Exercise WS16.1

1. (a) For each of the following functions, find the optimum value of z (that is, maximum or minimum) subject to the given constraint, first by the method of direct substitution, and then by setting the slope of the iso-z section equal to the slope of the constraint.

   (b) In each case, attempt to assess informally the shapes of the two surfaces and whether the optimum is a maximum or minimum of z.

(i) $z = xy$ subject to the constraint $x^2 + y^2 = 16$. (Note that there are four solutions.)

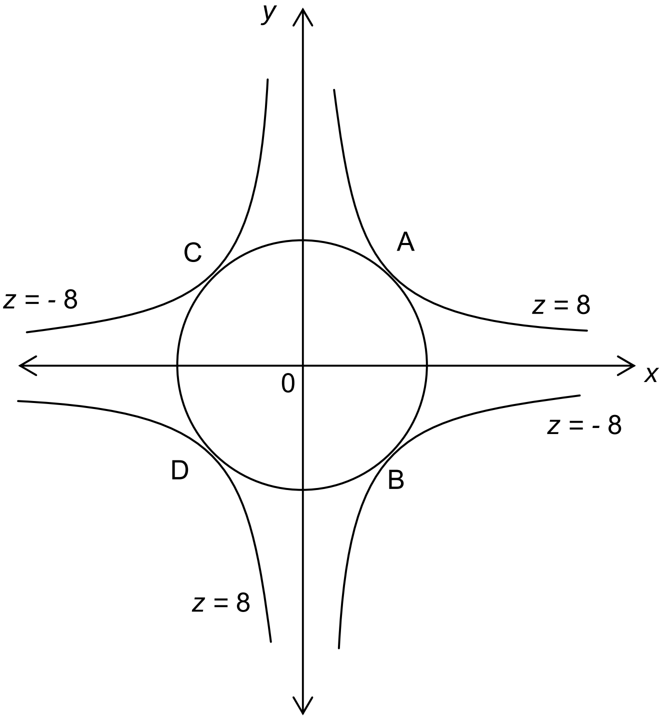
Answer to (a): by direct substitution. The constraint re-arranges as $y = \left(16 - x^2\right)^{0.5}$. Substituting this into the objective function gives $z = x(16 - x^2)^{0.5}$. As the constraint is now "built in" to the objective function, the constraint is automatically satisfied, and we can simply look for the unconstrained stationary value z by setting the first derivative equal to zero. This gives $\dfrac{dz}{dx} = x(0.5)(16 - x^2)^{-0.5}(-2x) + (16 - x^2)^{0.5} = 0$ (using product rule). This does not look very easy to solve, but if we multiply both sides by $(16 - x^2)^{0.5}$ it becomes $-x^2 + (16 - x^2) = 0$, from which $x^2 = 8$ and therefore $x = \pm\sqrt{8} = +2.828$ or -2.828 (to 3 dp). Substituting $x^2 = 8$ into the constraint gives $y = \left(16 - 8\right)^{0.5} = +2.828$ or -2.828. So the four solutions, with their values of z, are:

(1) $x = +2.828,\ y = +2.828,\ z = xy = +8$
(2) $x = +2.828,\ y = -2.828,\ z = xy = -8$
(3) $x = -2.828,\ y = +2.828,\ z = xy = -8$
(4) $x = -2.828,\ y = -2.828,\ z = xy = +8$

Clearly, solutions (1) and (4) are the joint maximum values, and (2) and (3) the joint minima.

Answer by setting slopes equal. Given any function $z = f(x, y)$, the slope of any iso-z contour is given by $-\dfrac{\partial z/\partial x}{\partial z/\partial y}$; in this case $-\dfrac{\partial z/\partial x}{\partial z/\partial y} = -\dfrac{y}{x}$. By implicit differentiation, the slope of any implicit function $g(x, y)$ = a constant is given by the ratio of partial derivatives, $-\dfrac{g_x}{g_y}$. In this case, $-\dfrac{g_x}{g_y} = -\dfrac{2x}{2y}$. Setting these two slopes equal to one another gives $-\frac{y}{x} = -\frac{x}{y}$, from which $x^2 = y^2$. Using this to eliminate y from the constraint gives $2x^2 = 16$, therefore $x = \pm\sqrt{8}$ as before.

Answer to (i)(b)



In the sketch, the circle represents the constraint, $x^2 + y^2 = 16$. (In three dimensions, this would be a cylinder parallel to the *z* axis.) The four curves are iso-*z* contours of $z = xy$ for $z = 8$ and $z = -8$. (Note that $z= xy$ is positive in the north-east and south-west quadrants, where *x* and *y* have the same sign; and negative in the north-west and south-east quadrants where *x* and *y* have opposite signs.) The tangencies at A, B, C and D correspond respectively to the solutions (1) – (4) above.

(ii) $z = 3x^2 - 10xy + 12y^2$ subject to the constraint $y = 20 - \frac{1}{2}x$

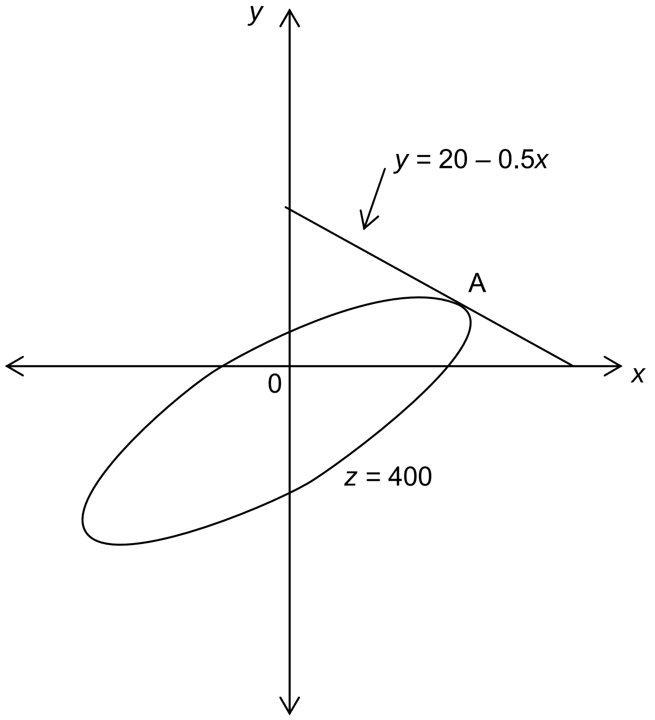
Answer to (a): by direct substitution. Substitution for *y* gives

$z = 3x^2 - 10x(20 - \frac{1}{2}x) + 12(20 - \frac{1}{2}x)^2 = 11x^2 - 440x + 4800$. So:

$\frac{dy}{dx} = 22x - 440 = 0$, with solution $x = 20$. Substituting this into the constraint gives $y = 10$.

Answer by setting slopes equal. In this case the slope of an iso-*z* contour of the objective function is: $-\frac{\partial z/\partial x}{\partial z/\partial y} = -\frac{6x - 10y}{24y - 10x}$, and the slope of the constraint is $\frac{dy}{dx} = -\frac{1}{2}$. Setting these equal gives $\frac{6x - 10y}{24y - 10x} = \frac{1}{2}$, which simplifies to *x* = 2*y*. Substituting this into the constraint gives *y* = 20 – *y*, from which *y* = 10 and therefore *x* = 2*y* = 20. The value of *z* is 400.

Answer to (b). The equation of the iso-*z* contour when $z = z_0$ is $3x^2 - 10xy + 12y^2 = z_0$. This is probably an ellipse (see section 5.8 of the book). The constraint is a downward sloping straight line. See sketch below. The coordinates of point A are *x* = 20, *y* = 10.



