**Exercise WS15.1**

1. Show that the function $z = 30x + 50y - 3y^2 - 3x^2 - 4xy + 200$ has a single stationary point (SP) at $x = -1,\ y = 9$ and that this SP is a maximum.

Answer: We first find the two first-order partial derivatives, set them equal to zero, and solve as simultaneous equations. This gives:

$\frac{\partial z}{\partial x} = 30 - 6x - 4y = 0$ and $\frac{\partial z}{\partial y} = 50 - 6y - 4x = 0$. The solution is $y$ = 9, $x$ = –1

Next we take the two direct second derivatives, which are $\frac{\partial^2 z}{\partial x^2} = -6$ and $\frac{\partial^2 z}{\partial y^2} = -6$. As these are both negatives, the point $y$ = 9, $x$ = –1 is probably a maximum of $z$ (see rule 15.1 in the book). To be sure, we must compare $\frac{\partial^2 z}{\partial x^2}\frac{\partial^2 z}{\partial y^2}$ with $\frac{\partial^2 z}{\partial y \partial x}\frac{\partial^2 z}{\partial x \partial y}$. We find $\frac{\partial^2 z}{\partial y \partial x} = -4$ and $\frac{\partial^2 z}{\partial x \partial y} = -4$ (note incidentally that these are equal, due to Young's theorem). So we have

$\frac{\partial^2 z}{\partial x^2}\frac{\partial^2 z}{\partial y^2} = (-6)(-6) = 36$ and $\frac{\partial^2 z}{\partial y \partial x}\frac{\partial^2 z}{\partial x \partial y} = (-4)(-4) = 16$. Thus

$\frac{\partial^2 z}{\partial x^2}\frac{\partial^2 z}{\partial y^2} > \frac{\partial^2 z}{\partial y \partial x}\frac{\partial^2 z}{\partial x \partial y}$, as required for a maximum (see rule 15.3).

2. Show that $z = 30.005x^2 - xy + 50y^2 - 0.25x^3$ has a minimum at x = y = 0 and a saddle point at x = 80, y = 0.8.

Answer: repeating the steps in (1) above, we get

$\frac{\partial z}{\partial x} = 60.01x - y - 0.75x^2 = 0$ and $\frac{\partial z}{\partial y} = -x + 100y = 0$. Because of the $x^2$ term in $\frac{\partial z}{\partial x}$, there are two solutions: (i) $x$ = $y$ = 0 and (ii) $x$ = 80, $y$ = 0.8.

We also have $\frac{\partial^2 z}{\partial x^2} = 60.01 - 1.5x$ ; $\frac{\partial^2 z}{\partial y^2} = 100$ ; $\frac{\partial^2 z}{\partial y \partial x} = \frac{\partial^2 z}{\partial x \partial y} = -1$. Therefore, at $x$ = $y$ = 0, we have $\frac{\partial^2 z}{\partial x^2} = 60.01 > 0$ and $\frac{\partial^2 z}{\partial y^2} = 100 > 0$. So this is probably a minimum (see rule 15.2). To confirm this, we have $\frac{\partial^2 z}{\partial x^2}\frac{\partial^2 z}{\partial y^2} = 60.01(100) = 6001$ and $\frac{\partial^2 z}{\partial y \partial x}\frac{\partial^2 z}{\partial x \partial y} = (-1)(-1) = 1$. Thus $\frac{\partial^2 z}{\partial x^2}\frac{\partial^2 z}{\partial y^2} > \frac{\partial^2 z}{\partial y \partial x}\frac{\partial^2 z}{\partial x \partial y}$, as required for a minimum


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(see rule 15.3).

Similarly, at the second solution, $x = 80$, $y = 0.8$, we have $\frac{\partial^2 z}{\partial x^2} = 60.01 - 120 < 0$ and $\frac{\partial^2 z}{\partial y^2} = 100 > 0$. So this is probably a saddle point (see rule 15.5). To confirm this, we have $\frac{\partial^2 z}{\partial x^2}\frac{\partial^2 z}{\partial y^2} = -59.99(100) = -5999$ and $\frac{\partial^2 z}{\partial y \partial x}\frac{\partial^2 z}{\partial x \partial y} = (-1)(-1) = 1$. Thus $\frac{\partial^2 z}{\partial x^2}\frac{\partial^2 z}{\partial y^2} < \frac{\partial^2 z}{\partial y \partial x}\frac{\partial^2 z}{\partial x \partial y}$, as required for a saddle point (see rule 15.5).

3. Find the stationary points of the following functions, and determine whether each is a maximum, minimum or saddle point.

(a) $z = 3x^2 + 2y^2 - 15x + 12y + 100$

Answer: using the method of (1) and (2) above, we get $\frac{\partial z}{\partial x} = 6x - 15 = 0$, from which $x = {}^5/_2$ ; and $\frac{\partial z}{\partial y} = 4y + 12 = 0$, from which $y = -3$. (Note incidentally that in this example these equations are not simultaneous; each can be solved separately.)

We also have $\frac{\partial^2 z}{\partial x^2} = 6$ ; $\frac{\partial^2 z}{\partial y^2} = 4$ ; $\frac{\partial^2 z}{\partial y \partial x} = \frac{\partial^2 z}{\partial x \partial y} = 0$. Thus at $x = {}^5/_2$, $y = -3$ we have $\frac{\partial^2 z}{\partial x^2} > 0$ ; $\frac{\partial^2 z}{\partial y^2} > 0$ ; and $\frac{\partial^2 z}{\partial x^2}\frac{\partial^2 z}{\partial y^2} > \frac{\partial^2 z}{\partial y \partial x}\frac{\partial^2 z}{\partial x \partial y}$. So this is a minimum.

(b) $z = -x^2 + 2.5y^2 + 10xy - 16x - 30y$

Answer: using the method of (1) and (2) above, we get $\frac{\partial z}{\partial x} = -2x + 10y - 16 = 0$, and $\frac{\partial z}{\partial y} = 5y + 10x - 30 = 0$, from which $y = 2$, $x = 2$.

