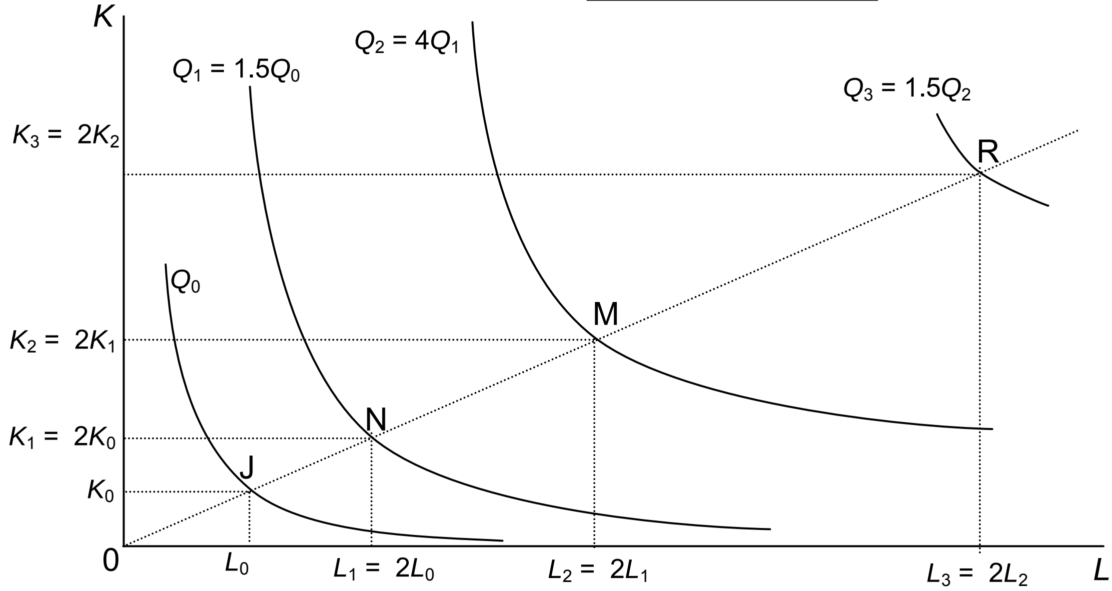
Exercise WS17.1

1. (a)

Figure for q. 1(a)



J

Figure for q.1(a). Because J, N, M and P lie on the same ray from the origin, they all have the same ratio of K to L; they differ only in the scale of output. The movement from J to N doubles both inputs, but less than doubles output (decreasing returns). The movement from N to M doubles both inputs, but more than doubles output (increasing returns). Finally, the movement from M to P doubles both inputs, but less than doubles output (decreasing returns).

1(b).

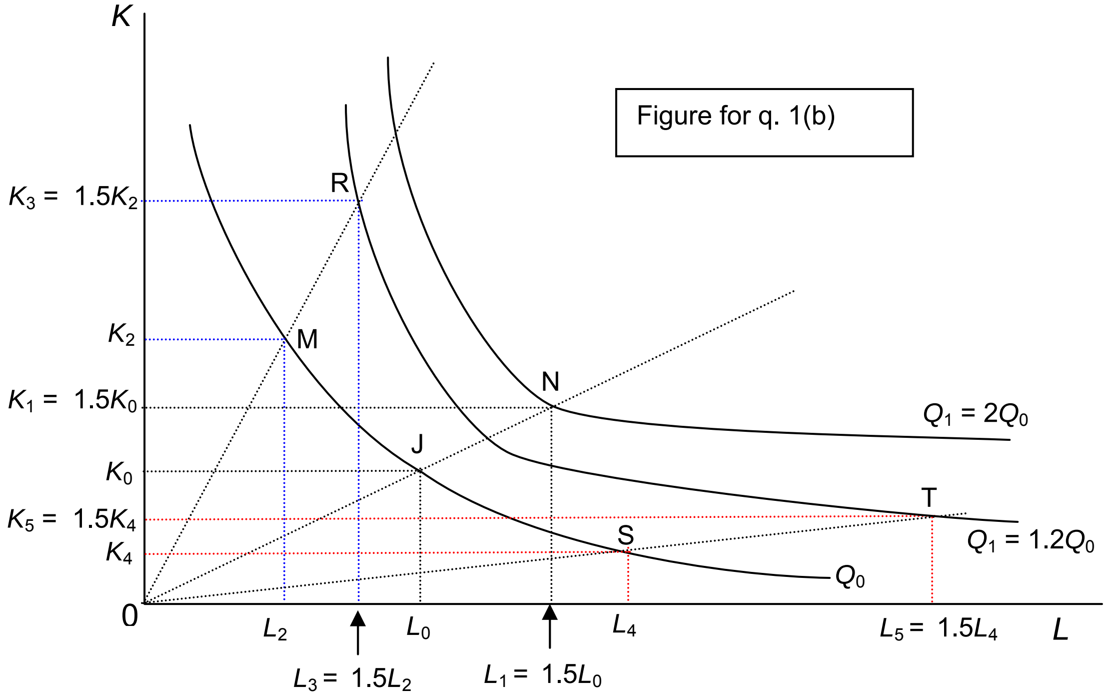


Figure for q.1(b). At M, K is large relative to L (capital intensive production). In moving from M to R, K and L increase by 50% but Q increases by only 20% (decreasing returns). At S, K is small relative to L (labour intensive production). In moving from S to T, K and L increase by 50% but Q increases by only 20% (decreasing returns). At J, production is neither capital intensive nor labour intensive. In moving from J to N, K and L increase by 50% but Q increases by 100% (increasing returns).

2. (a) $z = 3x^3 - 4xy^2 + 9x^{-3}y^6$

The initial value of z, z_0 (when $x = x_0$ and $y = y_0$) is

$z_0 = 3x_0^3 - 4x_0y_0^2 + 9x_0^{-3}y_0^6$

When x increases to $x = \lambda x_0$ and y to $y = \lambda y_0$, the new value of z, z_1, is

$z_1 = 3(\lambda x_0)^3 - 4(\lambda x_0)(\lambda y_0)^2 + 9(\lambda x_0)^{-3}(\lambda y_0)^6$

Removing the brackets from this, we get $z_1 = 3\lambda^3 x_0^3 - 4\lambda^{1+2}x_0y_0^2 + 9\lambda^{-3+6}x_0^{-3}y_0^6$

$= 3\lambda^3 x_0^3 - 4\lambda^3 x_0y_0^2 + 9\lambda^3 x_0^{-3}y_0^6 = \lambda^3\left(3x_0^3 - 4x_0y_0^2 + 9x_0^{0.5}y_0^6\right) = \lambda^3 z_0$

Therefore the function is homogeneous of degree 3 (the power to which λ is raised at the right hand end of the line above). This means for example that if λ = 2, then $\lambda^3 = 8$, so if x and y are both doubled, z increases eight-fold.

(b) $z = \sqrt[3]{x^2y}$

Using the method of (a) above, we first write the initial value of z. For simplicity we will from now on drop the "zero" subscripts so the initial values are denoted simply by x and y. So the initial value of z is simply $z_0 = \sqrt[3]{x^2y} \equiv \left(x^2y\right)^{\frac{1}{3}}$. The new value of z is $z_1 = \left((\lambda x)^2 \lambda y\right)^{\frac{1}{3}} = \left(\lambda^2 x^2 \lambda y\right)^{\frac{1}{3}} = \left(\lambda^3 x^2 y\right)^{\frac{1}{3}} = \lambda\left(x^2y\right)^{\frac{1}{3}} = \lambda^1 z_0$

Therefore the function is homogeneous of degree 1 (the power to which λ is raised at the right hand end of the line above). This means for example that if λ = 2, then $\lambda^1 = \lambda = 2$, so if x and y are both doubled, z also doubles.