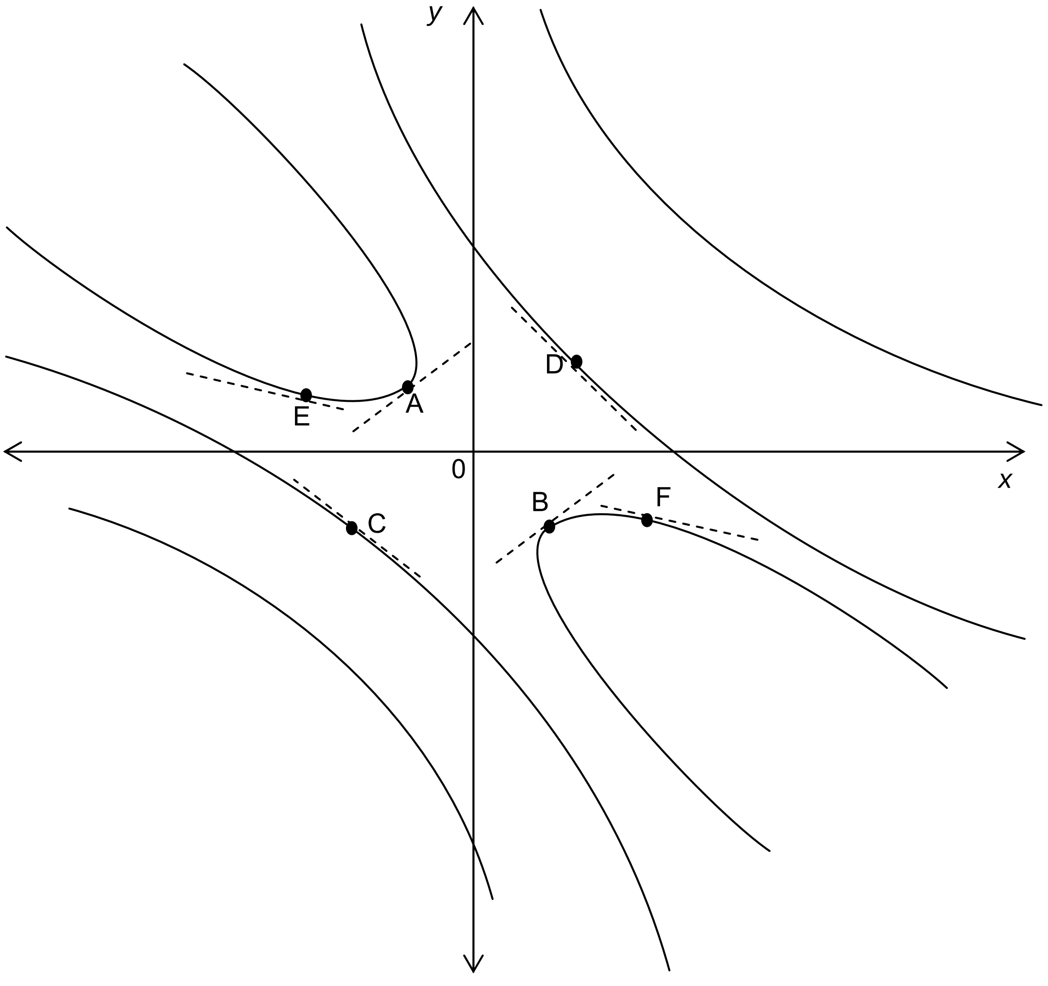
(iii) $z = x^2 + 3xy + y^2$ subject to the constraint $y = ax + b$, where a and b are parameters. How do the signs of a and b influence the solution(s)?

Answer to (a): by direct substitution. Substitution gives $z = x^2 + 3x(ax + b) + (ax + b)^2 = (1 + a^2 + 3a)x^2 + (2ab + 3b)x + b^2$ So: $\frac{dz}{dx} = 2(1 + a^2 + 3a)x + 2ab + 3b = 0$. Solving this gives $x = -\frac{2ab + 3b}{2(1 + a^2 + 3a)}$.

Substituting this into the constraint gives $y = -\frac{2a^2b + 3ab}{2(1 + a^2 + 3a)} + b = \frac{3ab + 2b}{2(1 + a^2 + 3a)}$

Answer by setting slopes equal: The slope of an iso-*z* contour of the objective function is: $-\frac{\partial z/\partial x}{\partial z/\partial y} = -\frac{2x+3y}{2y+3x}$, and the slope of the constraint is $\frac{dy}{dx} = a$. Setting these equal and re-arranging gives $x = \frac{-2a-3}{2+3a}y$. We then substitute for *y* using the constraint, and get $x = \frac{-2a-3}{2+3a}(ax+b)$. After some manipulation of this, we obtain $x = -\frac{2ab+3b}{2(1+3a+a^2)}$ and $y = \frac{3ab+2b}{2(1+a^2+3a)}$, the same solutions as above.

Answer to (b): it is hard to visualise the shape of the function $z = x^2 + 3xy + y^2$. As the functional form is similar to that of question (ii) above, it is tempting to imagine that the iso-*z* contours are ellipses, but in this case the parameter values produce iso-*z* contours like those sketched below. (See the last sentence of section 5.8 in the book).



The constraint is the linear function $y = ax + b$. Some possible slopes and positions for this function are shown as dotted lines in the sketch above. If *a* is positive, the constraint could be tangent to an iso-*z* contour at a point such as A or B. If *a* is negative, the tangency could be at a point such as C, D, E, or F.

Exercise WS16.2

Solve each of the problems in exercise WS16.1 using the Lagrange multiplier method. In question (i), find the value of λ and verify by recalculating the solution that the value of λ gives (with a small error) the change in z when the constraint is relaxed by 1 unit.

(i) Answer: the Lagrangean expression is:

$V = xy + \lambda(x^2 + y^2 - 16)$. (Note that the constraint is written as an implicit function).

The partial derivatives, set equal to zero, are

$V_x = y + \lambda 2x = 0$ (1)

$V_y = x + \lambda 2y = 0$ (2)

$V_\lambda = x^2 + y^2 - 16 = 0$ (3)

From (1), $\lambda = -\dfrac{y}{2x}$. From (2), $\lambda = -\dfrac{x}{2y}$. Combining these, $\dfrac{y}{2x} = \dfrac{x}{2y}$, which rearranges as $x^2 = y^2$. Using this in (3) we get $2x^2 = 16$, so $x = \pm\sqrt{8}$ as before. The rest of the answer is the same as in Ex WS16.1, with $y = \pm\sqrt{8}$ and $z = xy = (\pm\sqrt{8})(\pm\sqrt{8}) = \pm 8$.

Lambda and the effect of a relaxation of the constraint. From above, we have $\lambda = -\dfrac{y}{2x}$, and from our answer to Ex WS16.1 we have $x = y = \pm 2.828$ (four combinations of values). Therefore $\lambda = -\frac{1}{2}$ when x and y have the same sign, and $\lambda = \frac{1}{2}$ when x and y have opposite signs.

Now we suppose the constraint is relaxed by one unit, becoming $x^2 + y^2 - 17 = 0$. When we re-solve with this new constraint, the solution values are $x = y = \pm\sqrt{8.5}$. So the new value of z is $z = xy = (\pm\sqrt{8.5})(\pm\sqrt{8.5}) = \pm 8.5$. So the increase in z is 0.5, which equals the value of λ.

(ii) Answer: the Lagrangean expression is:

$V = 3x^2 + 12y^2 - 10xy + \lambda(20 - 0.5x - y)$. The partial derivatives, set equal to zero, are

$V_x = 6x - 10y - 0.5\lambda = 0$ (1)

$V_y = 24y - 10x - \lambda = 0$ (2)

$V_\lambda = 20 - 0.5x - y = 0$ (3)

From (1), λ = 12*x* – 20*y*. Using this in (2) gives $24y - 10x - (12x - 20y) = 0$ which simplifies to *y* = 0.5*x*. Using this in (3) gives $20 - 0.5x - 0.5x = 0$, from which *x* = 20. The rest of the answer is the same as in Ex WS16.1.

Lambda and the effect of a relaxation of the constraint. From our answer to Ex WS16.1 we have *x* = 20, *y* = 10, and therefore *z* = 440. From above, we have λ = 12*x* – 20*y*, and therefore $\lambda = 40$ at the solution values of *x* and *y*.

Now we suppose the constraint is relaxed by one unit, from $y = 20 - 0.5x$ to $y = 21 - 0.5x$. When we re-solve with this new constraint, the solution values are *x* =21, *y* =10.5, *z* =441. So the increase in *z* is 41, which is very close to the value of $\lambda = 40$.