Then we have $\frac{\partial^2 z}{\partial x^2} = -2 < 0$ ; $\frac{\partial^2 z}{\partial y^2} = 5 > 0$, so this is probably a saddle point. We also have $\frac{\partial^2 z}{\partial y \partial x} = \frac{\partial^2 z}{\partial x \partial y} = 10$, so $\frac{\partial^2 z}{\partial x^2}\frac{\partial^2 z}{\partial y^2} = -2(5) = -10 < \frac{\partial^2 z}{\partial y \partial x}\frac{\partial^2 z}{\partial x \partial y} = 100$, confirming that it is a saddle point.


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**Exercise WS15.2**

1. Find the total differential (d$z$) of the following functions:

(a) $z = (x+1)^2 + (y^2 - 2)^3$

Answer: By definition, $dz = \frac{\partial z}{\partial x}dx + \frac{\partial z}{\partial y}dy$. Here we have $\frac{\partial z}{\partial x} = 2(x+1)$ and $\frac{\partial z}{\partial y} = 3(y^2-2)^2(2y) = 6y(y^2-2)^2$. So $dz = 2(x+1)dx + 6y(y^2-2)^2 dy$

(b) $z = x^2 + y^3 + \frac{2x}{3y^2}$

Answer: Here we have $\frac{\partial z}{\partial x} = 2x + \frac{2}{3y^2}$ and $\frac{\partial z}{\partial y} = 3y^2 - \frac{4x}{3y^3}$. So

$$dz = \left(2x + \frac{2}{3y^2}\right)dx + \left(3y^2 - \frac{4x}{3y^3}\right)dy$$

(c) $z = \frac{x^2 + 2y}{x^3 - y^2}$

Answer: Here we have

$\frac{\partial z}{\partial x} = \frac{\left(x^3 - y^2\right)2x - (x^2+2y)3x^2}{\left(x^3 - y^2\right)^2}$ and $\frac{\partial z}{\partial y} = \frac{\left(x^3 - y^2\right)2 - (x^2+2y)(-2y)}{\left(x^3 - y^2\right)^2}$. So

$$dz = \frac{\left(x^3 - y^2\right)2x - (x^2+2y)3x^2}{\left(x^3 - y^2\right)^2}dx + \frac{\left(x^3 - y^2\right)2 - (x^2+2y)(-2y)}{\left(x^3 - y^2\right)^2}dy$$

(d) $z = x^{\alpha}\left(\frac{y}{x}\right)^{1-\alpha}$ (where $\alpha$ is a parameter)

We can write this function as $z = x^{\alpha}\left(\frac{y}{x}\right)^{1-\alpha} \equiv x^{\alpha-(1-\alpha)}y^{1-\alpha} \equiv x^{2\alpha-1}y^{1-\alpha}$. So

$\frac{\partial z}{\partial x} = (2\alpha - 1)x^{2\alpha-2}y^{1-\alpha}$ and $\frac{\partial z}{\partial y} = (1-\alpha)x^{2\alpha-1}y^{-\alpha}$. So

$$dz = \left((2\alpha-1)x^{2\alpha-2}y^{1-\alpha}\right)\text{dx} + \left((1-\alpha)x^{2\alpha-1}y^{-\alpha}\right)dy$$

2. I want to lay a concrete path in my garden which I calculate will require $3\frac{1}{3}$ cubic yards of concrete. However, when I telephone a local supplier of ready-mixed concrete he tells me that he supplies concrete only by the cubic metre.


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(Note: 1 yard = 36 inches; assume 1 metre = 39 inches). Using the total differential:

(a) Calculate how much concrete will be left over if I order $3\frac{1}{3}$ cubic metres.

(b) Whether, if I order 3 cubic metres, this will be enough.

Answer: in general, the volume ($V$) of a cube with sides of lengths $p$, $q$ and $r$ is:

$V = pqr$. The partial derivatives of this function are
$\frac{\partial V}{\partial p} = qr$ ; $\frac{\partial V}{\partial q} = pr$ ; and $\frac{\partial V}{\partial r} = pq$

If we now increase the length of the sides by the amounts $dp$, $dq$ and $dr$, then the differential formula gives the change in volume as

$$dV = \frac{\partial V}{\partial p}dp + \frac{\partial V}{\partial q}dq + \frac{\partial V}{\partial r}dr = qrdp + prdq + pqdr$$

In the case of a cubic yard, we have $p = q = r = 36$ inches, so the volume is $V_1 = 36 \times 36 \times 36 = 36^3 = 46656$ cubic inches. If we now increase the length of each side by 3 inches, we have $dp = dq = dr = 3$. The differential formula then gives the change in volume as $dV = qrdp + prdq + pqdr = (36)(36)3 + (36)(36)3 + (36)(36)3 = 11664$ cubic inches. (This is an approximation, due to the inherent error in the differential formula).

But adding 3 inches to each side means that we now have a cubic metre. So our estimate of the volume ($V_2$) of a cubic metre is

$V_2 = V_1 + dV = 46656 + 11664 = 58320$ cubic inches

Answer to (a): Therefore if I order $3^1/_3$ cubic metres of concrete, I will receive $3^1/_3 \times 58320 = 194380.56$ cubic inches. But my actual need is for $3^1/_3$ cubic yards, which equals $3^1/_3 \times 46656 = 155504.49$ cubic inches. So I will be left with 194380.56 – 155504.49 = 38876.11 cubic inches of surplus concrete. This equals $38876.11 \div 46656 = 0.833$ cubic yards.

Answer to (b): If I order 3 cubic metres, then I will receive $3 \times 58320 = 174960$ cubic inches. But my actual need is for $3^1/_3$ cubic yards, which equals $3^1/_3 \times 46656 = 155504.49$ cubic inches. So I will be left with 174960 – 155504.49 = 19455.51 cubic inches of surplus concrete. This equals $19455.51 \div 46656 = 0.417$ cubic yards. So 3 cubic metres is enough.