(c) $z = (x^2 + xy - y^2)^{0.5}$

The initial value is $z_0 = (x^2 + xy - y^2)^{0.5}$. The new value is

$$z_1 = \left[(\lambda x)^2 + (\lambda x)(\lambda y) - (\lambda y)^2\right]^{0.5} = \left[\lambda^2\left(x^2 + xy - y^2\right)\right]^{0.5}$$

$$= \lambda^1\left(x^2 + xy - y^2\right)^{0.5} = \lambda^1 z_0$$

So this function is homogeneous of degree 1.

(d) $z = \dfrac{x^2}{3x^2 + 4xy - y^2}$

The initial value is $z_0 = \dfrac{x^2}{3x^2 + 4xy - y^2}$. The new value is

$$z_1 = \frac{(\lambda x)^2}{3(\lambda x)^2 + 4(\lambda x)(\lambda y) - (\lambda y)^2} = \frac{\lambda^2 x^2}{3\lambda^2 x^2 + 4\lambda^2 xy - \lambda^2 y^2}$$

$$= \frac{\lambda^2}{\lambda^2}\frac{x^2}{3x^2 + 4xy - y^2} = \lambda^0 \frac{x^2}{3x^2 + 4xy - y^2} = \lambda^0 z_0 = z_0 .$$

So this function is homogeneous of degree 0. So if λ = 2, say, then *x* and *y* both double but *z* is left unchanged.

3. (a) $Q = 100K^{0.3}L^{0.7}$

Using the method of question (2) above, the initial value of *Q* is $Q_0 = 100K^{0.3}L^{0.7}$. The new value is
$Q_1 = 100(\lambda K)^{0.3}(\lambda L)^{0.7} = 100\lambda^{0.3}K^{0.3}\lambda^{0.7}L^{0.7} = 100\lambda^1 K^{0.3}L^{0.7} = \lambda^1 Q_0$. This production function is homogeneous of degree one (also known as linear homogeneous), and therefore has constant returns to scale. Any given proportionate change in both inputs changes output in the same proportion.

(b) $Q = 50K^{0.6}L^{0.5}$

The initial value of *Q* is $Q_0 = 50K^{0.6}L^{0.5}$. The new value is
$Q_1 = 100(\lambda K)^{0.6}(\lambda L)^{0.5} = 100\lambda^{1.1}K^{0.6}L^{0.5} = \lambda^{1.1}Q_0$ ..This production function is homogeneous of degree 1.1, and therefore has increasing returns to scale. If the proportionate change in both inputs is 2 (that is, λ = 2), the proportionate change in output is $\lambda^{1.1} = 2^{1.1} = 2.14$. In other words, if both inputs increase by 100%, output increases by 114%.

(c) $Q = AK^{\alpha}L^{\beta}$

The initial value of *Q* is $Q_0 = 50K^{\alpha}L^{\beta}$. The new value is
$Q_1 = 100(\lambda K)^{\alpha}(\lambda L)^{\beta} = 100\lambda^{\alpha+\beta}K^{\alpha}L^{\beta} = \lambda^{\alpha+\beta}Q_0$. This production function is homogeneous of degree α + β. Therefore there are constant, increasing or

decreasing returns to scale depending on whether $\alpha + \beta$ is equal to, greater than or less than 1.

(d) $Q = aKL - bK^2 - cL^2$ (where *a*, *b*, and *c* are parameters)

The initial value of *Q* is $Q_0 = aKL - bK^2 - cL^2$. The new value is $Q_1 = a(\lambda K)(\lambda L) - b(\lambda K)^2 - c(\lambda L)^2 = \lambda^2(aKL - bK^2 - cL^2) = \lambda^2 Q_0$. This production function is homogeneous of degree 2 (increasing returns to scale).

(e) $Q = aKL - bK^2 - cL^2 + dK + eL$ (where *a*, *b*, *c*, *d,* and *e* are parameters)

The initial value of *Q* is $Q_0 = aKL - bK^2 - cL^2 + dK + eL$. The new value is $Q_1 = a(\lambda K)(\lambda L) - b(\lambda K)^2 - c(\lambda L)^2 + d(\lambda K) + e(\lambda L)$
$= \lambda^2(aKL - bK^2 - cL^2) + \lambda(dK + eL)$. As there is no common factor involving λ, this production function is not homogeneous and there are no uniform constant, increasing or decreasing returns to scale.

Exercise WS17.2

1. Euler's theorem says that if a function $Q = f(K, L)$ is homogeneous of degree *v*, then: $Kf_K + Lf_L = vQ$, where f_L and f_K are the partial derivatives (marginal products). (see equation 17.1 in book).

(a) Given $Q = 100K^{0.3}L^{0.7}$, we have
$Kf_K + Lf_L = \left[(0.3)100K^{-0.7}L^{0.7}\right]K + \left[(0.7)100K^{0.3}L^{-0.3}\right]L$
$= (0.3)100K^{0.3}L^{0.7} + (0.7)100K^{0.3}L^{0.7} = 1 \times Q = Q$. Thus $v = 1$. Since we found in Ex WS17.1 question 3 that this function was homogenous of degree 1, Euler's theorem is verified.

Sketch of *AC* and *MC* functions. As there are constant returns to scale, sketches would be like figure 17.4(a) in book.

(b) Given $Q = 50K^{0.6}L^{0.5}$, we have
$Kf_K + Lf_L = \left[(0.6)50K^{-0.4}L^{0.5}\right]K + \left[(0.5)50K^{0.6}L^{-0.5}\right]L$
$= (0.6)50K^{0.6}L^{0.5} + (0.5)50K^{0.6}L^{0.5} = 1.1 \times Q$. Thus $v = 1.1$. Since we found in Ex WS17.1 question 3 that this function was homogenous of degree 1.1, Euler's theorem is verified.

Sketch of *AC* and *MC* functions. As there are increasing returns to scale, sketches would be like figure 17.4(c) in book.

(c) Given $Q = AK^{\alpha}L^{\beta}$, we have

$$Kf_K + Lf_L = \left[\alpha AK^{\alpha-1}L^{\beta}\right]K + \left[\beta AK^{\alpha}L^{\beta-1}\right]L$$

$= \alpha AK^{\alpha}L^{\beta} + \beta AK^{\alpha}L^{\beta} = (\alpha+\beta)AK^{\alpha}L^{\beta} = (\alpha+\beta)Q$. Thus $v = \alpha + \beta$. Since we found in Ex WS17.1 question 3 that this function was homogenous of degree $\alpha + \beta$, Euler's theorem is verified.

Sketch of *AC* and *MC* functions. If $\alpha + \beta = 1$, sketches would be like figure 17.4(a) in book. If $\alpha + \beta < 1$, sketches would be like figure 17.4(b). If $\alpha + \beta > 1$, sketches would be like figure 17.4(c).

(d) Given $Q = aKL - bK^2 - cL^2$, we have $Kf_K + Lf_L = aKL - 2bK^2 + aKL - 2cL^2 = 2Q$. Thus $v = 2$. Since we found in Ex WS17.1 question 3 that this function was homogenous of degree 2, Euler's theorem is verified.

As increasing returns, sketch of *AC* and *MC* functions would be like figure 17.4(c) in book.

(e) Since we found in Ex WS17.1 question 3 that this function was not homogeneous, it does not conform to Euler's theorem. *AC* and *MC* depend on level of output and capital intensity, with no regular pattern.