(iii) Answer: the Lagrangean expression is:

$V = x^2 + y^2 + 3xy + \lambda(ax + b - y)$. The partial derivatives, set equal to zero, are

$V_x = 2x + 3y + \lambda a = 0$ (1)

$V_y = 2y + 3x - \lambda = 0$ (2)

$V_\lambda = ax + b - y = 0$ (3)

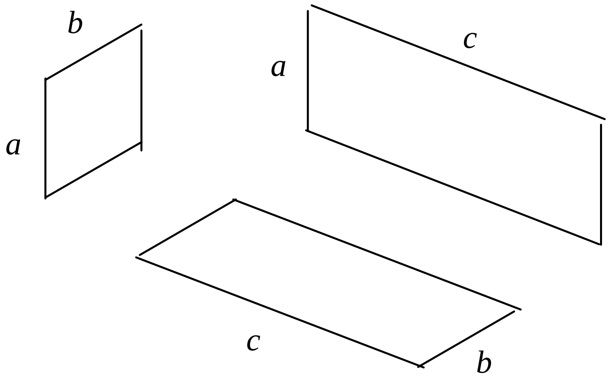
From (2), λ = 2*y* + 3*x*. Using this in (1) gives $2x + 3y + (2y + 3x)a = 0$. This rearranges as $y = -\frac{2+3a}{3+2a}x$. From (3), we have *y* = *ax* + *b*. Combining these two equations, we get $-\frac{2+3a}{3+2a}x = ax + b$. After some manipulation of this, we obtain $x = -\frac{2ab + 3b}{2(1 + 3a + a^2)}$, the same solution as in Ex WS16.1

Lambda and the effect of a relaxation of the constraint. In this example the (unspecified) values of both *a* and *b* affect the solution value of *z*, and therefore both *a* and *b* can be viewed as constraints. To keep the problem as simple as possible, we will consider only a simple numerical example. We assume initially that with *a* = 1 and *b* = 10. Substituting these values into our solution, we get *x* = –5, *y* = 5, *z* = –25 and λ = –5. If we now relax the constraint by increasing *b* from 10 to 11, our new solution values are *x* = –5.5, *y* = 5.5, *z* = –30.25. Thus the value of λ (–5) is very close to the change in the value of *z* (–5.25).

Exercise WS16.3

1. A carpenter plans to make a rectangular wooden box with a lid (that is, a closed box) and with a capacity of 27 cubic metres. The necessary wood costs 25 euros per square metre. Find the dimensions of the box that will minimise the cost of wood. Find also the minimised cost.

Answer:



The box will have 2 ends, each of area *ab*; 2 sides, each of area *ac*; and a top (the lid) and bottom each of area *bc* (see diagram above). So the total surface area of the box in square metres is $z = 2ab + 2ac + 2bc$, and the total cost of the wood is $C = 25z = 25(2ab + 2ac + bc)$ euros. The volume of the box, *x*, will be $x = abc = 27$

The task is to minimise *C* subject to the constraint of *x*. So the Lagrangean expression is

$V = 25(2ab + 2ac + 2bc) + \lambda(abc - 27)$.

(Note the constraint is written as an implicit function).

Here there are 4 variables: *a*, *b*, *c*, and λ. Setting the 4 partial derivatives equal to zero gives

$$\frac{\partial V}{\partial a} = 50b + 50c + \lambda bc = 0 \quad (1). \text{ Re-arranging, } -\lambda = \frac{50b + 50c}{bc} \quad (1a)$$

$$\frac{\partial V}{\partial b} = 50a + 50c + \lambda ac = 0 \quad (2). \text{ Re-arranging, } -\lambda = \frac{50a + 50c}{ac} \quad (2a)$$

$$\frac{\partial V}{\partial c} = 50a + 50b + \lambda ab = 0 \quad (3). \text{ Re-arranging, } -\lambda = \frac{50a + 50b}{ab} \quad (3a)$$

$$\frac{\partial V}{\partial \lambda} = abc - 27 = 0 \quad (4)$$

Combining (1a) and (2a), we get $\frac{50(b+c)}{bc} = \frac{50(a+c)}{ac}$. Then divide both sides by 50 and multiply both sides by *bc* and by *ac*, to get $abc + ac^2 = abc + bc^2$. After subtracting *abc* from both sides and dividing both sides by c^2 we are left with $a = b$.

Repeating the operations of the previous paragraph on equations (2b) and (2c) we get $b = c$. Thus $a = b = c$. Substituting this information into equation (4) we get $a^3 = 27$, from which $a = 3$. Thus $a = b = c = 3$. The minimised cost is $C = 25(2ab + 2ac + bc) = 25(2 \times 3 \times 3 + 2 \times 3 \times 3 + 2 \times 3 \times 3) = 1350$. As a rough check on whether this is a minimum, we can try varying the dimensions slightly; say, $a = 3$, $b = 3.0545$, $c = 2.946$. This results in a higher cost (1350.003) even though the volume is less than required (26.99567). This suggests we have found a minimum of cost.

2. A firm's production function is $Q = K^{0.6}L^{0.6}$ where K is capital input measured in machine-hours and L is labour input measured in worker-hours. The firm is perfectly competitive and hires its machines at a constant rental rate of $r = 5$ euros per hour and its workers at a constant wage rate of $w = 2$ euros per hour.

(a) Using the Lagrange multiplier method, find the maximum hourly output that the firm can produce, given a fixed budget of 1000 euros.

Answer: Let us look at this problem in general terms, then tackle the specific case. We have a production function $Q = f(K,L)$ and a budget $B = wL + rK$ that specifies that the firm's expenditure on labour, wL, plus its expenditure on hiring machines, rK, must equal the budget, B. (For simplicity we ignore the logical possibility that it might be optimal to spend less than the whole budget.)

Then we want to maximise output, $Q = f(K,L)$ subject to the constraint $B = wL + rK$. The Lagrangean expression is therefore:

$V = f(K,L) + \lambda(wL + rK - B)$ Setting the 3 partial derivatives equal to zero gives

$\frac{\partial V}{\partial L} = f_L + \lambda w = 0$ (1) (where $f_L \equiv \frac{\partial Q}{\partial L}$ is the marginal product of labour)

$\frac{\partial V}{\partial K} = f_K + \lambda r = 0$ (2) (where $f_K \equiv \frac{\partial Q}{\partial K}$ is the marginal product of capital)

$\frac{\partial V}{\partial \lambda} = wL + rK - B = 0$ (3)

From (1), $\frac{f_L}{w} = -\lambda$, and from (2), $\frac{f_K}{r} = -\lambda$. Combining these, we have

$\frac{r}{w} = \frac{f_K}{f_L}$ (4)

(As emphasised in the book, this is a key equation in economic analysis of the firm.)

Moving now to the specific example, we have $w = 2,\ r = 5,$ and $B = 1000$.

Given the production function $Q = K^{0.6}L^{0.6}$ we have $f_L = 0.6K^{0.6}L^{-0.4}$ and $f_K = 0.6K^{-0.4}L^{0.6}$. Substituting this information into equations (3) and (4) they become

$2L + 5K - 1000 = 0$ (3a) and

$\frac{5}{2} = \frac{0.6K^{-0.4}L^{0.6}}{0.6K^{0.6}L^{-0.4}}$ which simplifies to $\frac{5}{2} = \frac{L}{K}$ (4a)

From (4a) we get $K = \frac{2}{5}L$, and substituting this into (3a) gives

$2L + 5(\tfrac{2}{5}L) = 1000$ (5)

with solution L = 250. Therefore $K = \frac{2}{5}L = 100$ and $Q = K^{0.6}L^{0.6}$ = 435.2753.

(b) How is the solution changed if the budget is doubled? Is the maximum output also doubled?

Answer: from (a) above we see that the budget B appears only in equations (3) and (3a). So equation (3a) becomes

$2L + 5K - 2000 = 0$ (3a')

But as equations (3) and (3a) play no part in the derivation of equation (4a), this equation is therefore unaffected by the doubling of the budget.

So with the doubled budget, we now have equations (3a') and (4a). From (4a) we get, as before, $K = \frac{2}{5}L$, and substituting this into (3a') gives

$2L + 5(\tfrac{2}{5}L) = 2000$ (5')

with solution L = 500. Therefore $K = \frac{2}{5}L = 200$ and $Q = K^{0.6}L^{0.6}$ = 1000. Thus both inputs have doubled (labour from 250 to 500, and capital from 100 to 200), while output has more than doubled, from 435 to 1000.