Note: although it is not part of the question, it is interesting to check the size of the error in our estimate of 1 cubic metre. Our estimate is $V_2 = 58320$ cubic inches, when the true volume of 1 cubic metre is $(39)^3 = 59319$ cubic inches. So our error is 59319 – 58320 = 999 cubic inches. As our estimate is an *under*-estimate, this


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means that in practice the surpluses of concrete will be larger than estimated in (a) and (b).

3. (a) If $z = x^2 + 3y$, suppose x increases from 10 to 10.1, and *y* increases from 5 to 5.1. What percentage error occurs if we use the total differential, d*z*, to calculate the resulting change in *z*?

Answer: here $\frac{\partial z}{\partial x} = 2x$ and $\frac{\partial z}{\partial y} = 3$, so $dz = 2xdx + 3dy$. We have *x* = 10, *y* = 5, *dx* = *dy* = 0.1. So $dz = 20(0.1) + 3(0.1) = 2.3$. We calculate the true change in *z* as follows. The initial value is $z_0 = 10^2 + 15 = 115$. The new value is $z_0 = (10.1)^2 + 3(5.1) = 117.31$. So the true change is 117.31 – 115 = 2.31. Thus our estimated change is 0.01 less than the true change. As a percentage of the true change, our error is $\frac{0.01}{2.31} \times 100 = 0.4\%$ (that is, less than half of 1 percent).

(b) If $z = x^{\frac{1}{3}} y^{\frac{2}{3}}$, suppose initially x = y = 1000. Then x increases by 5% and y increases by 10%. What percentage error occurs if we use the total differential, d*z*, to calculate the resulting change in *z*?

Answer: here $\frac{\partial z}{\partial x} = \frac{1}{3} x^{-\frac{2}{3}} y^{\frac{2}{3}} = \frac{1}{3}\left(\frac{y}{x}\right)^{\frac{2}{3}}$ and $\frac{\partial z}{\partial y} = \frac{2}{3}\left(\frac{x}{y}\right)^{\frac{1}{3}}$. Also *dx* = 50, *dy* = 100. So

$$dz = \left[\frac{1}{3}\left(\frac{y}{x}\right)^{\frac{2}{3}}\right]dx + \left[\frac{2}{3}\left(\frac{x}{y}\right)^{\frac{1}{3}}\right]dy = \left[\frac{1}{3}\left(\frac{1000}{1000}\right)^{\frac{2}{3}}\right](50) + \left[\frac{2}{3}\left(\frac{1000}{1000}\right)^{\frac{1}{3}}\right](100)$$

$= \frac{1}{3}(50) + \frac{2}{3}(100) = 83.33$. To get the true change, the initial value is $z_0 = (1000)^{\frac{1}{3}}(1000)^{\frac{2}{3}} = 1000$. The new value is $z_1 = (1050)^{\frac{1}{3}}(1100)^{\frac{2}{3}} = 1082.6519$. So the true change is 82.6519, and our error is 83.33 – 82.65 = 0.68 (over-estimate). The percentage error is $\frac{0.68}{82.6519} \times 100 = 0.82\%$; that is, less than 1%. Note: to calculate, say, $(1000)^{\frac{1}{3}}$, you must key in to your calculator: 1000 ^ (1/3). If you key in: 1000 ^ 0.333 you will introduce a significant new error (because, of course, 0.333 is only an approximation to one-third).


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**Exercise WS15.3**

1. In each of the following, use the differential to find the specified derivative. Then check your answer by direction substitution.

(a) $z = 2x^3 + xy$, where $x = y^2 + 1$ ; find $\frac{dz}{dy}$.

Answer: $dz = \frac{\partial z}{\partial x}dx + \frac{\partial z}{\partial y}dy = (6x^2 + y)dx + xdy$. Dividing through by $dy$ gives

$\frac{dz}{dy} = \frac{\partial z}{\partial x}\frac{dx}{dy} + \frac{\partial z}{\partial y} = (6x^2 + y)\frac{dx}{dy} + x$. Given $x = y^2 + 1$, we have $\frac{dx}{dy} = 2y$.

Substituting this into our expression for $\frac{dz}{dy}$, we get

$\frac{dz}{dy} = \frac{\partial z}{\partial x}\frac{dx}{dy} + \frac{\partial z}{\partial y} = (6x^2 + y)(2y) + x$. Replacing $x$ using $x = y^2 + 1$ this

becomes $\frac{dz}{dy} = 12y(y^2 + 1)^2 + 3y^2 + 1$

Checking by direct substitution of $x = y^2 + 1$ into $z = 2x^3 + xy$, we get

$z = 2(y^2 + 1)^3 + (y^2 + 1)y = 2(y^2 + 1)^3 + y^3 + y$. The derivative is

$\frac{dz}{dy} = 12y(y^2 + 1)^2 + 3y^2 + 1$ as above.

(b) $z = \frac{x^2}{2u - 1}$, where $u = (1 - x)^2$ ; find $\frac{dz}{dx}$.

Answer: $dz = \frac{\partial z}{\partial x}dx + \frac{\partial z}{\partial y}dy = \frac{2x}{2u - 1}dx - \frac{2x^2}{(2u - 1)^2}du$. Dividing through by $dx$ gives

$\frac{dz}{dx} = \frac{2x}{2u - 1} - \frac{2x^2}{(2u - 1)^2}\frac{du}{dx}$. Given $u = (1 - x)^2$ we have $\frac{du}{dx} = -2(1 - x)$ Substituting

this into our expression for $\frac{du}{dx}$, we get

$\frac{dz}{dx} = \frac{2x}{2u - 1} - \frac{2x^2(-2(1 - x))}{(2u - 1)^2} = \frac{2x}{2u - 1} + \frac{4x^2(1 - x)}{(2u - 1)^2}$. Using $u = (1 - x)^2$ this becomes

$\frac{dz}{dx} = \frac{2x}{2(1 - x)^2 - 1} + \frac{4x^2(1 - x)}{(2(1 - x)^2 - 1)^2}$.