2. A firm's production function is $Q = KL$

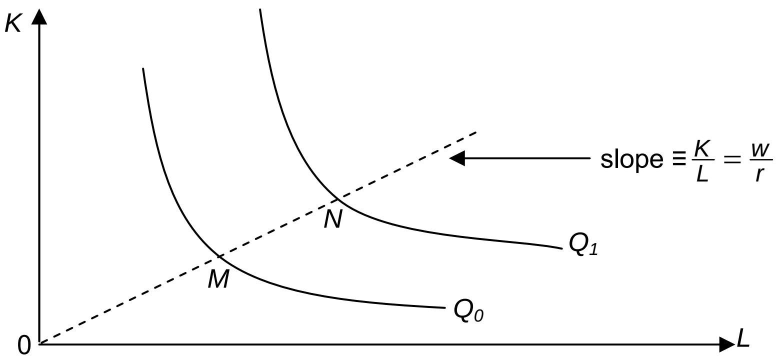
(a) Assess whether this function is homogeneous; and, if so, of what degree. Are returns to scale constant, increasing, or decreasing? What happens to output if both inputs are, say, doubled?

Answer: using the method of Ex.WS17.1 question 2, we have $Q_1 = (\lambda K)(\lambda L) = \lambda^2 KL = \lambda^2 Q_0$, so the production function is homogeneous of degree 2. Because the degree of homogeneity is greater than 1, there are increasing returns to scale. If both inputs are doubled, we have $\lambda = 2$, hence $Q_1 = \lambda^2 Q_0 = 4Q_0$; that is, output quadruples.

(b) If the prices of capital and labour are *r* and *w* respectively, find the cost-minimising capital/labour ratio. Show that, for this particular production function, this ratio depends only on $\frac{w}{r}$, the ratio of input prices, and is independent of the level of output. (Assume the firm is perfectly competitive in the markets for capital and labour, so that w and r are exogenous; that is, for the firm they are given constants.)

Answer: we know that cost minimisation requires the firm to adjust its use of K and L so as to satisfy the equation $\frac{MPL}{MPK} = \frac{w}{r}$. In this example we have $MPL \equiv \frac{\partial Q}{\partial L} = K$ and $MPK \equiv \frac{\partial Q}{\partial K} = L$, so cost minimisation requires $\frac{K}{L} = \frac{w}{r}$

(c) From (b), show diagrammatically how K and L vary as output varies, assuming $\frac{w}{r}$ constant.



The firm will always produce at some point on the dotted line, because at every point on that line we have $\frac{K}{L} = \frac{w}{r}$ and costs are therefore minimised. For example, if the firm wishes to produce Q_0 units of output, it will produce at M; if it wishes to produce Q_1 units of output, it will produce at N. Thus as output varies, K and L vary in the same proprotion, so as to maintain $\frac{K}{L}$ constant.

(d) Using your answer to (b), derive an expression for total cost, TC, as a function of Q alone, with w and r as parameters.

Answer: from (b), assuming cost minimisation by the firm, we have $\frac{K}{L} = \frac{w}{r}$, hence $K = \frac{w}{r}L$. Substituting this into the total cost function $TC = wL + rK$ gives $TC = wL + rK = wL + r\frac{w}{r}L = 2wL$.

Next, we substitute $K = \frac{w}{r}L$ into the production function and get $Q = KL = \frac{w}{r}L^2$. This rearranges as $L = \left(\frac{r}{w}\right)^{0.5} Q^{0.5}$. Substituting this into our TC function gives $TC = 2wL = 2w\left(\frac{r}{w}\right)^{0.5} Q^{0.5} = 2w^{0.5}r^{0.5}Q^{0.5}$. This gives TC as a function of Q, with w and r as parameters, as required.

(e) By differentiating your answer to (d), examine how *TC* varies with output. What does this tell us about the shape of the average (AC) and marginal cost (MC) curves associated with this production function? How is your answer related to your answer to (a)? (Hint: You will find it interesting and instructive to find the elasticity of the *TC* function, though this is not essential.)

Answer: from (d) we have $TC = 2w^{0.5}r^{0.5}Q^{0.5}$, so *MC* is the derivative of this; that is $MC \equiv \frac{dTC}{dQ} = w^{0.5}r^{0.5}Q^{-0.5}$. (Recall that *w* and *r* are constants). This may be written as $MC = w^{0.5}r^{0.5}\frac{1}{Q^{0.5}}$ and thus we see that *MC* decreases as *Q* increases. (More rigorously, we can take the derivative $\frac{dMC}{dQ}$ and show that this is negative; this is left to you.)

<u>The elasticity of total cost with respect to output</u>: Applying the standard definition of an elasticity, we define this as

$$E^{TC} \equiv \frac{Q}{TC}\frac{dTC}{dQ} = \frac{Q}{TC}w^{0.5}r^{0.5}Q^{-0.5} = \frac{1}{TC}w^{0.5}r^{0.5}Q^{0.5} = \frac{w^{0.5}r^{0.5}Q^{0.5}}{2w^{0.5}r^{0.5}Q^{0.5}} = 0.5$$

(because $TC = 2w^{0.5}r^{0.5}Q^{0.5}$). Thus the elasticity of total cost with respect to output is less than 1, meaning that a 10% increase in output increases total cost by less than10%. (This last remark is not rigorous, as E^{TC} as defined here is a point elasticity, whereas an arc elasticity is required to consider a change as large as 10%. However in this case this last remark is accurate, because the elasticity is constant and therefore the point elasticity and the arc elasticity are equal.)

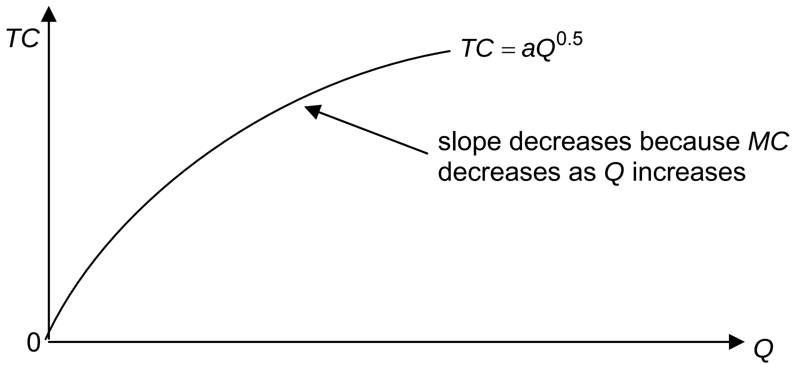
The key point is that the elasticity of total cost with respect to output is less than 1 because the production function exhibits increasing returns to scale. In other words, doubling inputs doubles total cost but more than doubles output due to increasing returns to scale. So marginal cost is below average cost and average cost is falling as output increases. Decreasing *MC* as output increases means that the *TC* curve is concave from below, reflecting increasing returns to scale.

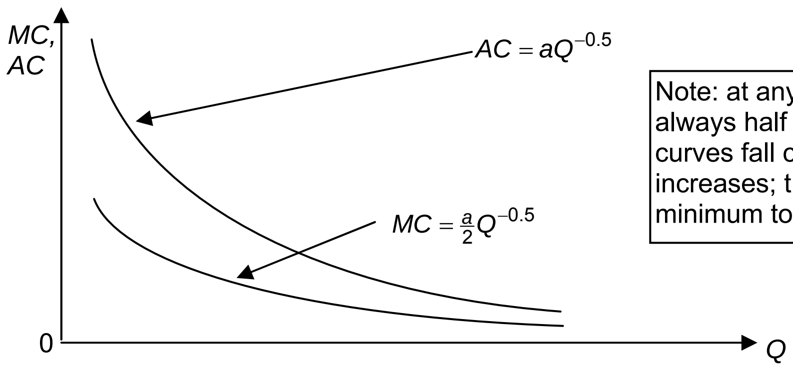
(f) Sketch the graph of the *TC*, *AC* and *MC* functions derived in (d) and (e). (As usual, don't aim at a high degree of accuracy; just try to get the general shape right.)

