue of the east shop (23500), and so on.

- (c) Does it matter in (b) whether \mathbf{p} is defined as a row vector or a column vector?

Answer: In part (a) we could have constructed the following table:

	cigarettes	newspapers	ice creams
south shop	1000	2000	3000
east shop	2000	500	1000
west shop	4000	1000	500

Then we could have defined a matrix \mathbf{B} such that element b_{ij} gives quantity sold by shop i of product j . (That is, each row gives the quantities sold of a shop, and each column gives the quantity sold of a product.) Therefore

$$\mathbf{B} = \begin{bmatrix} 1000 & 2000 & 3000 \\ 2000 & 500 & 1000 \\ 4000 & 1000 & 500 \end{bmatrix} \quad (\text{Note that } \mathbf{B} \text{ is the transpose of } \mathbf{A}.)$$

In this case, we would need to define a **column** vector of prices, \mathbf{p}^* , where

$$\mathbf{p}^* = \begin{bmatrix} p_c \\ p_n \\ p_i \end{bmatrix} = \begin{bmatrix} 10 \\ 3 \\ 2 \end{bmatrix} \quad (\text{Note that } \mathbf{p}^* \text{ is the transpose of } \mathbf{p}.)$$

and then **post**-multiply \mathbf{B} by \mathbf{p}^* to get

$$\mathbf{Bp}^* = \begin{bmatrix} 1000 & 2000 & 3000 \\ 2000 & 500 & 1000 \\ 4000 & 1000 & 500 \end{bmatrix} \begin{bmatrix} 10 \\ 3 \\ 2 \end{bmatrix} = \begin{bmatrix} 22000 \\ 23500 \\ 44000 \end{bmatrix}$$

giving the total sales revenue of each shop, as before, but now in the form of a column vector rather than a row vector.

Thus whether we need to define our price vector as a row vector or a column vector, and whether we need to pre-multiply or post-multiply our matrix of quantities by it, in order to arrive at a vector of total sales revenue of each shop, depends on whether we choose to define our quantities sold matrix as in \mathbf{A} or as in \mathbf{B} .

- (d) What operation on the vector of total revenues would result in a scalar that gave the combined total revenue of all three shops? (Hint: Consider a vector [1, 1, 1].)

Answer: if we define a unit column vector as $\mathbf{i} = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$, then

$$\mathbf{pAi} = \begin{bmatrix} 22000 & 23500 & 44000 \end{bmatrix} \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} = 89500 \text{ (a scalar)}$$

Thus 89500 is the combined sales revenue of the three shops.

Alternatively, if we write the transpose of \mathbf{i} as $\mathbf{i}^* = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$, then

$$\mathbf{i}^* \mathbf{Bp}^* = \begin{bmatrix} 1 & 1 & 1 \end{bmatrix} \begin{bmatrix} 22000 \\ 23500 \\ 44000 \end{bmatrix} = 89500 \text{ (a scalar)}$$

5. Consider the following reduced-form macroeconomic model:

$$Y = C + I^* + G^* \quad \text{(equilibrium condition)} \quad (1)$$

$$C = a(Y - T) + b \quad \text{(consumption behavioural relationship)} \quad (2)$$

$$T = tY \quad \text{(income tax revenue, } T, \text{ as a function of the income tax rate, } t) \quad (3)$$

The unknown or endogenous variables are Y = aggregate output and income, C = consumption by households (assumed equal to their planned or desired consumption \hat{C}) and T = income tax revenue. The pre-determined or exogenous variables are I^* and G^* (private investment and government expenditure respectively) and b . The parameters are a and t .

- (a) Show that, after suitable rearrangement, this set of simultaneous equations can be written as $\mathbf{Ax} = \mathbf{b}$, where:

$$\mathbf{A} = \begin{bmatrix} 1 & -1 & 0 \\ a & -1 & -a \\ t & 0 & -1 \end{bmatrix} ; \quad \mathbf{x} = \begin{bmatrix} Y \\ C \\ T \end{bmatrix} ; \quad \mathbf{b} = \begin{bmatrix} I^* + G^* \\ -b \\ 0 \end{bmatrix}$$

Note that \mathbf{A} is a matrix containing the parameters of the model, \mathbf{x} is a vector containing the endogenous variables and \mathbf{b} is a vector containing the exogenous variables.

Answer: we can rearrange the given equations as follows.

$$\begin{aligned} Y - C &= I^* + G^* \\ aY - C - aT &= b \\ tY - T &= 0 \end{aligned}$$

When these equations are written in matrix form, they become $\mathbf{Ax} = \mathbf{b}$, where \mathbf{A} , \mathbf{x} and \mathbf{b} are as given above.

(b) By calculating \mathbf{A}^{-1} , find the equilibrium values of the endogenous variables as a function of the values of the exogenous variables and the parameter values.

Answer: given $\mathbf{Ax} = \mathbf{b}$, we have $\mathbf{x} = \mathbf{A}^{-1}\mathbf{b}$. In general this gives us the values of the endogenous variables (Y , C and T) as a function of the exogenous variables (I , G and b) and the parameters (a and t). (Pedantic aside: a purist might question whether it is appropriate to treat b as exogenous and t as a parameter. We have done this to simplify the question.)

The matrix of transposed cofactors, \mathbf{C} , is $\mathbf{C}' = \begin{bmatrix} 1 & -1 & a \\ a(1-t) & -1 & a \\ t & -t & a-1 \end{bmatrix}$.