(In chapter 17 of the book we explain why, in this case, a doubling of both inputs more than doubles output. This occurs because the production function $Q = K^{0.6}L^{0.6}$ is homogeneous of degree greater than 1. In economists' language, with this production function there are increasing returns to scale.)

(c) How is the solution changed if the market wage rate rises to 4 euros?

Answer: Equations (3a) and (4a) now become

$4L + 5K - 1000 = 0$ (new 3a) and

$\frac{5}{4} = \frac{L}{K}$ (new 4a)

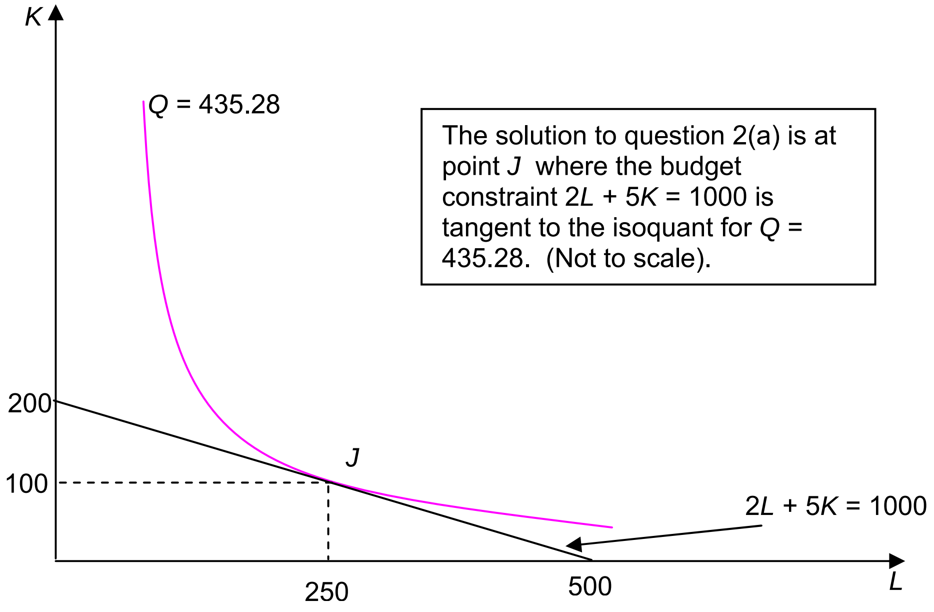
From (new 4a) we get $K = \frac{4}{5}L$, and substituting this into (new 3a) gives

$$4L + 5(\tfrac{4}{5}L) = 1000 \qquad \text{(new 5)}$$

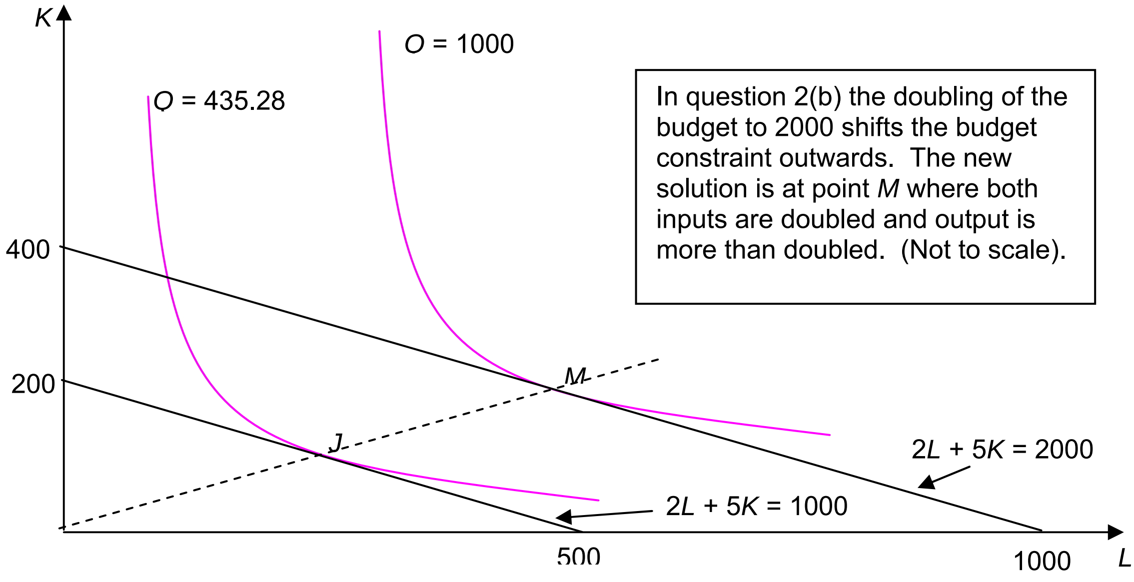
with solution L = 125. Therefore $K = \frac{4}{5}L = 100$ and $Q = K^{0.6}L^{0.6}$ = 287.1746.

(d) Illustrate your solutions graphically.

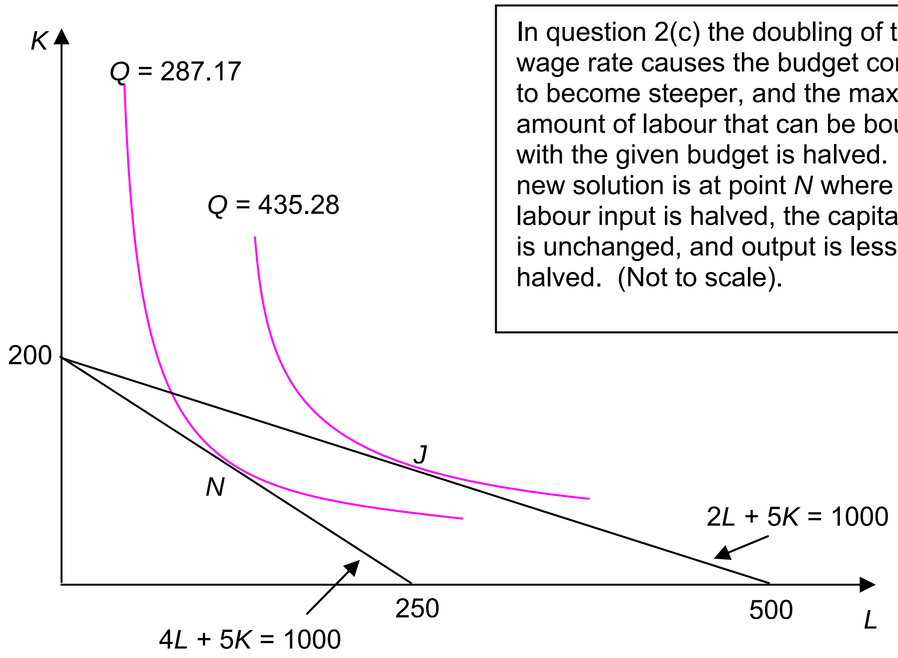
Answer:



The solution to question 2(a) is at point *J* where the budget constraint 2*L* + 5*K* = 1000 is tangent to the isoquant for *Q* = 435.28. (Not to scale).



In question 2(b) the doubling of the budget to 2000 shifts the budget constraint outwards. The new solution is at point *M* where both inputs are doubled and output is more than doubled. (Not to scale).



In question 2(c) the doubling of the wage rate causes the budget constraint to become steeper, and the maximum amount of labour that can be bought with the given budget is halved. The new solution is at point *N* where the labour input is halved, the capital input is unchanged, and output is less than halved. (Not to scale).

3. A firm's production function is $Q = K^{0.6}L^{0.5}$ where *K* is capital input measured in machine-hours and *L* is labour input measured in worker-hours. The firm is perfectly competitive and hires its machines at a constant rental rate of *r* = 2 euros per hour and its workers at a constant wage rate of *w* = 6 euros per hour.

(a) Using the Lagrange multiplier method, find the minimum cost of producing 100 units of output.

As we did in question 2(a) above, we will first look at this problem in general terms, then tackle the specific case. We have a production function $Q = f(K, L)$ and a total cost function $TC = wL + rK$, where the input prices *w* and *r* are given parameters. Our task is to minimise total cost, subject to the constraint that a specified level of output, $\overline{Q}$, must be produced.