Answer: we have, from above, $TC = 2w^{0.5}r^{0.5}Q^{0.5}$ and $MC = w^{0.5}r^{0.5}Q^{-0.5}$. We can quickly derive *AC* as $AC \equiv \frac{TC}{Q} = 2w^{0.5}r^{0.5}Q^{-0.5}$. We can simplify the notation by defining $a \equiv 2w^{0.5}r^{0.5}$, where *a* is, of course, a positive constant. Then we have:

$TC = aQ^{0.5}$; $MC = \frac{a}{2}Q^{-0.5}$; and $AC = aQ^{-0.5}$ (Note that *MC* = 0.5*AC*)

Sketch graphs are shown below.





Note: at any output *MC* is always half of *AC*. Both curves fall continuously as *Q* increases; that is, there is no minimum to either.

3. Repeat the previous question for the production function $Q = 16KL - 4K^2 - 5L^2$, with the following minor modifications:

In (b), show that the cost-minimising *K* to *L* ratio now depends on the parameters of the production function as well as on $\frac{w}{r}$, but remains independent of the level of output. Do you think it likely that this feature (that is, the cost-minimising *K* to *L* ratio being independent of the level of output) is likely to be found in the real world?

Also in (b), find the cost-minimising *K* to *L* ratio if *w* = 3 and *r* = 4.

If you find (d) difficult, simplify by continuing to assume that *w* = 3 and *r* = 4. You should then find that *TC* as a function of *Q* is given by

$$TC = (w + r)\frac{Q^{0.5}}{7^{0.5}}$$

and therefore, for example, that when *K* = *L* = 10, *Q* = 700 and *TC* = 70. What are the values of *Q* and *TC* when *K* = *L* = 20?

Answers:

(a) using the method of Ex.WS17.1, the production function is homogeneous of degree 2. Because the degree of homogeneity is greater than 1, there are increasing returns to scale. If both inputs are doubled, we have λ = 2, hence $Q_1 = \lambda^2 Q_0 = 4Q_0$; that is, output quadruples.

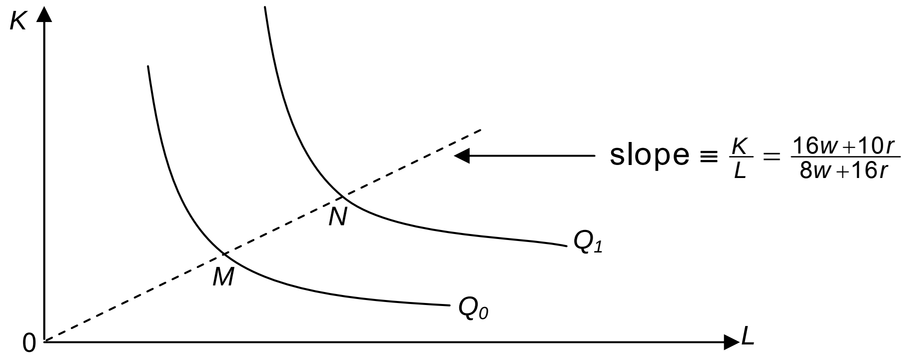
(b) we know that cost minimisation requires the firm to adjust its use of *K* and *L* so as to satisfy the equation $\frac{MPL}{MPK} = \frac{w}{r}$. In this example we have $MPL \equiv \frac{\partial Q}{\partial L} = 16K - 10L$ and $MPK \equiv \frac{\partial Q}{\partial K} = 16L - 8K$, so cost minimisation requires $\frac{16K - 10L}{16L - 8K} = \frac{w}{r}$. If we multiply both sides by *r* and by 16*L* – 8*K*, and rearrange, we get $\frac{K}{L} = \frac{16w + 10r}{8w + 16r}$, the cost-minimising capital to labour ratio.

For convenience we will define a new constant, *b*, as $b = \frac{16w + 10r}{8w + 16r}$. Then we can write the cost-minimising condition as $\frac{K}{L} = b$, or $K = bL$.

As well as *w* and *r*, *b* contains the constants 16, 10 and 8, which are derived from the parameters of the given production function. But *b* does not contain *Q*, and therefore the cost-minimising capital to labour ratio is independent of the level of output. This independence is not very likely in the real world, as it is common for labour-intensive techniques of production to be used when the scale of production is small, and capital-intensive techniques when the scale is large.

If $w = 3$ and $r = 4$, then $\frac{K}{L} = \frac{16w + 10r}{8w + 16r} = b = 1$

(c)



The firm will always produce at some point on the dotted line, because at every point on that line we have $\frac{K}{L} = \frac{16w + 10r}{8w + 16r}$ and costs are therefore minimised. For example, if the firm wishes to produce Q_0 units of output, it will produce at *M*; if it wishes to produce Q_1 units of output, it will produce at *N.*

(d) From (b), assuming cost minimisation by the firm, we have $K = bL,$ where $b = \frac{16w + 10r}{8w + 16r}$. Substituting this into the total cost function $TC = wL + rK$ gives $TC = wL + rK = wL + rbL = (w + rb)L.$ Next, we substitute $K = bL$ into the production function and get $Q = 16KL - 4K^2 - 5L^2 = 16bL^2 - 4b^2L^2 - 5L^2 = (16b - 4b^2 - 5)L^2$. This rearranges as $L = (16b - 4b^2 - 5)^{-0.5}Q^{0.5}$. Substituting this into our *TC* function gives $TC = \frac{(w + rb)}{(16b - 4b^2 - 5)^{0.5}}Q^{0.5}$. This gives *TC* as a function of *Q*, with *w*, *r* and the production function parameters (in *b*) as parameters (as discussed in (b) above).

From (b), if $w = 3$ and $r = 4$, then $\frac{K}{L} = \frac{16w + 10r}{8w + 16r} = b = 1$. Therefore $TC = \frac{(w + r)}{(7)^{0.5}}Q^{0.5}$.

When K = L =10, from the production function we have $Q = 16L^2 - 4L^2 - 5L^2 = 7L^2 = 700$. Therefore in the TC function, with b = 1, w = 3 and r = 4, we have $TC = \frac{(w+r)}{(7)^{0.5}} Q^{0.5} = \frac{7}{(7)^{0.5}} 700^{0.5} = 70$.

Similarly, when K = L = 20, Q = 2800 and TC = 140. Note the strength of the increasing returns to scale in this production function: doubling both inputs from 10 to 20 has doubled TC from 70 to 140 (obviously, as input prices have not changed); but has quadrupled output, from 700 to 2800.

(e) By differentiating your answer to (d), examine how TC varies with output. What does this tell us about the shape of the average (AC) and marginal cost (MC) curves associated with this production function? How is your answer related to your answer to (a)? (Hint: You will find it interesting and instructive to find the elasticity of the TC function, though this is not essential.)