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Checking by direct substitution we get $z = \dfrac{x^2}{2(1-x)^2 - 1}$, with derivative

$$\frac{dz}{dx} = \frac{\left[2(1-x)^2 - 1\right]2x - x^2\left[-4(1-x)\right]}{\left[2(1-x)^2 - 1\right]^2} = \frac{2x}{2(1-x)^2 - 1} + \frac{4x^2(1-x)}{\left[2(1-x)^2 - 1\right]^2}$$ as above.

2. Use the function of a function rule to find the total differential, dz, where $z = u^{\frac{1}{2}}$ and $u = \dfrac{y}{x}$.

Answer: Given $z = u^{\frac{1}{2}}$ we can write its differential as $dz = \dfrac{dz}{du}du = \dfrac{1}{2}u^{-\frac{1}{2}}du$ (see section 15.4 in the book). Given $u = \dfrac{y}{x}$ its differential is $du = -\dfrac{y}{x^2}dx + \dfrac{1}{x}dy$ .
Substituting this into $dz$ gives $dz = \dfrac{1}{2}u^{-\frac{1}{2}}(-\dfrac{y}{x^2}dx + \dfrac{1}{x}dy)$. Using $u = \dfrac{y}{x}$ to eliminate $u$ this becomes $dz = \dfrac{1}{2}\left(\dfrac{x}{y}\right)^{\frac{1}{2}}(-\dfrac{y}{x^2}dx + \dfrac{1}{x}dy)$

3. Use the differential (dz) to find the derivative ($\dfrac{dy}{dx}$) of each of the following implicit functions:

(a) $x^2 + 2xy^2 + y^3 = 0$

Answer: if we write $f(x,y) = x^2 + 2xy^2 + y^3$, then the partial derivatives are $f_x = 2x + 2y^2$ and $f_y = 4xy + 3y^2$ (see section 15.7 of the book). Then the derivative, $\dfrac{dy}{dx}$, of the implicit function $x^2 + 2xy^2 + y^3 = 0$ is given by $\dfrac{dy}{dx} = -\dfrac{f_x}{f_y} = -\dfrac{2x + 2y^2}{4xy + 3y^2}$ (see rule 15.9).

(b) $0 = (x^2 + y^3)^{\frac{1}{3}}$

Answer: using the same method as (a) above, we have $f(x,y) = (x^2 + y^3)^{\frac{1}{3}}$ with partial derivatives $f_x = \dfrac{1}{3}(x^2 + y^3)^{-\frac{2}{3}}(2x)$ and $f_y = \dfrac{1}{3}(x^2 + y^3)^{-\frac{2}{3}}(3y^2)$. So

$$\frac{dy}{dx} = -\frac{f_x}{f_y} = -\frac{\frac{1}{3}(x^2 + y^3)^{-\frac{2}{3}}(2x)}{\frac{1}{3}(x^2 + y^3)^{-\frac{2}{3}}(3y^2)} = -\frac{2x}{3y^2}$$


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(c) $100 - x^{\frac{1}{2}}y^{\frac{3}{4}} = 0$

Answer: $\dfrac{dy}{dx} = -\dfrac{f_x}{f_y} = -\dfrac{-\frac{1}{2}x^{-\frac{1}{2}}y^{\frac{3}{4}}}{-\frac{3}{4}x^{\frac{1}{2}}y^{-\frac{1}{4}}} = -\dfrac{2}{3}x^{-\frac{1}{2}-\frac{1}{2}}y^{\frac{3}{4}-(-\frac{1}{4})} = -\dfrac{2}{3}\dfrac{y}{x}$

(d) $\dfrac{2x+1}{x^2+y^3} = 0$

Answer: $\dfrac{dy}{dx} = -\dfrac{f_x}{f_y} = -\dfrac{\dfrac{(x^2+y^3)2-(2x+1)2x}{(x^2+y^3)^2}}{\dfrac{(x^2+y^3)0-(2x+1)3y^2}{(x^2+y^3)^2}} = \dfrac{2(x^2+y^3)-2x(2x+1)}{3y^2(2x+1)}$

4. For each of the functions in question 1 above, use the differential to find the slope of an iso-*z* section.

Answer: (a) if we are given the function $z = x^2 + 2xy^2 + y^3$, then the equation of the iso-*z* contour for the fixed value $z = z_0$ is given by $z_0 = x^2 + 2xy^2 + y^3$. We can write this as $x^2 + 2xy^2 + y^3 - z_0$. The slope of this iso *z* contour is then given by $\dfrac{dy}{dx} = -\dfrac{f_x}{f_y} = -\dfrac{2x+2y^2}{4xy+3y^2}$; that is, the answer to 3(a) above (see section 15.8 of the book). That is, if we are given the *x* and *y* coordinates of the point in question, we can substitute these values into our expression for $\dfrac{dy}{dx}$ and thereby find the slope of the isoquant passing through that point. The answers to (b), (c) and (d) are found in the same way; that is, they are simply the answers to (3) (b), (c) and (d).


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**Exercise WS15.4**

1. A firm's production function is $Q = K^{0.5}L^{0.5}$

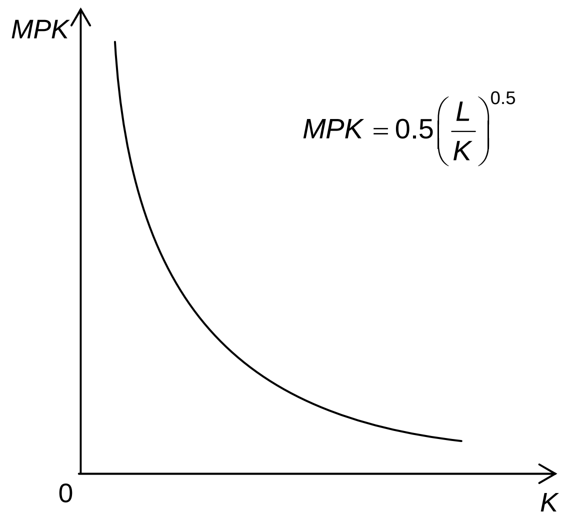
   (a) Find the marginal products of capital and labour. Are they always positive? Sketch their graphs.

