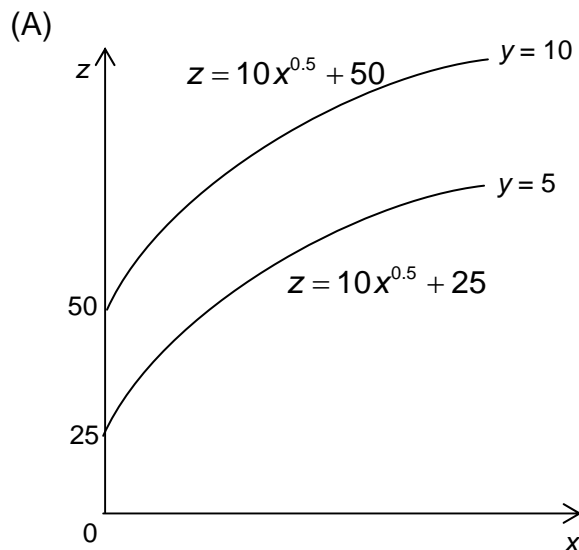


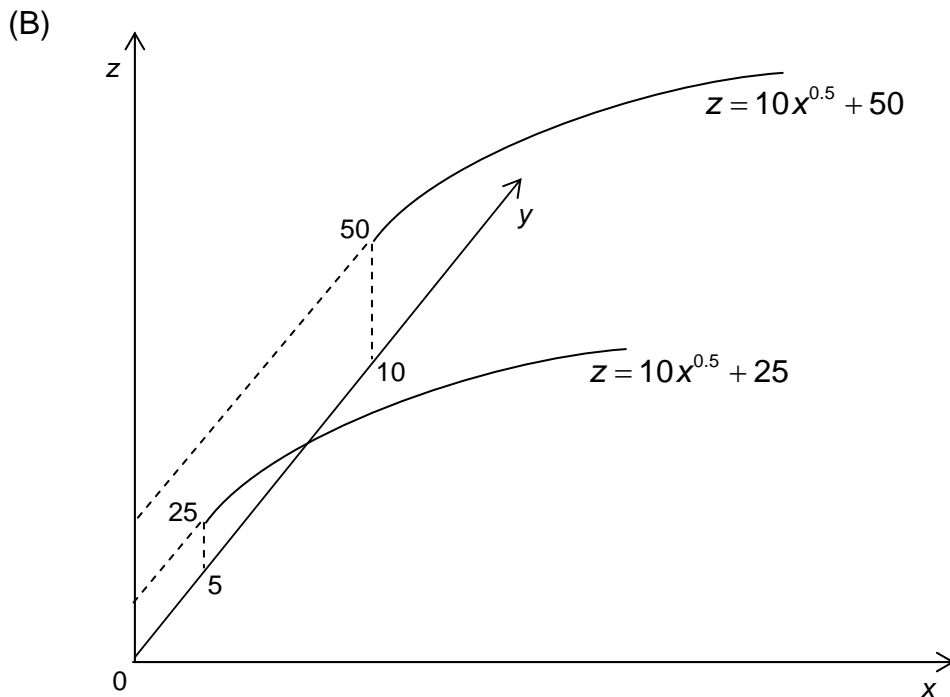
Exercise WS14.1

1. Consider the surface $z = 10x^{0.5} + 5y$, with x and y assumed not negative.
- (a) Assign any fixed positive value to y , and thence sketch the graph of the resulting relationship between x and z . Repeat this for further fixed values of y until you have built up a picture of what the iso- y sections of this surface look like. (Hint: In chapter 4 we considered the graph of $y = x^{0.5}$.)