Answer: from (d) we have $TC = \frac{(w+rb)}{(16b - 4b^2 - 5)^{0.5}} Q^{0.5}$, so MC is the derivative of this; that is $MC \equiv \frac{dTC}{dQ} = 0.5 \frac{(w+rb)}{(16b - 4b^2 - 5)^{0.5}} Q^{-0.5}$. This may be written as $MC = 0.5 \frac{(w+rb)}{(16b - 4b^2 - 5)^{0.5}} \frac{1}{Q^{0.5}}$ and thus we see that MC decreases as Q increases. (More rigorously, we can take the derivative $\frac{dMC}{dQ}$ and show that this is negative; this is left to you.) Decreasing MC as output increases means that the TC curve is concave from below, reflecting increasing returns to scale.

<u>The elasticity of total cost with respect to output</u>: Applying the standard definition of an elasticity, we define this as

$E^{TC} \equiv \frac{Q}{TC} \frac{dTC}{dQ} =$

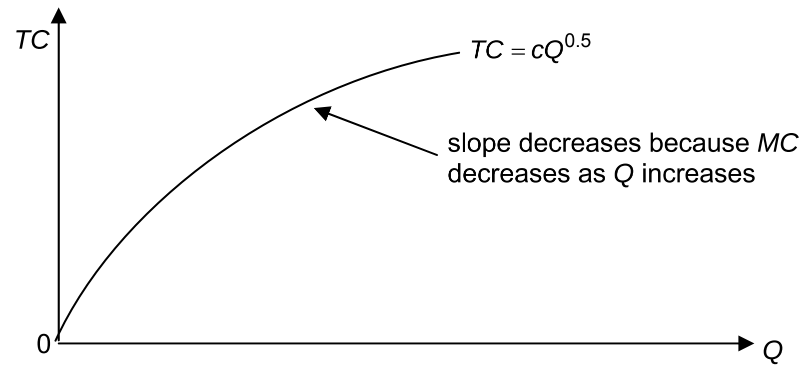
$\frac{Q}{TC} 0.5 \frac{(w+rb)}{(16b - 4b^2 - 5)^{0.5}} Q^{-0.5} = 0.5 \frac{1}{TC} \frac{(w+rb)}{(16b - 4b^2 - 5)^{0.5}} Q^{0.5} = 0.5$ (because $TC = \frac{(w+rb)}{(16b - 4b^2 - 5)^{0.5}} Q^{0.5}$). Thus the elasticity of total cost with respect to output is less than 1, meaning that a 10% increase in output increases total cost by less than10%. (This last remark is not rigorous, as E^{TC} as defined here is a point elasticity, whereas an arc elasticity is required to consider a change as large as 10%. However in this case this last remark is accurate, because the elasticity is constant and therefore the point elasticity and the arc elasticity are equal.)

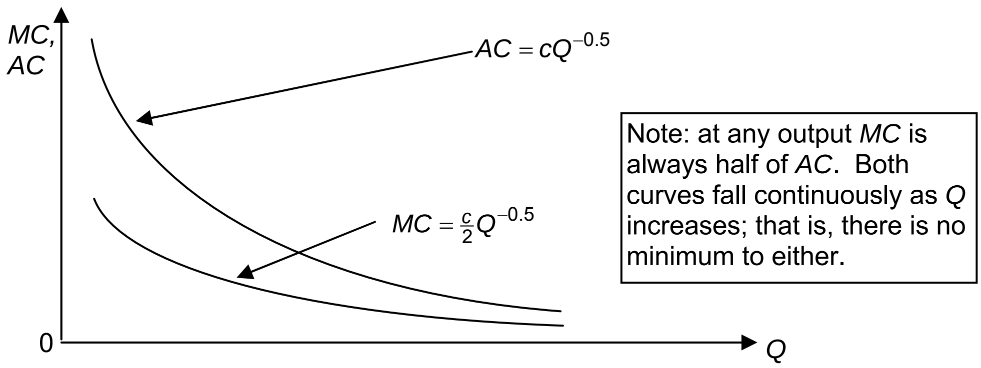
The key point is that the elasticity of total cost with respect to output is less than 1 because the production function exhibits increasing returns to scale. In other words, doubling inputs doubles total cost but more than doubles output due to increasing returns to scale. So marginal cost is below average cost and average cost is falling as output increases.

(f) We have, from above, $TC = \frac{(w+rb)}{(16b-4b^2-5)^{0.5}}Q^{0.5}$ and $MC \equiv \frac{dTC}{dQ} = 0.5\frac{(w+rb)}{(16b-4b^2-5)^{0.5}}Q^{-0.5}$. We can quickly derive AC as $AC \equiv \frac{TC}{Q} = \frac{(w+rb)}{(16b-4b^2-5)^{0.5}}Q^{-0.5}$. We can simplify the notation by defining a new constant, $c \equiv \frac{(w+rb)}{(16b-4b^2-5)^{0.5}}$. Then we have:

$TC = cQ^{0.5}$; $MC = \frac{c}{2}Q^{-0.5}$; and $AC = cQ^{-0.5}$ (Note that $MC = 0.5AC$)

Sketch graphs are shown below.





4. What is meant by the adding-up problem? Illustrate it for a firm operating under conditions of perfect competition and with the production function of question (2) above; that is, $Q = KL$.

Euler's theorem says that if a production function (or any other function) $Q = f(K, L)$ is homogeneous of degree *v*, then $K\frac{\partial Q}{\partial K} + L\frac{\partial Q}{\partial L} = vQ$. Under perfect competition, cost minimisation requires $\frac{\partial Q}{\partial K} = \frac{r}{P}$ and $\frac{\partial Q}{\partial L} = \frac{w}{P}$, with $\frac{r}{P}$ and $\frac{w}{P}$ assumed exogenous to the firm. Substituting these into the equation above gives $K\frac{r}{P} + L\frac{w}{P} = vQ$, which rearranges as $rK + wL = vPQ$. If $v > 1$ (increasing returns to scale) we have $rK + wL = vPQ > PQ$; that is, total payments to factors of production exceed total revenue, implying on-going losses for firms. This is the case In this example, as the production function $Q = KL$ is homogeneous of degree *v*.

In chapter 17 we saw that a firm with increasing returns to scale and operating under perfect competition has negatively sloped *AC* and *MC* curves. Therefore the firm in this example can increase its profits (reduce its losses) by increasing output without limit. There is no finite equilibrium output.

To put the same conclusion in a slightly different way, the assumption of (i) perfect competition (which requires $\frac{\partial Q}{\partial K} = \frac{r}{P}$ and $\frac{\partial Q}{\partial L} = \frac{w}{P}$, with $\frac{r}{P}$ and $\frac{w}{P}$ assumed exogenous to the firm) and (ii) increasing returns to scale are incompatible with a finite equilibrium output.

Exercise WS17.3

1. The demand functions for two goods, X and Y, are

$$X = 100P_X^{-1}P_Y^2 \qquad \text{and} \qquad Y = 500P_X^{0.1}P_Y^{-2}$$

(a) Find the own-price and cross-price elasticities.