Answer: $MPK \equiv \frac{\partial Q}{\partial K} = 0.5K^{-0.5}L^{0.5} = 0.5\left(\frac{L}{K}\right)^{0.5}$ ;

$MPL \equiv \frac{\partial Q}{\partial L} = 0.5K^{0.5}L^{-0.5} = 0.5\left(\frac{K}{L}\right)^{0.5}$

The above says that *MPK* is given by the square root of the capital to labour ratio. We can safely assume that *K* and *L* are themselves positive. By convention we always take the positive root (see section 4.9 of the book), so *MPK*, and similarly *MPL*, are both always positive.

To sketch the graph of *MPK*, we see that, with *L* constant, $\frac{L}{K}$ approaches zero as *K* approaches infinity, so the graph is asymptotic to the *K* (horizontal) axis. At the other extreme, as *K* approaches zero, $\frac{L}{K}$ approaches infinity. So the graph is asymptotic to the *MPK* (vertical) axis. See sketch below. The graph of *MPL* is identical but with *K* and *L* interchanged.





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(b) Find the equations of the isoquants for (i) 10 units, and (ii) 15 units of output. In each case, express the isoquant both as an implicit function and also with $K$ as an explicit function of $L$.

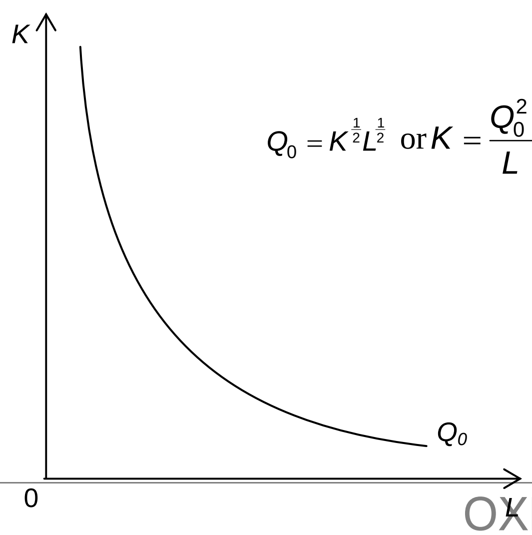
Answer: (i) The isoquant for $Q$ = 10 is $10 = K^{0.5}L^{0.5}$. Raising both sides to the power 2, we get $100 = KL$, from which $K = \frac{100}{L}$. (ii) $K = \frac{225}{L}$ in the same way.

(c) By implicit differentiation, find the slope of any isoquant and show that this slope is given by the ratio of the marginal products of capital and labour. Is the slope always negative?

Answer: Given the production function $Q = K^{0.5}L^{0.5}$, the differential is $dQ = \frac{\partial Q}{\partial K}dK + \frac{\partial Q}{\partial L}dL$. Along any isoquant, we have $dQ$ = 0 by definition, therefore $\frac{\partial Q}{\partial K}dK + \frac{\partial Q}{\partial L}dL = 0$. This re-arranges as $\frac{dK}{dL} = -\frac{\frac{\partial Q}{\partial K}}{\frac{\partial Q}{\partial L}}$ (see rules 15.9 and 15.11), giving us the slope of any isoquant at any point. But by definition $\frac{\partial Q}{\partial K}$ and $\frac{\partial Q}{\partial L}$ are the marginal products of $K$ and $L$. So we have $\frac{dK}{dL} = -\frac{MPK}{MPL}$. The slope is always negative for this production function because $MPK$ and $MPL$ were shown in (a) above to be always positive.

(d) Use the information from (b) and (c) above to sketch the isoquants.

Answer: From (b) the equation of any isoquant can be written as $K = \frac{Q_0^2}{L}$, where $Q_0$ is the fixed level of output. With $K$ on the vertical and $L$ on the horizontal axis, we see that $K$ approaches zero as $L$ approaches infinity, therefore the isoqant is asymptotic to the $L$ axis. At the other extreme, as $L$ approaches zero, $K$ approaches infinity, therefore the isoqant is also asymptotic to the $K$ axis. From (c) the slope is always negative. So the isoquant must have the shape of the sketch below.





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(e) Suppose a firm is producing on the $Q$ = 10 isoquant using $L_0$ units of labour and $K_0$ units of capital. Then it decides to employ one more unit of labour while keeping output constant. Use the differential, dQ, to calculate the required reduction in the capital input. Compare this with the reduction calculated from the production function itself, if $K_0 = L_0 = 10$. Explain with the aid of a diagram why the two answers differ.

Answer: For any production function $Q = f(K,L)$, the differential is $dQ = \frac{\partial Q}{\partial K}dK + \frac{\partial Q}{\partial L}dL$. First the firm employs 1 more unit of labour, with unchanged capital input. Thus we have $dL$ = 1 and $dK$ = 0, so the resulting change in output is $dQ = 0 + \frac{\partial Q}{\partial L}dL = \frac{\partial Q}{\partial L}$. Second, the firm changes the capital input by $dK$, with unchanged labour input, so the resulting change in output is $dQ = \frac{\partial Q}{\partial K}dK + 0 = \frac{\partial Q}{\partial K}dK$. To keep output constant these two output changes must be equal in absolute magnitude but opposite in sign, so we must have $\frac{\partial Q}{\partial L} = -\frac{\partial Q}{\partial K}dK$. This rearranges as $dK = -\frac{\frac{\partial Q}{\partial L}}{\frac{\partial Q}{\partial K}}$, where $dK$ is the change in capital input required to keep output constant when the labour input increases by 1 unit. In this example we have from (a) $-\frac{\frac{\partial Q}{\partial L}}{\frac{\partial Q}{\partial K}} = -\frac{0.5\left(\frac{K}{L}\right)^{0.5}}{0.5\left(\frac{L}{K}\right)^{0.5}} = -\frac{K}{L}$. So at the point on the isoquant where $K = L = 10$, we have $dK = -\frac{K}{L} = -1$. That is, a reduction in capital input of 1 unit is required to off-set the 1 unit increase in labour input.

