Exercise WS18.1

1. Find the following indefinite integrals.

(a) $\int (3x^{\frac{1}{2}} + x^{-2})\,\mathrm{d}x = 2x^{\frac{3}{2}} - \frac{1}{x} + c$

(b) $\int (4x^3 - x^{-1})\,\mathrm{d}x = x^4 - \ln(x) + c$

(c) $\int \frac{1}{2}\left(\frac{2x + e^x}{x^2 + e^x}\right)\mathrm{d}x = \frac{1}{2}(x^2 + e^x)^{-1} + c$

(d) $\int \left(2xe^{x^2}\right)\mathrm{d}x = e^{x^2} + c$

2. Find the following indefinite integral (harder example; some guesswork needed):

$$\int x^2 e^x \mathrm{d}x$$

Answer: as we know that the integral of e^x is e^x, there seems a faint hope that the integral of x^2e^x might be x^2e^x. If so, then by differentiating x^2e^x, we should get back to x^2e^x again. Let's try differentiating x^2e^x. We get $\frac{d}{dx}\left[x^2e^x\right] = x^2e^x + 2xe^x$ (Power rule of differentiation). This is not right, because of the unwanted term $2xe^x$. So we have to get rid of this term, by thinking of something of which the derivative is $-2xe^x$. After some experimentation, we stumble on the fact that that something is $2e^x - 2xe^x$, because the derivative of this is

$$\frac{d}{dx}\left[2e^x - 2xe^x\right] = 2e^x - (2xe^x + 2e^x) = -2xe^x .$$

So we conclude that $\int x^2e^x\mathrm{d}x = x^2e^x + 2e^x - 2xe^x$. As usual, we check by differentiating the right hand side of this equation to see whether we then get the left hand side:

$$\frac{d}{dx}\left[x^2e^x + 2e^x - 2xe^x\right] = \left(x^2e^x + 2xe^x\right) + 2e^x - (2xe^x + 2e^x) = x^2e^x$$

so our answer is correct. (Note, there is a method for solving some examples of this type of problem, but it was not covered in the book.)

Exercise WS18.2

Evaluate the following definite integrals

1. $\int_0^5 (3x^2 + x^{-2})\mathrm{d}x$ Answer: $\int (3x^2 + x^{-2})\mathrm{d}x = x^3 - \frac{1}{x} + c$. Therefore

$$\int_0^5 (3x^2 + x^{-2})\mathrm{d}x = \left[x^3 - \frac{1}{x}\right]_{x=5} - \left[x^3 - \frac{1}{x}\right]_{x=0} = 124\tfrac{4}{5} + \infty = \infty$$

[Note, you might think that this definite integral is undefined, because $\frac{1}{x}$ is undefined when x = 0. However, the definite integral is a limiting value approached as the number of rectangles drawn under the curve increases (and the base of each rectangle gets smaller and smaller). As this limiting value is approached, the height of one of these rectangles (the one closest to the *y* axis) approaches infinity.]

2. $\int_0^1 (1-x)^2\mathrm{d}x$ Answer: $\int (1-x)^2\mathrm{d}x = -\tfrac{1}{3}(1-x)^3 + c$; $\int_0^1 (1-x)^2\mathrm{d}x = \tfrac{1}{3}$

3. $\int_0^{20} 10e^{-0.05x}dx$ Answer: $\int 10e^{-0.05x}dx = -200e^{-0.05x} + c$;

$$\int_0^{20} 10e^{-0.05x}dx = -200e^{-0.05x}$$

4. $\int_1^e \frac{2}{x}\mathrm{d}x$ Answer: $\int \frac{2}{x} = 2\ln x + c$; $\int_1^e \frac{2}{x}\mathrm{d}x = 2$

Exercise WS18.3

1. The inverse demand function for a good is $p = 100 - \frac{1}{4}q$.

(a) Show that total revenue when $q = 100$ can be measured as the area under the marginal revenue curve between $q = 0$ and $q = 100$.

Answer: since *MR* is the derivative of *TR*, and finding the indefinite integral reverses differentiation, it follows that in general $\int MR\,dq = TR$

(b) Calculate total revenue by this method, and check your answer by directly obtaining total revenue from the total revenue function.

Answer: total revenue is $TR \equiv pq = \left(100 - \frac{1}{4}q\right)q = 100q - \frac{1}{4}q^2$. Therefore

$MR \equiv \dfrac{dTR}{dq} = 100 - \frac{1}{2}q$. To find the area under the MR curve between $q = 0$ and $q = 100$, we find the definite integral

$$\int_0^{100} \left(100 - \tfrac{1}{2}q\right)dq = \left[\left(100q - \tfrac{1}{4}q^2\right)\right]_{q=100} - \left[\left(100q - \tfrac{1}{4}q^2\right)\right]_{q=0}$$

$$= 100(100) - \tfrac{1}{4}(100)^2 - 0 = 7500$$

We can find TR directly from the TR function when q = 100 using, we evaluate $TR = 100q - \frac{1}{4}q^2 = 100(100) - \frac{1}{4}(100)^2 = 7500$. When we compare the two calculations, it is obvious that the area under the MR curve between 0 and some quantity q_0 will always give us the TR for that quantity.

(c) Illustrate your answers to (a) and (b) by means of a sketch graph.

Answer: see figure 18.7 in the book.

2. We know that under perfect competition a firm's marginal cost curve is also its supply curve (see section 8.18 of the book). Suppose there are 100 perfectly competitive firms in an industry, each with the marginal cost function $MC = \frac{1}{4}q + 5$.

(a) Find the producer's surplus of each firm when $p = 20$, and the producers' surplus of all firms in aggregate.

Answer: first, we set $p = MC$ to find the typical firm's most profitable output. This gives $p = 20 = MC = \frac{1}{4}q + 5$, with solution q = 60. So $TR \equiv pq$ = 1200.