Answer: Own-price elasticity for good X, $E_{P_X}^X$: we have $\frac{\partial X}{\partial P_X} = (-1)100P_X^{-2}P_Y^2$,

$$\text{so } E_{P_X}^X \equiv \frac{P_X}{X}\frac{\partial X}{\partial P_X} = \frac{P_X}{X}(-1)100P_X^{-2}P_Y^2 = (-1)\frac{100P_X^{-1}P_Y^2}{X} = -1$$

Similarly the cross-price elasticity for good X is

$$E_{P_Y}^X \equiv \frac{P_Y}{X}\frac{\partial X}{\partial P_Y} = \frac{P_Y}{X}(2)100P_X^{-1}P_Y = (2)\frac{100P_X^{-1}P_Y^2}{X} = 2$$

Own-price elasticity for good Y, $E_{P_Y}^Y$:

$$E_{P_Y}^Y \equiv \frac{P_Y}{Y}\frac{\partial Y}{\partial P_Y} = \frac{P_Y}{Y}(-2)500P_X^{0.1}P_Y^{-3} = (-2)\frac{500P_X^{0.1}P_Y^{-2}}{Y} = -2$$

Similarly the cross-price elasticity for good Y is

$$E_{P_X}^Y \equiv \frac{P_X}{Y}\frac{\partial Y}{\partial P_X} = \frac{P_X}{Y}(0.1)500P_X^{-0.9}P_Y^{-2} = (0.1)\frac{500P_X^{0.1}P_Y^{-2}}{Y} = 0.1$$

(b) Comment on their numerical values.

Both own-price elasticities are negative, which is normally the case. Both cross-price elasticities are positive, indicating that the goods are substitutes rather than complements. However it is odd that the cross-price elasticities are very different in numerical value (0.1 and 2), as we would expect a small percentage increase in the price of X to have the same effect on demand for both goods as a small percentage decrease in the price of Y.

(c) Sketch the graphs of these demand functions when log scales are taken on both axes. Show that, with log scales on the axes, the slopes measure the elasticities.

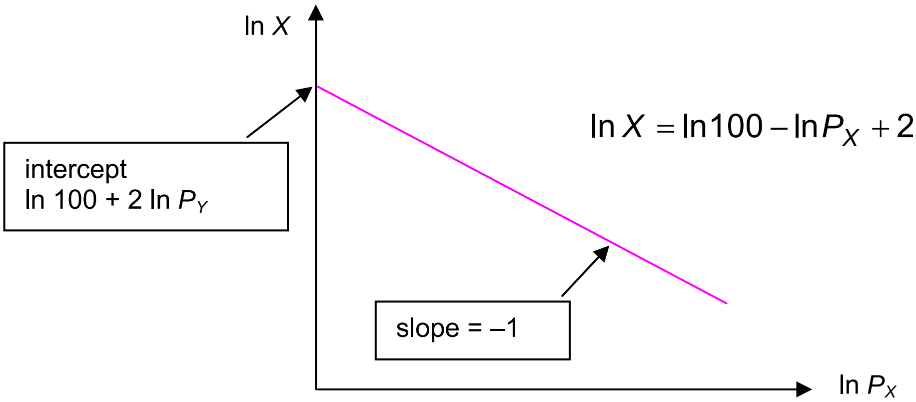
(i) Given $X = 100P_X^{-1}P_Y^2$, taking natural logs on both sides gives

$\ln X = \ln 100 - \ln P_X + 2\ln P_Y$ (1)

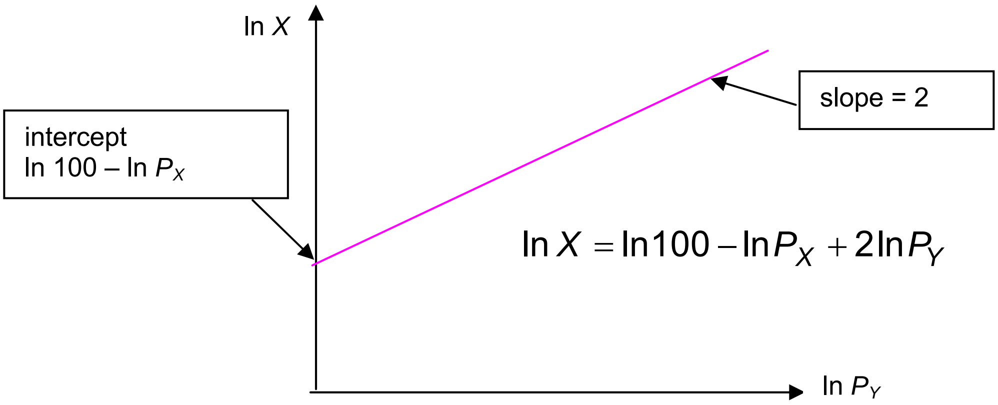
The partial derivatives are $\dfrac{\partial \ln X}{\partial \ln P_X} = -1$ (= the own-price elasticity, from above);

and $\dfrac{\partial \ln X}{\partial \ln P_Y} = 2$ (= the cross-price elasticity, from above)

Holding P_Y constant, equation (1) is a linear relationship between ln X and ln P_X with a slope of –1 and an intercept of $\ln 100 + 2\ln P_Y$. (For example, if P_Y = 55, ln P_Y = 4, and ln 100 = 4.6, so the intercept is 4.6 + (2)4 = 12.6). This means that a rise in P_Y causes a shift up of the demand function shown in the graph, to a higher section through the demand surface. See sketch graph below.



Similarly, holding P_X constant, equation (1) is a linear relationship between ln X and ln P_Y with a slope of 2 and an intercept of $\ln 100 - \ln P_X$. This means that a rise in P_X causes a shift down of the demand function shown in the graph, to a lower section through the demand surface. See sketch graph below.



(ii) Given $Y = 500 P_X^{0.1} P_Y^{-2}$, then exactly as in (i) we have

$\ln Y = \ln 500 + 0.1 \ln P_X - 2\ln P_Y$ (2)

The partial derivatives are $\frac{\partial \ln Y}{\partial \ln P_Y} = -2$ (= the own-price elasticity, from above);

and $\frac{\partial \ln Y}{\partial \ln P_X} = 0.1$ (= the cross-price elasticity, from above)

With P_X held constant, equation (2) is a linear relationship between ln Y and ln P_Y, with a slope of –2 and an intercept of $\ln 500 + 0.1 \ln P_X$. With P_Y held constant, equation (2) is a linear relationship between ln Y and ln P_X, with a slope of 0.1 and an intercept of $\ln 50 - 2\ln P_Y$. The graphs of these relationships are similar to the two graphs of equation (1),

$\ln X = \ln 100 - \ln P_X + 2\ln P_Y$ immediately above.

2. In year 0, government debt was D_0 and GDP was Y_0. In the same year the debt ratio, R, (the ratio of government debt to GDP) was $R_0 \equiv \frac{D_0}{Y_0} = 0.35$. Each year, government debt grows due to new borrowing, which equals 2% of the previous year's GDP. GDP itself is also growing at 3.5% per year.

(a) Show that, in general, the debt ratio will increase through time if the growth rate of debt exceeds the growth rate of GDP. (Hint: Example 17.12 in the book is relevant.)

Answer: $R = \dfrac{D}{Y}$ gives us the debt ratio in any (unspecified) year. The proportionate differential is $\dfrac{dR}{R} = \dfrac{dD}{D} - \dfrac{dY}{Y}$. (See rule 17.4 and example 17.10 in the book.) From this we see that the proportionate change, or proportionate growth, in the debt ratio is given by the proportionate growth of debt, *minus* the proportionate growth of income (GDP). So the growth of the debt ratio is positive (and therefore the debt ratio will increase through time) if the growth rate of debt exceeds the growth rate of GDP.

(b) Treating growth of all variables as occurring discretely in annual jumps, find an expression for the growth rate of the debt ratio and calculate after how many years the debt ratio will reach 40%.