The above calculation is an approximation, because the differential always gives us an approximation to the true change. In the example, in the initial position we have $K = L = 10$, so $Q_0 = K^{0.5}L^{0.5} = 10^{0.5}10^{0.5} = 10$. In the new position, the capital input is $K_1$ and the labour input is 11, so the new output is $Q_1 = K_1^{0.5}11^{0.5} = K_1^{0.5}(3.3166)$. But we want output to be unchanged; that is, $Q_1 = Q_o = 10$. Combining these last two equations, we get $10 = K_1^{0.5}11^{0.5}$ By rearrangement this becomes $K_1 = \frac{10^2}{11} = 9.0909$. Thus the required change in capital input is 9.9090 – 10 = -0.9090. Comparing our two answers we see that the differential formula in this case over-states the required reduction in capital input by 1 – 0.9090 = 0.0910, which is 10.011% of the true change required.


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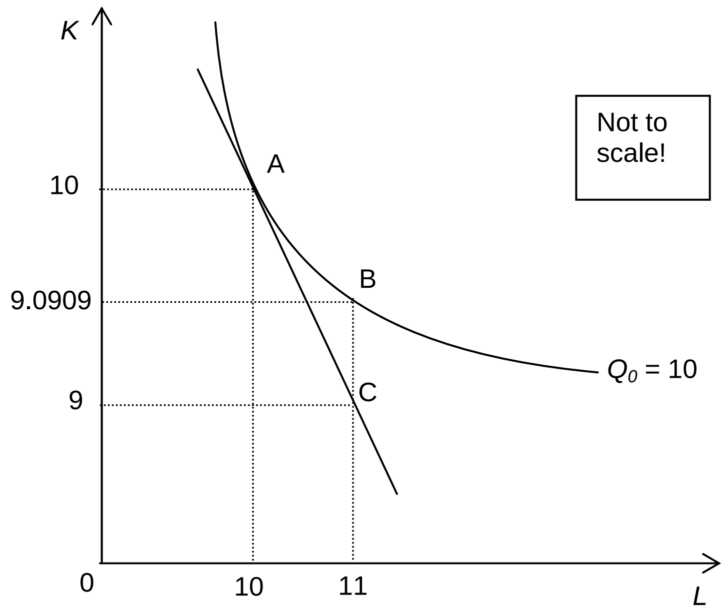


The true shift is from A to B along the isoquant. The differential formula effectively treats the isoquant as if it were linear, with a slope given by the tangent at A. Thus the differential formula calculates the shift from A to C, giving the new required capital input as 9, when the true value is 9.090. For small changes in *L* this approximation as acceptable. Its attraction is that it is easy to compute, both algebraically and arithmetically.

2. A firm's production function is $Q = 15KL - 4K^2 - 5L^2 + 6K + 4L$

(a) Find the marginal products of capital and labour. Are they always positive? Sketch their graphs.

Answer: $MPK \equiv \dfrac{\partial Q}{\partial K} = 15L - 8K + 6$ ; $MPL \equiv \dfrac{\partial Q}{\partial L} = 15K - 10L + 4$

*MPK* is positive when $15L - 8K + 6 > 0$ ; true when $K < \dfrac{15}{8}L + \dfrac{6}{8}$.

*MPL* is positive when $15K - 10L + 4 > 0$ ; true when $K > \dfrac{10}{15}L - \dfrac{4}{15}$.

(b) Use the differential of the production function to find the slope of any isoquant and show that this slope is given by the ratio of the marginal products of capital and labour. Is the slope always negative?

Answer: following question 1(c) above, and using our answer to (a) above, the slope of an isoquant is given by $\dfrac{dK}{dL} = -\dfrac{MPL}{MPK} = -\dfrac{15K - 10L + 4}{15L - 8K + 6}$. The isoquants are not


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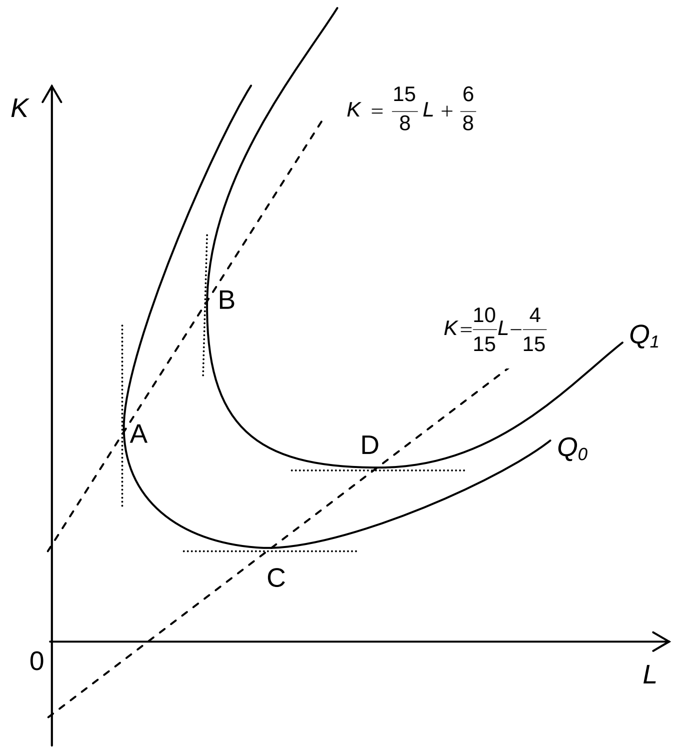
always negatively sloped. They are positively sloped if either *MPK* or *MPL* is positive; that is, from (a) above, respectively when $K > \frac{15}{8}L + \frac{6}{8}$ or $K < \frac{10}{15}L - \frac{4}{15}$.