Thus we want to minimise $TC = wL + rK$, subject to the constraint $\overline{Q} = f(K, L)$. The Lagrangean expression is therefore:

$V = wL + rK + \lambda\left(f(K,L) - \overline{Q}\right)$ Setting the 3 partial derivatives equal to zero gives

$\frac{\partial V}{\partial L} = w + \lambda f_L = 0$ (1) (where $f_L \equiv \frac{\partial Q}{\partial L}$ is the marginal product of labour)

$\frac{\partial V}{\partial K} = r + \lambda f_K = 0$ (2) (where $f_K \equiv \frac{\partial Q}{\partial K}$ is the marginal product of capital)

$\frac{\partial V}{\partial \lambda} = f(K,L) - \overline{Q} = 0$ (3)

From (1), $\frac{w}{f_L} = -\lambda$, and from (2), $\frac{r}{f_K} = -\lambda$. Combining these, we have

$\frac{r}{w} = \frac{f_K}{f_L}$ (4)

(Note that this is the same key equation as we obtained in question 2 above. However, in question 2 we were trying to achieve the maximum possible output with a fixed budget; while in this question we are trying to achieve the minimum total cost of producing a fixed output.)

Moving now to the specific example, we have $w = 6$, $r = 2$, and $\overline{\mathrm{Q}} = 100$. Given the production function $Q = K^{0.6}L^{0.5}$ we have $f_L = 0.5K^{0.6}L^{-0.5}$ and $f_K = 0.6K^{-0.4}L^{0.5}$. After substituting this information into equations (3) and (4) they become

$K^{0.6}L^{0.5} - 100 = 0$ (3a) and

$\frac{2}{6} = \frac{0.6K^{-0.4}L^{0.5}}{0.5K^{0.6}L^{-0.5}}$ (4a)

Equations (3a) and (4a) give us a pair of simultaneous equations which we can solve for K and L. One method of attack is to start with the right hand side of equation (4a). This can be manipulated as follows:

$\frac{0.6K^{-0.4}L^{0.5}}{0.5K^{0.6}L^{-0.5}} = \tfrac{6}{5}\left(K^{-0.4}L^{0.5}\right)\left(K^{-0.6}L^{0.5}\right) = \tfrac{6}{5}\left(K^{-1}L^{1}\right) = \tfrac{6}{5}\frac{L}{K}$. Thus (4a) becomes:

$\tfrac{2}{6} = \tfrac{6}{5}\frac{L}{K}$, from which $K = \tfrac{36}{10}L$. We can then substitute this into (3a) and get:

$\left(\tfrac{36}{10}\right)^{0.6} L^{0.6}L^{0.5} = 100$. So $L^{1.1} = 100\left(\tfrac{10}{36}\right)^{0.6} = 46.368$. Therefore

$L = \left(46.368\right)^{\frac{1}{1.1}} = 32.71$ and hence $K = \tfrac{36}{10}L = \tfrac{36}{10}32.71 = 117.77$. Minimised total cost is thus $TC = wL + rK = 6(32.71) + 2(117.77) = 431.8$. We can also check that the required level of output is produced. Output is $Q = K^{0.6}L^{0.5} = (117.77)^{0.6}(32.71)^{0.5} = 99.99104$. (Note the small rounding error).

(b) In the same way, find the minimum cost of producing 200 units of output. Comparing your answers to (a) and (b), does increasing output by 100% increase costs by more or less than 100% in this example?

Answer: The increase in the required level of output from 100 to 200 means that equation (3a) in part (a) above now becomes

$$K^{0.6}L^{0.5} - 200 = 0 \qquad \text{(new 3a)}$$

while equation (4a) remains unchanged as:

$$\frac{2}{6} = \frac{0.6K^{-0.4}L^{0.5}}{0.5K^{0.6}L^{-0.5}} \qquad \text{(4a)}$$

As in part (a) above, from equation (4a) we obtain: $\frac{2}{6} = \frac{6}{5}\frac{L}{K}$, from which $K = \frac{36}{10}L$.

We can then substitute this into (new 3a) and get: $\left(\frac{36}{10}\right)^{0.6} L^{0.6}L^{0.5} = 200$. So $L^{1.1} = 200\left(\frac{10}{36}\right)^{0.6} = 92.736$. Therefore $L = (92.736)^{\frac{1}{1.1}} = 61.434$ and hence $K = \frac{36}{10}L = \frac{36}{10}61.434 = 221.162$. Minimised total cost is thus $TC = wL + rK = 6(61.434) + 2(221.162) = 810.93$. We can also check that the required level of output is produced. Output is $Q = K^{0.6}L^{0.5} = (221.162)^{0.6}(61.434)^{0.5} = 200.0003$. (Note the small rounding error).

Thus to double output from 100 to 200, while minimising total cost, requires in this example that inputs of capital and labour be less than doubled; from 32.71 to 61.434 in the case of labour, and from 117.77 to 221.162 in the case of capital. Consequently, with unchanged input prices (*w* and *r*), total cost is less than doubled, from 431.8 to 810.93. (See chapter 17 of the book for further analysis.)

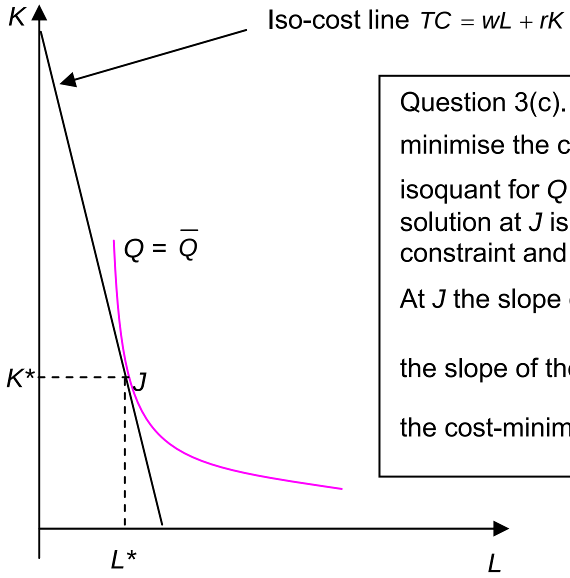
(c) Show algebraically and diagrammatically that to achieve minimum cost for a given output, $Q = \overline{Q}$, the firm must choose values for *L* and *K* such that the isoquant for $Q = \overline{Q}$, is tangent to an isocost line.

Answer: In part (a) above we obtained the key equation:

$$\frac{r}{w} = \frac{f_K}{f_L} \qquad (4)$$

This equation must be satisfied by a firm seeking to produce a given output at minimum total cost. The left hand side of this equation is derived from the iso-cost line $\overline{TC} = wL + rK$, where $\overline{TC}$ is any given level of total cost. With *TC* fixed, and *w* and *r* given, the equation $\overline{TC} = wL + rK$ is an implicit function relating the two variables *K* and *L*. By implicit differentiation, its slope is $-\frac{r}{w}$. Thus the left hand side of this equation is the absolute value (that is, ignoring sign) of the slope of any iso-cost line.

Similarly, the right hand side of equation (4) is the absolute slope of an isoquant of the production function. Thus equation (4) says that the firm, in order to minimise the total cost of any given output, must produce at a point of tangency between an iso-cost line and the relevant production function isoquant. (See section 16.6 of the book if you are unsure about this.) See the diagram below.



Question 3(c). When the firm is seeking to minimise the cost of a given output $\overline{Q}$, the isoquant for $Q = \overline{Q}$ is the constraint. The solution at *J* is the tangency between the constraint and the lowest possible iso-cost line. At *J* the slope of the iso-cost line, $-\frac{r}{w}$, equals the slope of the isoquant, $-\frac{f_K}{f_L}$. *K** and *L** are the cost-minimising levels of *K* and *L*.

(d) Show that, whatever level of output the firm produces, it will always employ capital and labour in the proportion $\frac{K}{L} = \frac{18}{5}$. Show how this proportion is related to the input prices and the parameters of the production function.