Second, we find the indefinite integral of the MC function, which is $\int \left(\frac{1}{4}q + 5\right)dq = \frac{1}{8}q^2 + 5q + c$.

Third, we find the definite integral of the MC function between q = 0 and q = 60. This is $\int_0^{60} \left(\frac{1}{4}q + 5\right)dq = \left[\frac{1}{8}q^2 + 5q\right]_{q=60} - \left[\frac{1}{8}q^2 + 5q\right]_{q=0} = 150$. This definite integral measures total variable cost (TVC); that is, the difference between the total cost of producing 60 units and the total cost of producing zero units. (The latter equals fixed costs, FC.) (See section 18.10 of the book.)

Finally, by definition producer's surplus (*PS*) equals *TR* – *TVC*. So when $q = 60$, *PS* = *TR* – *TVC* = 1200 – 150 = 1050. As there are 100 identical firms in the industry, aggregate *PS* is 100×1050 = 105000.

(b) What inferences can we make about profits, from information about producer's surplus?

Answer: we have three definitions: (i) profit is given by $\Pi \equiv TR - TC$; (ii) *PS* is given by $PS \equiv TR - TVC$; (iii) $TC \equiv TVC + FC$. Combining these, we get
$\Pi \equiv TR - TC \equiv TR - TVC - FC \equiv PS - FC$
Thus profits are divided between producer's surplus and fixed costs. In this example we are given no information about fixed costs, so in this case we can make no inference regarding profits from our calculation of producer's surplus. If however we had found that producer's surplus was negative, then we could safely infer that profits were negative (assuming that fixed costs are zero or positive, which seems reasonable).

3. Given the inverse demand function $p = 9 - q - 0.1q^2$.

(a) Find the consumers' surplus when quantity purchased is $q = 5$.

Answer: first, when $q = 5$, $p = 9 - q - 0.1q^2 = 1.5$. So total expenditure by consumers' is $p_0q_0 = 1.5\times5 = 7.5$.

Second, if the inverse demand function is $p = f(q)$, then the consumers' surplus (*CS*) when the price is p_0. and quantity purchased is q_0 is given by $CS \equiv \int_0^{q_0}[f(q)]dq - p_0q_0$ (see rule 18.4 in book).

In this example we have $f(q) = 9 - q - 0.1q^2$, so the indefinite integral is
$\int[f(q)]dq = \int\left[9 - q - 0.1q^2\right]dq = 9q - \frac{1}{2}q^2 - \frac{1}{30}q^3 + c$.
Therefore the definite integral when $q = 5$ is

$\int_0^5[f(q)]dq = \left[9q - \frac{1}{2}q^2 - \frac{1}{30}q^3\right]_{q=5} - \left[9q - \frac{1}{2}q^2 - \frac{1}{30}q^3\right]_{q=0} = 28\frac{1}{3}$. Thus the consumers' surplus is $\int_0^{q_0}[f(q)]dq - p_0q_0 = 28\frac{1}{3} - 7.5 = 20\frac{5}{6}$. (Note that *CS* is very large ($28\frac{1}{3}$) relative to consumers' total expenditure (7.5). This is partly because the price is very low (1.5).)

(b) How does consumers' surplus change if *q* increases to $q = 5.5$?

Answer: repeating (a) with q = 5.5 gives CS = 28.829.

4. John rents a flat for 1000 euros per month. The owner offers to sell the flat to John at a price of x euros. If John can borrow or lend as much as he wishes at a nominal rate of 5% per year, and interest is continuously compounded, what is the largest value of x such that John would wish to buy the flat? (Assume the future is perfectly certain and that there will be no change in prices, incomes or in any other relevant factor.)

Answer: if John buys the flat for x euros, he will be better off in future by the amount of future rent payments that he will no longer have to make. Therefore the maximum that John should be willing to pay to buy the flat is given by the present value (PV) of this series of rental payments. If he assumes the rental payments will last for t years, and discounting is continuous, the PV is given by

$$PV = \int_0^t \left[ae^{rt}\right]dt = \frac{a}{r}\left(1 - e^{-rt}\right) = \frac{12000}{0.05}\left(1 - e^{-0.05t}\right)$$ (See rule 18.6a in book.

Note that we have used t to denote time in this example, as x has already been used in the question to denote the selling price of the flat. Note too that as r is the annual interest rate, we have re-expressed the rent as 12000 per year instead of 1000 per month.)

Now we have to decide what the value of t should be. One approach is to say that t should go to infinity, since neither John nor his heirs who will inherit the flat when he dies, nor his heirs' heirs, and so on forever, will have to pay any rent. If we adopt this approach, we see that in the expression above, as t approaches infinity so $e^{-0.05t}$ approaches zero, so the PV becomes

$$PV \frac{12000}{0.05}\left(1 - 0\right) = 240000$$

However the above approach may be criticised, partly because the flat will not last forever but will eventually become useless as accommodation, due to depreciation. And even if the flat will last forever, John may not necessarily be interested in, or willing to pay a higher price for, the benefits that his heirs will enjoy. Therefore for one or both of these reasons John might set t equal to, say, 50 (years). The PV is then

$$PV = \frac{12000}{0.05}\left(1 - e^{-0.05(5)}\right) = 220300$$

So if John expects the flat to last forever and cares as much about his heirs as he does about himself, x will be 240,000 euros. If his time horizon is only 50 years (either because he expects the flat to have physically collapsed by then, or because he expects to be dead and doesn't care about his heirs), then x will be 220,300 euros. Note how relatively small is the difference between these two PVs.

Obviously other assumptions will lead to a different value for t and therefore to different values for x. (Try calculating x if John's planning horizon is only, say, 5 years.)