(c) Use the information from (a) and (b) above to sketch some typical isoquants. Is their shape plausible?

Answer:




In the sketch, the broken line with equation $K = \frac{15}{8}L + \frac{6}{8}$ is the locus of points where *MPK* equals zero. Since the slope of any isoquant is $\frac{dK}{dL} = -\frac{MPL}{MPK}$, it follows that the slope of any isoquant is vertical when *MPK* = 0, as at points A and B. At these points, the isoquants are turning back and becoming positively sloped. Above the broken line that includes points A and B, *K* is so high relative to *L* that *MPK* is negative.

Similarly, the broken line with equation $K = \frac{10}{15}L - \frac{4}{15}$. is the locus of points where *MPL* equals zero. Since the slope of any isoquant is $\frac{dK}{dL} = -\frac{MPL}{MPK}$, it follows that the slope of any isoquant is zero when *MPL* = 0, as at points C and D. Again at these points, the isoquants are turning back and becoming positively sloped. Below the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broken line that includes points C and D, $L$ is so high relative to $K$ that $MPL$ is negative.

For the plausibility of this shape of isoquant, see answer to Ex WS14 4, question 1(f).

(d) Write down the equation of a typical short run production function; say, for $K = 10$. What determines its slope? What is true at its maximum value? Sketch its graph.

Answer: With $K$ = 10, the short run production function is $Q = 150L - 400 - 5L^2 + 60 + 4L = 154L - 5L^2 - 340$. To see what determines its slope, it is better to go back to the general form, $Q = 15KL - 4K^2 - 5L^2 + 6K + 4L$. With $K$ fixed (as in the short run) we have $MPL \equiv \frac{\partial Q}{\partial L} = 15K - 10L + 4$, and this measures the slope of the short run production function. This depends positively on $K$ and negatively on $L$.

The maximum of the short run production function occurs where $MPL \equiv \frac{\partial Q}{\partial L} = 15K - 10L + 4 = 0$. That is, where the marginal product of labour has fallen to zero. For sketch graph, see answer to Ex WS14 4, question 1(f).


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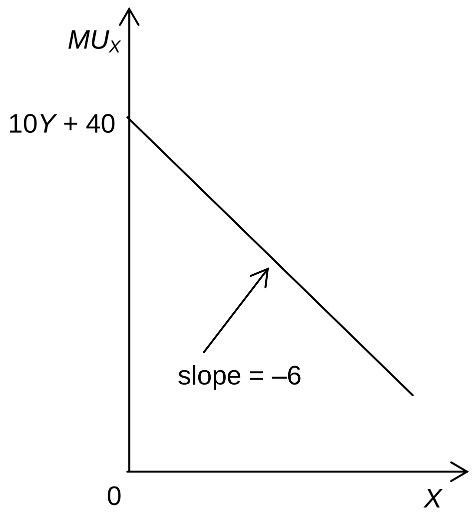
3. Consider the utility function $U = 10XY - 3X^2 - 2Y^2 + 40X + 50Y$

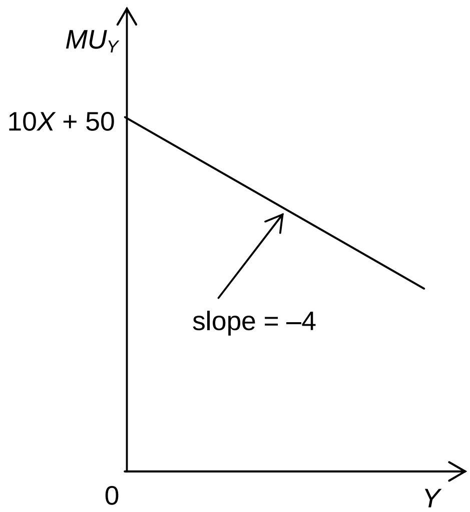
(a) Find the marginal utilities of the goods *X* and *Y*. Are they always positive? Sketch their graphs.

Answer: (Note mathematical similarity of this question to the previous one.)

$$MU_X \equiv \frac{\partial U}{\partial X} = 10Y - 6X + 40 \quad ; \quad MU_Y \equiv \frac{\partial U}{\partial Y} = 10X - 4Y + 50$$

$MU_X$ is positive when $MU_X = 10Y - 6X + 40 > 0$; true when $Y > \frac{6}{10}X - \frac{40}{10}$. Similarly $MU_Y$ is positive when $MU_Y = 10X - 4Y + 50 > 0$ ; true when $Y < \frac{10}{4}X + \frac{50}{4}$




Note similarity of above to Ex WS14.4 question 1(a)

(b) By implicit differentiation, find the slope of any indifference curve and show that it is given by the ratio of marginal utilities of the two goods. Are the indifference curves always negatively sloped? Sketch their graphs.

Answer: following question 1(c) above, and using our answer to (a) above, the slope of an indifference curve is given by $\frac{dY}{dX} = -\frac{MU_X}{MU_Y} = -\frac{10Y - 6X + 40}{10X - 4Y + 50}$. The indifference curves are not always negatively sloped. They are positively sloped if either *MPK* or *MPL* is negative; that is, from (a) above, respectively when $K > \frac{15}{8}L + \frac{6}{8}$ or $K < \frac{10}{15}L - \frac{4}{15}$. The indifference curves have the same general shape as the isoquants in the sketch answer to 2(c) above.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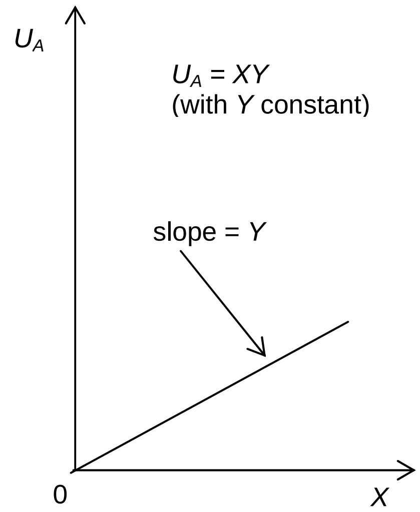
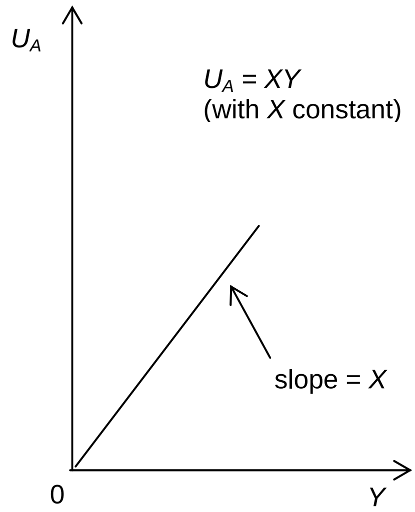
4. Suppose there are two individuals, Ann and Bernard. Ann's utility function is $U_{\text{A}} = XY$, while Bernard's is $U_{\text{B}} = \ln(XY)$.