Answer: In part (a) above we obtained the equation:

$$\frac{2}{6} = \frac{0.6K^{-0.4}L^{0.5}}{0.5K^{0.6}L^{-0.5}} \qquad \text{(4a)}$$

From this, by manipulation, we obtain: $\frac{2}{6} = \frac{6}{5}\frac{L}{K}$, from which $K = \frac{36}{10}L = \frac{18}{5}L$. If we look back at how we derived equation (4a), we see that the left hand side, the ratio $\frac{2}{6}$, is given by $\frac{r}{w}$, the ratio of the price of capital to the price of labour. On the right hand side, the ratio $\frac{6}{5}$ is given by the ratio of the exponents of *K* and *L* respectively in the production function $Q = K^{0.6}L^{0.5}$. (That is, $\frac{6}{5} = \frac{0.6}{0.5}$.) So unless the ratio of input prices changes, or the ratio of exponents in the production function changes, the firm will always use a capital to labour ratio given by $\frac{K}{L} = \frac{36}{10} = \frac{18}{5}$ in order to minimise the total cost of any given output.

Exercise WS16.4

1. A firm's production function is $Q = K^{0.4}L^{0.5}$. The firm is perfectly competitive and hires its machines at a constant rental rate of r = 4 euros per hour and its workers at a constant wage rate of w = 5 euros per hour. It can also sell as much output as it wishes at the ruling market price of P = 20 euros.

(a) Find the most profitable output, the profits at this output, and the capital and labour inputs.

Answer: This question is similar to questions 2 and 3 in Ex WS16.3 except that both output and total cost are now variables. We will first look at this problem in general terms, then tackle the specific case. We have a production function $Q = f(K, L)$ and a profit function $\Pi \equiv TR - TC = PQ - (wL + rK)$. Because we assume perfect competition, the output price P and the input prices w and r are given parameters. Our task is to maximise profit, subject to the constraint that the combination of Q, K and L chosen must satisfy the constraint of the production function. (Without this constraint, there would be no unique maximum to profit as we could always increase profit by increasing Q and decreasing K and L.)

Thus we want to maximise $\Pi = PQ - (wL + rK)$, subject to the constraint $Q = f(K, L)$. The Lagrangean expression is therefore:

$$V = PQ - wL - rK + \lambda\left(f(K, L) - Q\right)$$

This expression has four variables: Q, L, K, and λ. Setting the 4 partial derivatives equal to zero gives

$$\frac{\partial V}{\partial Q} = P - \lambda = 0 \qquad (1)$$

$$\frac{\partial V}{\partial L} = -w + \lambda f_L = 0 \qquad (2) \quad \text{(where } f_L \equiv \frac{\partial Q}{\partial L} \text{ is the marginal product of labour)}$$

$\dfrac{\partial V}{\partial K} = -r + \lambda f_K = 0$ (3) (where $f_K \equiv \dfrac{\partial Q}{\partial K}$ is the marginal product of capital)

$\dfrac{\partial V}{\partial \lambda} = f(K, L) - Q = 0$ (4)

From (1) we immediately get $P = \lambda$. Substituting this into equations (2) and (3), and re-arranging, we get:

$f_L = \dfrac{w}{P}$ (2a) and

$f_K = \dfrac{r}{P}$ (3a)

(Equations (2a) and (3a) are the key equations, known as the marginal productivity conditions; see rule 16.2 in the book.) Equations (2a) and (3a), together with equation (4), give us a set of three simultaneous equations in the three unknowns, Q, K and L.

Moving now to the specific example, we have $Q = K^{0.4}L^{0.5}$, so

$f_L \equiv \dfrac{\partial Q}{\partial L} = 0.5K^{0.4}L^{-0.5}$ and $f_K \equiv \dfrac{\partial Q}{\partial K} = 0.4K^{-0.6}L^{0.5}$.

We also have $P = 20$, $w = 5$ and $r = 4$.

Substituting all of this information into equations (2a), (3a), and (4) we get:

$0.5K^{0.4}L^{-0.5} = \frac{5}{20}$ (2b)

$0.4K^{-0.6}L^{0.5} = \frac{4}{20}$ (3b)

$K^{0.4}L^{0.5} - Q = 0$ (4a)

Equations (2b) and (3b) contain only K and L (but not Q) and can therefore be solved as a pair of simultaneous equations to get K and L. To do this, we can divide (2b) by (3b), giving $\dfrac{0.5K^{0.4}L^{-0.5}}{0.4K^{-0.6}L^{0.5}} = \dfrac{\frac{5}{20}}{\frac{4}{20}}$. Rearranging this gives $\dfrac{5}{4}\dfrac{K}{L} = \dfrac{\frac{5}{20}}{\frac{4}{20}}$, which simplifies to $K = L$. We can then use this to substitute for, say, K in equation (2b), and get

$0.5L^{0.4}L^{-0.5} = \frac{5}{20}$. Therefore $L^{-0.1} = \frac{10}{20} = \frac{1}{2} \Rightarrow$
$\frac{1}{L^{0.1}} = \frac{1}{2}$, so $L^{0.1} = 2$, so $L = 2^{10} = 1024$.

Therefore $K = L = 1024$. We can now use these values in (4a) to find Q as: $Q = (1024)^{0.4}(1024)^{0.5} = 512$.

Maximised profits are therefore:

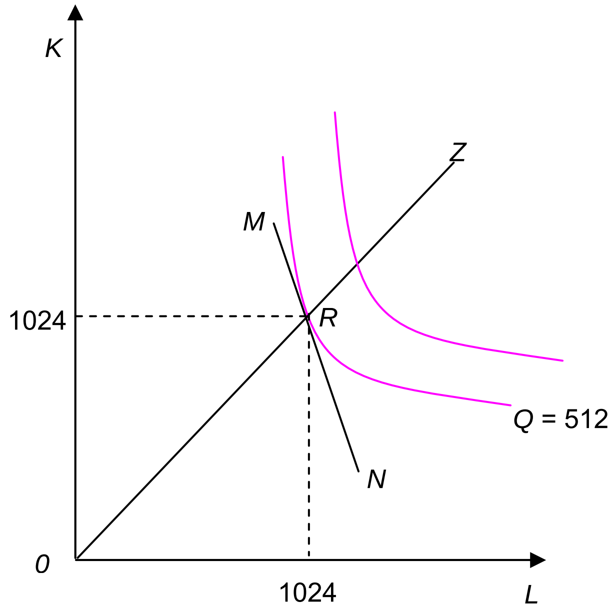
$$\Pi = PQ - (wL + rK) = 20(512) - 5(1024) - 4(1020) = 1024$$

(b) How would the most profitable output change if P rose from 20 to, say, 21 (with input prices unchanged)? What does this tell us about the firm's supply function (that is, Q as a function of P)? Does supply appear to be elastic or inelastic?

Answer: We re-solve equations 2a and 2b, with $P = 21$ and all other parameter values unchanged. The new solution values are $K = L = 1668$, $Q = 794$, and $\Pi = 1668$. Thus supply is very elastic, as an increase in P from 20 to 21 (5%) has led to an increase in Q from 512 to 794 (more than 50%). So the arc elasticity of supply on this section of the supply curve (that is, between $P = 20$ and $P = 21$) is more than 10 (more than 50, divided by 5).

(c) Find the relationship between K and L when total costs are minimised. Illustrate this graphically, showing the isoquants and iso-cost lines. Show the point of maximum profit (from (a) above) on the same diagram.

Answer: When we solved equations (2b) and (3b) in (a) above, we obtained $K = L$. So profit maximisation (or, more precisely, cost minimisation, which is a necessary condition for profit maximisation) requires using equal quantities of K and L, whatever the output level. See diagram below.



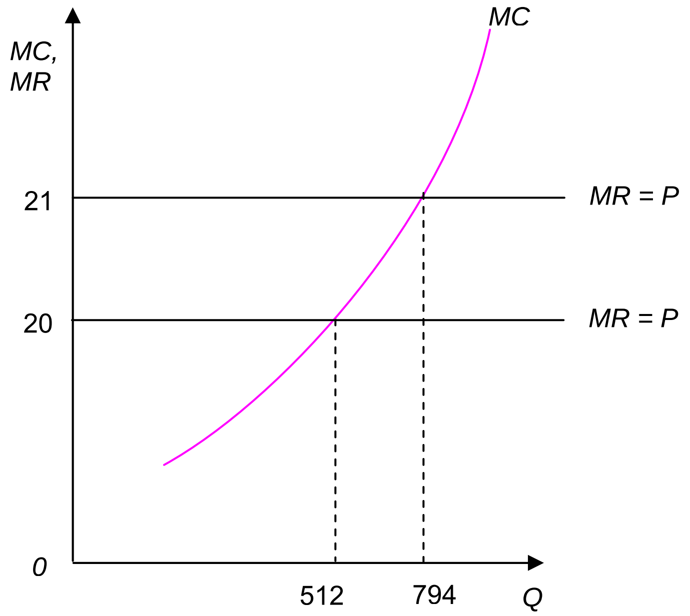
Question 1(d). To minimise the cost of any given output, the firm will always produce on the line *0Z* in the graph. This line has a slope of 1, so at every point on it we have *K* = *L*. Thus at every point on *0Z* equations 2b and 3b are satisfied. The point of maximum profit when *P* = 20 is at point *R*. The line *MN*, which is tangent to the isoquant at *R*, has a slope of $-\frac{w}{r} = -\frac{5}{4}$. (Of course, if the values of *w* and *r* were different, we would get a different solution and the slopes of *0Z* and *MN* would be different – either steeper or flatter.)