The determinant of \mathbf{A} is $|\mathbf{A}| = 1 - a(1-t)$. So \mathbf{A}^{-1} is

$$\mathbf{A}^{-1} = \frac{1}{|\mathbf{A}|} \mathbf{C}' = \begin{bmatrix} \frac{1}{1-a(1-t)} & \frac{-1}{1-a(1-t)} & \frac{a}{1-a(1-t)} \\ \frac{a(1-t)}{1-a(1-t)} & \frac{-1}{1-a(1-t)} & \frac{a}{1-a(1-t)} \\ \frac{t}{1-a(1-t)} & \frac{-t}{1-a(1-t)} & \frac{a-1}{1-a(1-t)} \end{bmatrix}. \text{ Therefore}$$

$$\text{So } \mathbf{x} = \mathbf{A}^{-1}\mathbf{b} \text{ implies } \begin{bmatrix} Y \\ C \\ T \end{bmatrix} = \begin{bmatrix} \frac{1}{1-a(1-t)} & \frac{-1}{1-a(1-t)} & \frac{a}{1-a(1-t)} \\ \frac{a(1-t)}{1-a(1-t)} & \frac{-1}{1-a(1-t)} & \frac{a}{1-a(1-t)} \\ \frac{t}{1-a(1-t)} & \frac{-t}{1-a(1-t)} & \frac{a-1}{1-a(1-t)} \end{bmatrix} \begin{bmatrix} I^* + G^* \\ -b \\ 0 \end{bmatrix}$$

(c) Give an economic interpretation to the elements of \mathbf{A}^{-1} .

Answer: Multiplying out our expression for $\mathbf{x} = \mathbf{A}^{-1}\mathbf{b}$ above, we get (in ordinary algebra):

$$Y = \frac{1}{1-a(1-t)}(I^* + G^*) + \frac{1}{1-a(1-t)}b ;$$

$$C = \frac{a(1-t)}{1-a(1-t)}(I^* + G^*) + \frac{1}{1-a(1-t)}b ;$$

$$T = \frac{t}{1-a(1-t)}(I^* + G^*) + \frac{t}{1-a(1-t)}b$$

Thus we see that the first element of the first row, $\frac{1}{1-a(1-t)}$, gives us the relationship between Y and $I^* + G^*$. Similarly the second element of the first row, $\frac{-1}{1-a(1-t)}$, gives us the relationship between $-b$ and Y . (The third element of the first row drops out as it is multiplied by 0.)

The other elements can be interpreted similarly. For example the first element of the second row, $\frac{a(1-t)}{1-a(1-t)}$, gives us the relationship between C and $I^* + G^*$.

- (d) Using your answer to (b), find the equilibrium values of the endogenous variables when $a = 0.75$, $t = 0.2$, $b = 500$, and $I^* + G^* = 1000$. Check your answer by solving the model by simultaneous equation methods.

Answer: we substitute these values for the parameters a and t , and the exogenous variables I^* , G^* and b , into our expression for $\mathbf{x} = \mathbf{A}^{-1}\mathbf{b}$ in (b) above. It then becomes

$$\begin{bmatrix} Y \\ C \\ T \end{bmatrix} = \begin{bmatrix} 2.5 & -2.5 & 1.875 \\ 1.5 & -2.5 & -1.875 \\ 0.5 & -0.5 & -0.625 \end{bmatrix} \begin{bmatrix} 1000 \\ -500 \\ 0 \end{bmatrix}. \text{ Multiplying out, we get}$$

$$Y = 3750, C = 2750, T = 750$$

We can check this by solving the three simultaneous equations given in the question, using ordinary algebra.

- (e) Use your answer to (b) to calculate the effect on Y , C , and T of a small increase in each of the exogenous variables, I^* , G^* , and b .

Answer: As we saw in our answer to (c), the first element of the first row, $\frac{1}{1-a(1-t)}$, of \mathbf{A}^{-1} gives us the relationship between Y and $I^* + G^*$. It tells us that if either I^* or G^* increases by 1 unit, Y will increase by $\frac{1}{1-a(1-t)}$ units. Here $\frac{1}{1-a(1-t)} = 2.5$. Thus $\frac{1}{1-a(1-t)}$ is the private investment and government expenditure multiplier.

Similarly the first second of the first row, $\frac{-1}{1-a(1-t)}$, of \mathbf{A}^{-1} gives us the relationship between Y and $-b$. It tells us that if b increases by 1 unit, Y will increase by $\frac{-1}{1-a(1-t)}(-b) = \frac{1}{1-a(1-t)}$ units. Here $\frac{1}{1-a(1-t)} = 2.5$.

Repeating the previous paragraph for the second row of \mathbf{A}^{-1} , we see that if I^* or G^* increase by 1 unit, C will increase by $\frac{a(1-t)}{1-a(1-t)} = 1.5$ units. If b increases by 1 unit, C will increase by $\frac{-1}{1-a(1-t)}(-b) = \frac{1}{1-a(1-t)} = 2.5$ units.

Finally, from the elements of the third row of \mathbf{A}^{-1} we can see that if $I^* + G^*$ increases by 1 unit, tax revenue, T , will increase by $\frac{t}{1-a(1-t)}$ units = 0.5 units. If b increases by 1 unit, T will increase by $\frac{-t}{1-a(1-t)}(-1) = (-0.5)(-1) = 0.5$ units.

(Note that we have been, in effect, differentiating our matrix expression to get these results. This technique of matrix differentiation is quite straightforward but we did not have space for it in the book.)