(a) Find the marginal utilities of the two goods for Ann and Bernard, and sketch them. What does this tell us about their tastes?

Answer: for Ann, the marginal utilities are $\frac{\partial U_{\text{A}}}{\partial X} = Y$ and $\frac{\partial U_{\text{A}}}{\partial Y} = X$. For Bernard, his utility function can be written as $U_{\text{B}} = \ln X + \ln Y$ (using rule 11.2a), so we have $\frac{\partial U_{\text{B}}}{\partial X} = \frac{1}{X}$ and $\frac{\partial U_{\text{B}}}{\partial Y} = \frac{1}{Y}$ (using rule 13.3).

Before sketching the marginal utility curves it may be useful to look at the partial utility functions. For example, Ann's partial utility function for *X* shows how her total utility varies as *X* varies, with *Y* constant. Similarly her partial utility function for *Y* shows how her total utility varies as *Y* varies, with *X* constant. These partial utility functions are sketched below.




From these graphs we can see clearly that the slopes give us the marginal utilities.

When we repeat this for Bernard, we get the graphs below. These graphs have the general shape of the logarithmic function $y = \ln x$ (see figs. 11.3 and 12.9 of the book). At point J in the left hand graph we have $X = 1$, and therefore $U_{\text{B}} = \ln X + \ln Y$ $= 0 + \ln Y$. Thus the position of the left hand graph depends on the fixed value assigned to ln *Y*. Similarly at point K in the right hand graph we have $Y = 1$, and therefore $U_{\text{B}} = \ln X + \ln Y$ $= \ln X + 0$, so again the position of the curve depends on the fixed value of ln *X*. From these graphs we can see that the marginal utilities are given by the slopes of the two curves.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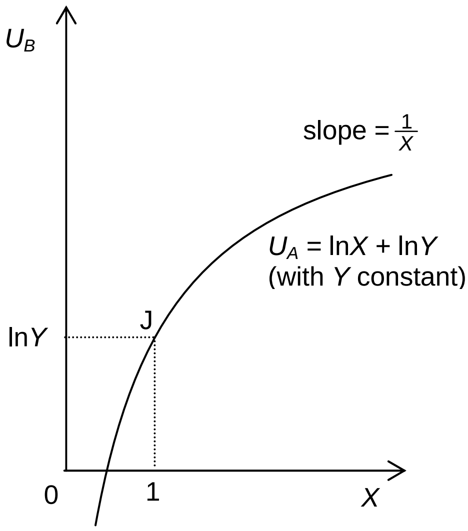


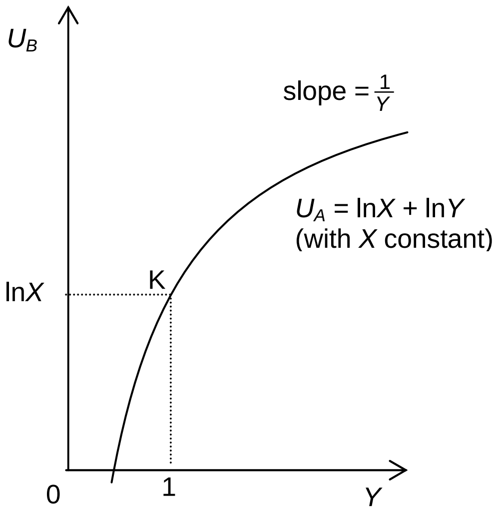


We see that Bernard's utility function is, apparently, very different from Ann's. For Ann, the fixed quantity of $Y$ determines the *slope* of the partial utility function for $X$. For Bernard, the fixed quantity of $Y$ determines instead the *intercept* (ln$Y$) of the partial utility function for $X$. And similarly for $Y$.
For Ann, the slope of the partial utility function for $X$ is constant, while for Bernard the slope decreases as $X$ increases. And similarly for $Y$.

(b) Show by implicit differentiation that, for any given combination of $X$ and $Y$, Ann's indifference curve has the same slope as Bernard's. What does this imply about their tastes?

Answer: By implicit differentiation, the slope of one of Ann's indifference curves is given by $\frac{dY}{dX} = -\frac{MU^A{}_X}{MU^A{}_Y} = -\frac{Y}{X}$.
Similarly the slope of one of Bernard's indifference curves is given by
$$\frac{dY}{dX} = -\frac{MU^B{}_X}{MU^B{}_Y} = -\frac{\frac{1}{X}}{\frac{1}{Y}} = -\frac{1}{X}\frac{Y}{1} = -\frac{Y}{X}.$$

Thus despite the apparent difference in their utility functions, Ann and Bernard's indifference curves have the same slope at any given point (that is, when Ann and Bernard are consuming the same quantities of $X$ and $Y$), and are therefore identical. This is because it is the *relative* marginal utilities which determine the slope of an indifference curve. Absolute marginal utilities are irrelevant and, indeed, meaningless since we have no way of measuring utility.


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