(d) Find the relationship between total cost and output when profits are maximised. (Hint: By definition, *TC* = *wL* + *rK*, and we can eliminate *K* and *L* using our answer to (c). It may help to recall that, by definition, a *TC* function gives the *minimum* total cost of a given output) What can you deduce about marginal cost, as a function of output? How does marginal cost compare with marginal revenue?

Answer: We know that cost minimisation requires *K* = *L*. (for the given parameters values in this example). Using this to substitute for *K* in the production function, we get $Q = K^{0.4}L^{0.5} = L^{0.4}L^{0.5} = L^{0.9}$. Raising both sides to the power $\frac{1}{0.9}$ we get $L = Q^{\frac{1}{0.9}} = Q^{1.111}$. Next, total cost is $TC \equiv wL + rK = (w + r)L = 9L$ (since we know *K* = *L* and *w* = 5, *r* = 4). Thus we have

$L = Q^{1.111}$ and $TC = 9L$.

Combining these gives $TC = 9Q^{1.111}$, which is the relationship that we were seeking between (minimum) total cost and output. We differentiate this (power rule) to get marginal cost, which is therefore $MC = 9(1.111)Q^{0.111} = 10Q^{0.111}$. As *Q* is raised to a positive power, we can deduce that *MC* increases as *Q* increases. As to marginal revenue, since price is constant (perfect competition) we have *MR* = *P* = constant. So the *MC* and *MR* curves (for *P* = 20 and *P* = 21) are as shown below. (Note: diagram is not to scale. As we saw in part (b), supply is very elastic, so the *MC* curve should have a much flatter slope than that shown. However, the *MC* curve is correctly drawn as non-linear. This is because $\frac{dMC}{dQ}$, the slope of the *MC* curve, increases as *Q* increases; check for yourself.)



2. A firm's production function is $Q = 5KL - 2K^2 - 3L^2$. The firm is perfectly competitive and the relevant prices, as defined in question 1 above, are $w = r = P = 1$.

(a) Find the function (equation) giving the combinations of *K* and *L* such that $MPK = \frac{r}{P}$. Sketch the graph of this function with *K* as the dependent variable.

Answer: We use equation 3a from question 1(a) above, which was:

$$f_K = \frac{r}{P} \qquad \text{(3a)}$$

In this example we have $f_K = 5L - 4K = \dfrac{r}{P} = 1$. Rearranging this gives $K = \frac{5}{4}L - \frac{1}{4}$. It is the relationship between *K* and *L* which must hold in order to satisfy the profit maximising condition, $MPK = \frac{r}{P}$.

(b) Similarly, find the function giving the combinations of K and L such that $MPL = \frac{w}{P}$. Sketch the graph of this function with K as the dependent variable. (This equation and its counterpart in (a) are known as the first-order conditions for maximum profit.)

Answer: This time we use equation 2a from question 1(a) above:

$$f_L = \frac{w}{P} \qquad \text{(2a)}$$

In this example we have $f_K = 5K - 6L = \dfrac{w}{P} = 1$. Rearranging this gives $K = \frac{6}{5}L + \frac{1}{5}$. It is the relationship between K and L which must hold in order to satisfy the profit maximising condition, $MPL = \frac{w}{P}$.

(c) Show that the first-order conditions for a maximum (or minimum) of profits are satisfied when $L = 9$ and $K = 11$. Find the output and profits. (Hint: You can do this either by the Lagrange multiplier method or by solving the two equations from (a) and (b).

Answer: From (a) and (b) above we have the two profit-maximising conditions: $K = \frac{5}{4}L - \frac{1}{4}$ and $K = \frac{6}{5}L + \frac{1}{5}$. Since both equations must be simultaneously satisfied if profits are to be maximised, we can solve them simultaneously. The solution is $K = 11$, $L = 9$. (By substituting these values into the production function we can find the profit-maximising level of output, $Q = 10$, though we are not asked to do so in this question.)

Alternatively, we can solve this problem using the Lagrange multiplier method. As in question 1, we want to maximise profit, $\Pi = PQ - (wL + rK)$, subject to the constraint of the production function $Q = f(K, L)$. The Lagrangean expression is therefore: $V = PQ - wL - rK + \lambda\big(f(K, L) - Q\big)$. In this example the production function is $Q = f(K, L) = 5KL - 2K^2 - 3L^2$ and the prices are $P = w = r = 1$. If we substitute this information into the Lagrangean expression, then follow the procedure of question 1, we will arrive at the same solution: $K = 11$, $L = 9$, and $Q = 10$.

(d) Consider whether $L = 9$, $K = 11$ is a maximum or a minimum of profits. Consider also whether costs are minimised, given the output, at this point.

Answer: this question cannot be answered rigorously without going beyond the scope of the book. But we can form an impression by using a calculator to examine points in the neighbourhood of our solution; see table below.
The middle row of the table gives our solution values: $K = 11$, $L = 9$, $Q = 10$. We have also calculated total revenue, $TR = PQ = 1 \times 10 = 10$; total cost, $TC = wL + rK = 1 \times 9 + 1 \times 11 = 20$; and finally profits as $\Pi = TR - TC = -10$. (Profit is negative, but this could nevertheless be maximum profit if losses are greater at any other combination of K and L.)

The first row examines the effect of a small reduction in *K* together with a small increase in *L*, so as to maintain output (approximately) constant. We see that this both increases *TC* and reduces profits (that is, losses increase). Total revenue is also lower, but this is merely because output is slightly lower.
Similarly, the last row examines the effect of a small increase in *K* together with a small decrease in *L*, again so as to maintain output (approximately) constant. Again we see that this both increases *TC* and reduces profits (that is, losses increase). Total revenue is also lower, but this is merely because output is slightly lower. So it appears (but is not certain) that any combination of *K* and *L,* other than $K = 11$, $L = 9$, with output unchanged, results in both higher costs and smaller profits (larger losses).

K	L	Q	TR	TC	Π
10.9901	9.011	9.999996	9.999996	20.0011	-10.0011
11	9	10	10	20	-10
11.0095	8.991	9.999649	9.999649	20.0005	-10.0009

Exercise WS16.5

1. Ann's utility function is $U = 3\ln X + 2\ln Y$ where *X* and *Y* are weekly consumption levels of goods *X* and *Y*. The market prices are $P_X = 2$ euros and $P_Y = 1$ euro, and her weekly budget is $B = 100$ euros.

(a) Find the quantities that Ann should buy each week in order to maximise her utility.

Answer: We want to maximise Ann's utility, $U = 3\ln X + 2\ln Y$, subject to the constraint of her budget, $B = P_X X + P_Y Y$, with $B = 100$, $P_X = 2$, $P_Y = 1$. The Lagrangean expression is therefore:

$V = 3\ln X + 2\ln Y + \lambda(B - P_X X - P_Y Y)$. The partial derivatives, set equal to zero, are:

$$V_X = \frac{3}{X} - \lambda P_X = 0 \;\; ; \;\; V_Y = \frac{2}{Y} - \lambda P_Y = 0 \;\; ; \;\; V_\lambda = B - P_X X - P_Y Y = 0$$

(recall rules 13.3 and 13.4 for the derivative of ln*X*).