Answer: if we let $y = 5$, we have $z = 10x^{0.5} + 25$. This gives the curve labelled $y = 5$ in sketch A below (see fig. 4.9 in the book). Similarly if we let $y = 10$, we have $z = 10x^{0.5} + 50$, giving the curve labelled $y = 10$ in sketch A. These are two curves of a family of curves, each showing how z varies with x for a fixed value of y . In this example, the fixed value of y determines the intercept on the z axis of each of these curves. Thus the function $z = 10x^{0.5} + 25$ gives a curved surface in the x direction, and the surface rises as y increases.



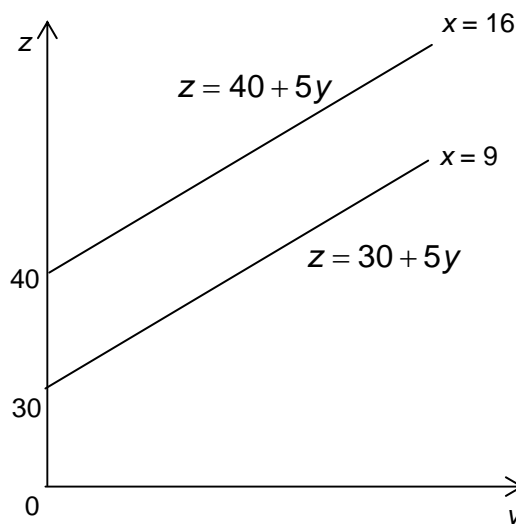
Sketch B on the next page gives a 3 dimensional perspective drawing of sketch A.



(b) Repeat (a) for various fixed values of x and thus build up a picture of what the iso- x sections look like. (Hint: this is easier than (a).)

Answer: if we let $x = 9$, we have $x^{0.5} = 3$ and therefore $z = 30 + 5y$. We see that z and y is here a linear function of y , giving the curve labelled $x = 9$ in sketch C below. Similarly if we let $x = 16$, we have $x^{0.5} = 4$ and therefore $z = 40 + 5y$, giving the linear function labelled $x = 16$ in sketch C. These are two curves of a family of curves, each showing how z varies with y for a fixed value of x . In this example, the fixed value of x determines the intercept on the z axis of each of these curves. Thus the function $z = 10x^{0.5} + 25$ gives a linear surface in the y direction, and the surface rises as x increases.

(C)



- (c) Find the intercepts on the three axes and, using this information together with what you have discovered in (a) and (b), make a three dimensional sketch graph of the surface.

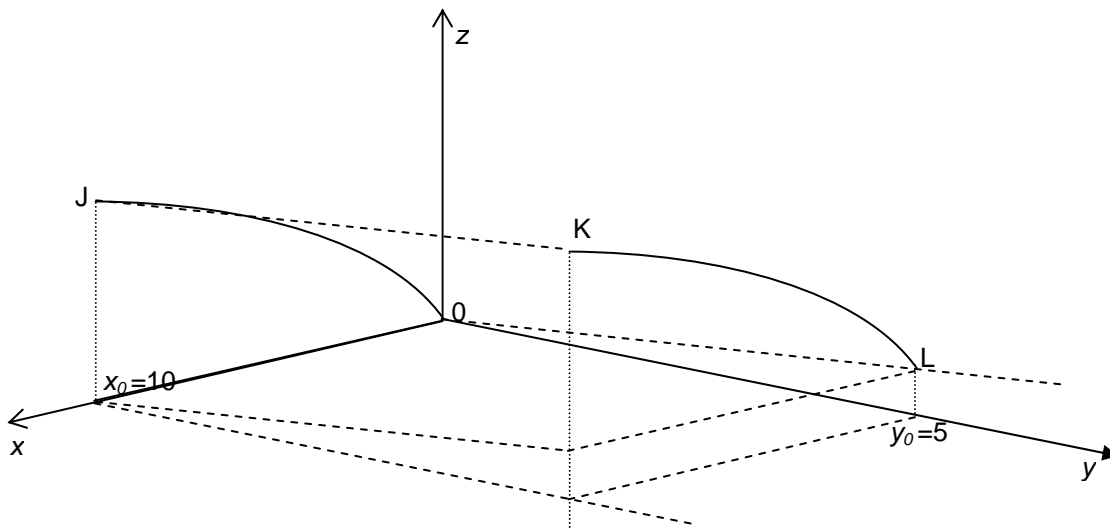
Answer: Sketch D below combines the information of sketches A and C above. The curve OJ gives the relationship between z and x with y held constant at $y = 0$. The equation of this line is $z = 10x^{0.5}$.