Answer: Let us write ΔR, ΔD, ΔY denote annual changes in R, D and Y. Then $\dfrac{\Delta R}{R}$, $\dfrac{\Delta D}{D}$ and $\dfrac{\Delta Y}{Y}$ correspondingly denote annual growth rates. We will assume that rule 17.4 is valid for these annual changes (though this is not valid, as considered below), so we have

$\dfrac{\Delta R}{R} = \dfrac{\Delta D}{D} - \dfrac{\Delta Y}{Y}$, from which $\Delta R = R\dfrac{\Delta D}{D} - R\dfrac{\Delta Y}{Y} = \dfrac{D}{Y}\dfrac{\Delta D}{D} - R\dfrac{\Delta Y}{Y} = \dfrac{\Delta D}{Y} - R\dfrac{\Delta Y}{Y}$.

In the question we are given $\Delta D = 0.02Y_{-1}$ (where Y_{-1} denotes the previous year's income) and $\dfrac{\Delta Y}{Y} = 0.035$. Substituting these values into the expression above, we get $\Delta R = \dfrac{0.02Y_{-1}}{Y} - R(0.035) = \dfrac{0.02}{1.035} - R(0.035)$ equation (1)

(since $\dfrac{Y}{Y_{-1}} = 1.035$)

Unfortunately we cannot use equation (1) to calculate ΔR, as it contains the unknown R on the right hand side. However, if we replace R with R_{-1} (which does not introduce a huge error. because this year's debt ratio is not likely to differ greatly from last year's), we then have

$$\Delta R = \frac{0.02}{1.035} - R_{-1}(0.035) \qquad \text{equation (1a)}$$

Because we are given $R_0 = 0.35$, we can calculate ΔR_1 (the change in R in year 1) from equation (1a) as

$$\Delta R_1 = \frac{0.02}{1.035} - R_0(0.035) = \frac{0.02}{1.035} - (0.35)(0.035) = 0.007074$$

From this we can get R_1 as $R_0 + \Delta R_1 = 0.35 + 0.007074 = 0.357074$. Then in the same way we can calculate ΔR_2 from equation (1a) as

$$\Delta R_2 = \frac{0.02}{1.035} - R_1(0.035) = \frac{0.02}{1.035} - (0.357074)(0.035) = 0.006826$$

From this we can get R_2 as $R_1 + \Delta R_2 = 0.357074 + 0.006826 = 0.3639$.

We can continue in this way to calculate ΔR_3, ΔR_4, and so on. (This is known as an iterative process, or simulation – where we generate a series of values by repeatedly applying the same formula.) The result of this simulation for ten years are shown in the table below. (The calculations were done using Excel – see the book's website for how to do this.) We see that the budget deficit as a proportion of GDP reaches 0.4 (40%) after 8 years.

	Using equation (1a)	
Year	ΔR	R
0		0.35
1	0.007074	0.357074
2	0.006826	0.3639
3	0.006587	0.370487
4	0.006357	0.376844
5	0.006134	0.382978
6	0.005919	0.388897
7	0.005712	0.394609
8	0.005512	**0.400122**
9	0.005319	0.405441
10	0.005133	0.410574

However the calculations above are rather pointless except as mental exercise. For we could just as easily have calculated the actual values of all of the variables, starting from an arbitrary initial level of GDP, say, $Y = 100$. These calculations are shown in the table below. Values for Y are calculated simply by applying a growth factor of 3.5% to the previous year's Y. The budget deficit D in each year is simply the previous year's D plus 2% of the previous year's Y.

	Directly calculated values		
Year	*Y (Euros)*	*D (Euros)*	*R*
0	100	35	0.35
1	103.5	37	0.357488
2	107.1225	39.07	0.364723
3	110.8718	41.21245	0.371713
4	114.7523	43.42989	0.378466
5	118.7686	45.72493	0.384992
6	122.9255	48.1003	0.391296
7	127.2279	50.55882	0.397388
8	131.6809	53.10337	**0.403273**
9	136.2897	55.73699	0.40896
10	141.0599	58.46279	0.414454

Note that the values for *R* are almost identical in the two tables above, showing that the error in equation (1a) used in the first table is quite small (see (c) below).

(c) What is the nature of the approximation (if any) in your answers to (a) and (b)?

As is explained in the book, any proportionate differential such as $\frac{dR}{R} = \frac{dD}{D} - \frac{dY}{Y}$ in (a) above is necessarily an approximation, and the inaccuracy or error in the formula becomes larger as *dR, dD* and *dY* become larger. In (b) we take these to be annual changes, therefore the errors are potentially quite large. In (b) we have also introduced a new error by replacing *R* with R_{-1}. However, as the year-to-year changes in all the variables are quite small (a few percentage points), the errors are probably acceptable in size. This was confirmed in (b) when we calculated the *actual* growth of the debt ratio (second table) and found that this was quite close to our calculated values (first table). For example our calculated value for *R* after 8 years was 40.0122%, compared with an actual value of 40.3273%.

(d) Repeat (b) treating growth of all variables as continuous.

With time varying continuously, we can use the proportionate differential from part (a): $\frac{dR}{R} = \frac{dD}{D} - \frac{dY}{Y}$, while remembering that this is an approximation and the error becomes larger as the changes in the variables become larger. From this we can get the continuous-time equivalent of equation (1), as

$$dR = R\frac{dD}{D} - R\frac{dY}{Y} = \frac{D}{Y}\frac{dD}{D} - R\frac{dY}{Y} = \frac{dD}{Y} - R\frac{dY}{Y} \qquad \text{equation (2)}$$

In the question we are given $\Delta D = 0.02Y_{-1}$ (where Y_{-1} denotes the previous year's income). However in continuous time Y_{-1} denotes the value of Y one minute ago, or possibly one second ago, and is therefore virtually identical to current income, Y. Therefore we can assume $dD = 0.02Y$. We also have $\frac{dY}{Y} = 0.035$. Substituting these into equation (2) we get

$$dR = \frac{dD}{Y} - R\frac{dY}{Y} = \frac{0.02Y}{Y} - R(0.035) = 0.02 - R(0.035) \qquad \text{equation (2a)}$$

As in part (b), we have the problem of the unknown R on the right hand side. However in theoretical models working in continuous time, it is standard practice to use the initial value of R, R_0, as a proxy for the current value R. In this example we have $R_0 = 0.35$, so equation (2a) becomes

$$dR = 0.02 - (0.35)(0.035) = 0.00775 \qquad \text{equation (2b)}$$

From this equation we can quickly find any desired value of R. For example

$R_1 = R_0 + 0.00775 = 0.35 + 0.00775 = 0.35775$;

$R_2 = R_0 + 2 \times 0.00775 = 0.35 + 0.0155 = 0.3655$;

$R_7 = R_0 + 7 \times 0.00775 = 0.40425$; and so on.

We see that, using equation (2b), the budget deficit reaches 40% of GDP after only 7 years, rather than 8 years as in part (b) above. The reason for this is that in equation (2b) we have assumed that R is constant and equal to R_0; whereas in equation (1b) we allowed for R to vary from year to year. Because R is rising through time, and is preceded by a minus sign in equations (1a) and (2a), the use of R_0 instead of R in equation 2(b) causes the debt ratio to rise more rapidly, reaching 40% of GDP sooner.

(e) Does the debt ratio stabilise after a certain number of years?

Looking at our answer to (b), the first table shows the debt ratio R rising from year to year, so at first sight we might be tempted to say that R rises without limit as time passes. But when we look at the ΔR column, we see that ΔR declines as time passes; that is, the debt ratio, although rising, is rising at a diminishing rate, suggesting it will eventually stabilise (or at least converge asymptotically towards a constant value). This is also true in the second table, though we have to calculate ΔR using the column for R. (For example, ΔR in year 1 is 0.357488 – 0.35). If either the first or second table is extended to cover 100 years (quickly done in Excel) we find that R appears to be converging towards a constant value of about 55 – 57 percent.