Combining the first two equations, we get $\dfrac{3}{P_X X} = \lambda = \dfrac{2}{P_Y Y}$, from which

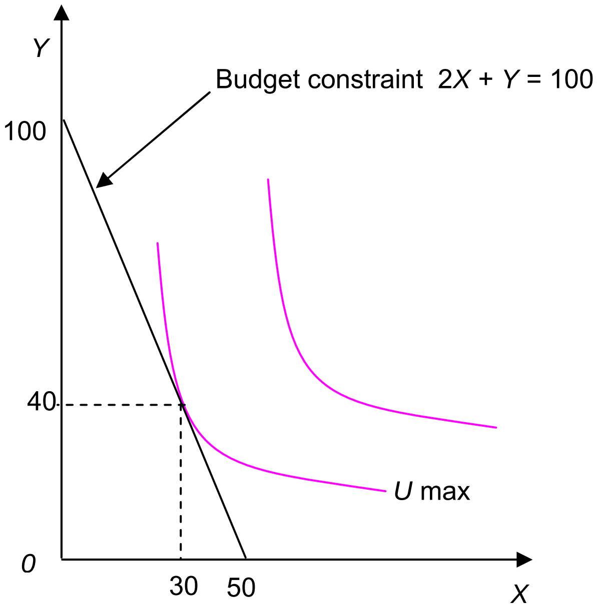
$\dfrac{Y}{X} = \tfrac{2}{3}\dfrac{P_X}{P_Y}$ and therefore $Y = \tfrac{2}{3}\dfrac{P_X}{P_Y} X$. Substituting this into the third equation, we get

$B = P_X X - P_Y \tfrac{2}{3}\dfrac{P_X}{P_Y} X = X(\tfrac{5}{3}P_X)$, and thus $X = \tfrac{3}{5}\dfrac{B}{P_X}$. Therefore

$Y = \tfrac{2}{3}\dfrac{P_X}{P_Y}X = \tfrac{2}{3}\dfrac{P_X}{P_Y}\left(\tfrac{3}{5}\dfrac{B}{P_X}\right) = \tfrac{2}{5}\dfrac{B}{P_Y}$. Thus the equilibrium (utility maximising) quantities purchased by Ann are $X = \tfrac{3}{5}\dfrac{B}{P_X}$ and $Y = \tfrac{2}{5}\dfrac{B}{P_Y}$.

In this question we have $B = 100$, $P_X = 2$, $P_Y = 1$. So her utility maximising quantities purchased are $X = 30$ and $Y = 40$.

(b) Show graphically her equilibrium as a tangency between an indifference curve and her budget line.



(c) What is her weekly expenditure on each good (i) in money terms, and (ii) as a proportion of her budget?

Answer: (i) using our answers to (a), expenditure on X and Y is

$P_X X = P_X \tfrac{3}{5}\dfrac{B}{P_X} = \tfrac{3}{5}B = 60$ and $P_Y Y = P_Y \tfrac{2}{5}\dfrac{B}{P_Y} = \tfrac{2}{5}B = 40$.

(ii) 60% and 40%

(d) Suppose P_X rises from 2 to 3 euros. Find the new utility-maximising quantities. Comment on the effect of a rise in P_X on the demand for good Y.

Answer: from (a) we have: $X = \tfrac{3}{5}\dfrac{B}{P_X}$ and $Y = \tfrac{2}{5}\dfrac{B}{P_Y}$. The new price of X is $P_X = 3$, with price of Y unchanged at $P_Y = 1$ and budget unchanged at $B = 100$. So the new solution for X is $X = \tfrac{3}{5}\dfrac{B}{P_X} = \tfrac{3}{5}\dfrac{100}{3} = 20$. Since our expression for the utility maximising quantity of *Y*, $Y = \tfrac{2}{5}\dfrac{B}{P_Y}$, does not contain P_X, the change in P_X has no effect on Ann's purchases of *Y*. This is unusual, as we would normally expect a rise in the price of one good to lead to changes in demand for other goods, especially those which are close substitutes for or complements to the good in question. In this example, Ann's utility function has the property that utility maximising quantities of each good depend only on the price of that good.

(e) Following this price increase, what are Ann's new weekly expenditures on each good (a) in money terms, and (b) as a proportion of her budget?

Answer: using the same method as in (c), $P_X X = P_X \tfrac{3}{5}\dfrac{B}{P_X} = \tfrac{3}{5}B = 60$. Thus expenditure on *X* remains the same, despite the higher price. This is because P_X cancels between numerator and denominator. As discussed in (d) above, neither price nor quantity of *Y* change. Expenditure remains at 40. Percentage shares remain unchanged, obviously.

(f) From your answer to (a), find Ann's demand function for good *X* (that is, the functional relationship between the quantity of *X* the she wishes to buy and the price of *X*, with other variable(s) treated as parameters). From the demand function, find the price elasticity of demand for *X*.

Answer: from (a) we have $X = \tfrac{3}{5}\dfrac{B}{P_X}$. This is by definition the demand function for *X*, as it gives quantity demanded as a function of P_X and *B*. (Thus a person's demand function for any good is derived from her utility maximising behaviour.) The derivative of the demand function, with respect to P_X, is $\dfrac{\partial X}{\partial P_X} = -\tfrac{3}{5}\dfrac{B}{(P_X)^2}$, so the own-price elasticity of demand is $E_X^D \equiv \dfrac{P_X}{X}\dfrac{\partial X}{\partial P_X} = -\tfrac{3}{5}\dfrac{B}{(P_X)^2}\dfrac{P_X}{X} = -\tfrac{3}{5}\dfrac{B}{P_X}\dfrac{1}{X}$. Since $X = \tfrac{3}{5}\dfrac{B}{P_X}$, this becomes $E_X^D = -\tfrac{3}{5}\dfrac{B}{P_X}\dfrac{1}{X} = -\tfrac{3}{5}\dfrac{B}{P_X}\tfrac{5}{3}\dfrac{P_X}{B} = -1$. The elasticity is constant and equal to minus one. This implies that total expenditure on *X* is constant, as we discovered in (e) above.

(g) Repeat (f) for good *Y*.

Answer: from (a) we have $Y = \tfrac{2}{5}\dfrac{B}{P_Y}$. Using the same method as in (f) above, we

find that $E_Y^D = -\tfrac{2}{5}\dfrac{B}{P_Y}\tfrac{5}{2}\dfrac{P_Y}{B} = -1$.

2. Ben's utility function is $U = X^3Y^2$. His weekly budget is 100 euros and he buys in the same markets as Ann in the previous question, and therefore faces the same prices; that is, P_X = 2 euros and P_Y = 1 euro.

   Show that Ben will buy the same quantities of each good as Ann, and that in general his demand functions for the two goods are identical to Ann's. Can you suggest why this is true, despite the difference in their utility functions?

Answer: We seek to maximise Ben's utility. The Lagrangean expression is:

$V = X^3Y^2 + \lambda(B - P_X X - P_Y Y)$. The partial derivatives, set equal to zero, are:

$V_X = 3X^2Y^2 - \lambda P_X = 0$; $V_Y = 2X^3Y - \lambda P_Y = 0$; $V_\lambda = B - P_X X - P_Y Y = 0$

Combining the first two equations, we get $\dfrac{3X^2Y^2}{P_X} = \lambda = \dfrac{2X^3Y}{P_Y}$, from which

$Y = \frac{2}{3}\dfrac{P_X}{P_Y}X$. Substituting this into the third equation, we get

$B = P_X X - P_Y \frac{2}{3}\dfrac{P_X}{P_Y}X = X(\frac{5}{3}P_X)$, and thus $X = \frac{3}{5}\dfrac{B}{P_X}$.

These demand functions for *X* and *Y* are the same as we found for Ann in question 1 above. Thus Ann and Ben will buy the same quantities of each good, because their demand functions are identical. This is puzzling, as their utility functions appear very different. However, if we take natural logs on both sides of Ben's utility function, we get $U = 3\ln X + 2\ln Y$, which is the same as Ann's utility function. This illustrates the important theoretical point that if one person's utility function is a monotonic transformation of another person's utility function (meaning that their preference ordering of bundles of goods is identical) then they will have identical demand functions. In this case, if Ann and Ben both consume identical bundles of *X* and *Y*, the utility enjoyed by Ann will be the natural log of the utility enjoyed by Ben. But this changes only the units in which her utility is measured, and we noted in the book that these units are inherently arbitrary.