This can be confirmed analytically using our basic formula $\frac{dR}{R} = \frac{dD}{D} - \frac{dY}{Y}$.
When the debt ratio R is constant, we will have $\frac{dR}{R} = 0$. Therefore $\frac{dD}{D} = \frac{dY}{Y}$.
In this example we have $dD = 0.02Y$ and $\frac{dY}{Y} = 0.035$. Substituting these into the condition $\frac{dD}{D} = \frac{dY}{Y}$, we get $\frac{0.02Y}{D} = 0.035$, which rearranges as $\frac{D}{Y} = \frac{0.02}{0.035} = 0.5714$. That is, the debt ratio converges asymptotically towards a long run equilibrium value of 57.14% of GDP.

(f) Does the debt ratio seem to you to be an important economic variable?

Answer: this question is one of economics (and politics) rather than maths for economics, and is therefore somewhat beyond the scope of the book; though hopefully not beyond your interest. The key point is that interest has to be paid on the government's debt. For example, if the interest rate is i, then interest on a debt of D is iD, and taxes of this amount to pay this interest must be levied. If the debt ratio $R = \frac{D}{Y}$ is rising through time, then interest payments as a proportion of GDP, $\frac{iD}{Y}$, must be rising too (unless I is falling, which seems unlikely). This is likely to have serious political and economic repercussions.

(Hints: You may find (c) easier than (b), in which case you should feel free to answer it first! If you get completely stuck or want to check your answer, you could simply construct a table of values of D and Y in successive years, starting with, say, $D = 35$ and $Y = 100$.)

3. Assume that the aggregate production function for the UK economy is $Q = AK^{0.4}L^{0.7}$ where A is a parameter.

(a) If capital and labour inputs both grow at the same rate (say, x%) per year, what growth rate of output should be expected? Explain the methodology underlying your method of calculation. How may we explain this ability of output to grow at a rate different from the growth rate of inputs?

The partial derivatives are: $\frac{\partial Q}{\partial K} = 0.4AK^{-0.6}L^{0.7} = 0.4AK^{-0.6}L^{0.7}\frac{K}{K} = 0.4\frac{Q}{K}$, and

by the same method, $\frac{\partial Q}{\partial L} = 0.7\frac{Q}{L}$. Therefore the differential is:

$$dQ = \frac{\partial Q}{\partial K}dK + \frac{\partial Q}{\partial L}dL = 0.4\frac{Q}{K}dK + 0.7\frac{Q}{L}dL$$

As *K* and *L* are both functions of time, *t*, we can divide through by *dt*. At the same time we also divide though by *Q*. The result is:

$$\frac{1}{Q}\frac{dQ}{dt} = 0.4\frac{1}{K}\frac{dK}{dt} + 0.7\frac{1}{L}\frac{dL}{dt} \qquad (1)$$

Here $\frac{1}{Q}\frac{dQ}{dt}$ is the (proportionate) rate of growth of *Q*, and similarly

$\frac{1}{K}\frac{dK}{dt}$ and $\frac{1}{L}\frac{dL}{dt}$ are the growth rates of *K* and *L*. We are told that these latter two growth rates are equal, so we can re-write (1) as

$$\frac{1}{Q}\frac{dQ}{dt} = 0.4\frac{1}{L}\frac{dL}{dt} + 0.7\frac{1}{L}\frac{dL}{dt} = 1.1\frac{1}{L}\frac{dL}{dt} \qquad (2)$$

Equation (2) says that the growth rate of output is 1.1 times the common growth rate of capital and labour inputs. For example, if *K* and *L* are both growing at 5% per year, *Q* will grow at 5.5% per year. (Note that by bringing in time as an explicit variable we have analysed this problem in a slightly different way from chapter 17.10 in the book, but the result is the same. See also chapter 13, especially rule 13.7). It is possible (indeed, inevitable) for output to faster than the common growth rate of capital and labour inputs because the production function $Q = AK^{0.4}L^{0.7}$ has increasing returns to scale.

The key methodological feature of this analysis is the assumption that there exists an aggregate production function for the whole economy, or at least that the behaviour of aggregate output can be analysed *as if* such a function exists.

(b) In recent years the labour input has been growing at about 0.5% per year (due mainly to immigration) and, due to net investment, the capital stock has been growing at a rate of about 1.5% per year. What consequent growth rate of aggregate output would you expect to observe?

Using equation (1) together with the information that

$\frac{1}{K}\frac{dK}{dt}$=0.015 and $\frac{1}{L}\frac{dL}{dt} = 0.005$ we get:

$$\frac{1}{Q}\frac{dQ}{dt} = 0.4(0.015) + 0.7(0.005) = 0.0095 \text{ (= 0.95\% per year)}$$

(c) Compare your answer to (b) with the actual growth rates of GDP in recent years. (Visit the Office of National Statistics, www.statistics.gov.uk if you don't have the data to hand.)

The following data, giving the annual growth (%) in real GDP (that is, after stripping out the effects of inflation) were downloaded from the ONS website. Clearly the actual growth since 1997 has been far greater than our calculation above.

1997	3.10
1998	3.35
1999	3.04
2000	3.80
2001	2.37
2002	2.05
2003	2.77
2004	3.26
2005	1.84
2006	2.84

(d) If the growth of the labour input ceased (perhaps due to restrictions on immigration) by how much would investment need to increase to maintain the growth rate of aggregate output?

From equation (1) above we have $\frac{1}{Q}\frac{dQ}{dt} = 0.4\frac{1}{K}\frac{dK}{dt} + 0.7\frac{1}{L}\frac{dL}{dt}$. We now have $\frac{1}{L}\frac{dL}{dt} = 0$ and need to find the value of $\frac{1}{K}\frac{dK}{dt}$ such that

$0.4\frac{1}{K}\frac{dK}{dt}+0.7(0)=0.0095$. Simple rearrangement gives

$\frac{1}{K}\frac{dK}{dt}=\frac{0.0095}{0.4}=0.02375=2.375\%$

(e) The parameter A may be regarded as a measure of overall productive efficiency, in the sense that any increase in A increases output with unchanged inputs of K and L. Suppose due to increased technological and managerial skills, A began to increase at y% per year. Show the effect of this on the growth rate of output.

In (a) above, in the production function $Q=AK^{0.4}L^{0.7}$ we now treat A as a variable instead of a parameter. Its partial derivative is

$\frac{\partial Q}{\partial A}=K^{0.4}L^{0.7}=K^{0.4}L^{0.7}\frac{A}{A}=\frac{Q}{A}$. With this additional variable, the differential is

$$dQ=\frac{\partial Q}{\partial A}dA+\frac{\partial Q}{\partial K}dK+\frac{\partial Q}{\partial L}dL=\frac{Q}{A}dA+0.4\frac{Q}{K}dK+0.7\frac{Q}{L}dL$$

Treating now A, as well as K and L, as functions of time, t, we divide through by dt. At the same time we also divide though by Q. The result is a modified version of equation (1) in (a) above:

$$\frac{1}{Q}\frac{dQ}{dt}=\frac{1}{A}\frac{dA}{dt}+0.4\frac{1}{K}\frac{dK}{dt}+0.7\frac{1}{L}\frac{dL}{dt} \qquad (5)$$

Comparing equations (5) and (1) we see that in (5) the growth rate of output is now increased by the full amount of the growth rate of A. So if A is growing at y% per year this will add y% per year to the growth rate of output. This is unlike growth of K and L where we see that the growth of output is only a fraction (0.4 and 0.7 respectively) of the growth of K and L.