Similarly the curve LK gives the relationship between z and x with y held constant at $y = y_0 = 5$. The equation of this line is $z = 10x^{0.5} + 25$. Note that the curve LK is 25 units higher than OJ, because z is now 25 units ($=5y_0$) larger for any given x .

The broken line OL gives the relationship between z and y with x held constant at $x = 0$. The equation of this line is $z = 5y$. Similarly the broken line JK gives the relationship between z and y with x held constant at $x = 10$. The equation of this line is $z = 31.62 + 5y$ (because $10x^{0.5} = 31.62$ when $x = 10$). The broken line JK is this 31.62 units higher than OL, for any given value of y .

Intercepts: the z intercept is found where $x = y = 0$. Substituting $x = y = 0$ into the function $z = 10x^{0.5} + 5y$, we obtain $z = 0$. Similarly the y intercept is found where $x = z = 0$. Substituting $x = z = 0$ into the function, we obtain $y = 0$. And in the same way, we obtain $x = 0$. So the surface passes through the origin; that is, the point $x = y = z = 0$.

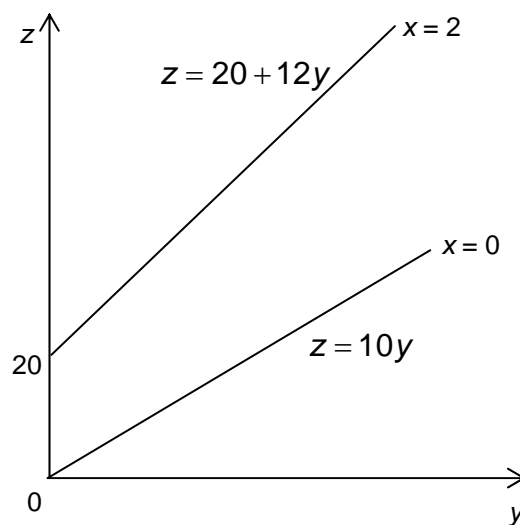
(D)



2. Repeat question 1 for the surface $z = xy + 10x + 10y$, with x and y assumed not negative. (Hints: In this case it might help to start by considering the iso- x section for $x = 0$, and similarly for $y = 0$. Then consider how z behaves when $x = y$.)

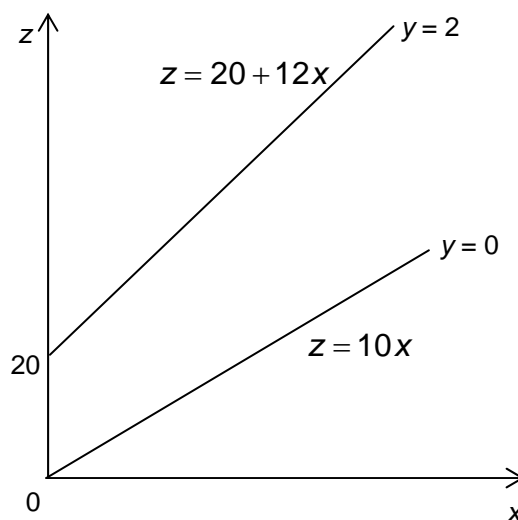
Answer: Setting $x = 0$, we get $z = 10y$, which is a straight line from the origin with a slope of 10. If we set $x = 2$, we get $z = 12y + 20$. Thus increasing the fixed value of x increases both the slope and the intercept of the iso- x contours; see sketch A below.

(A)



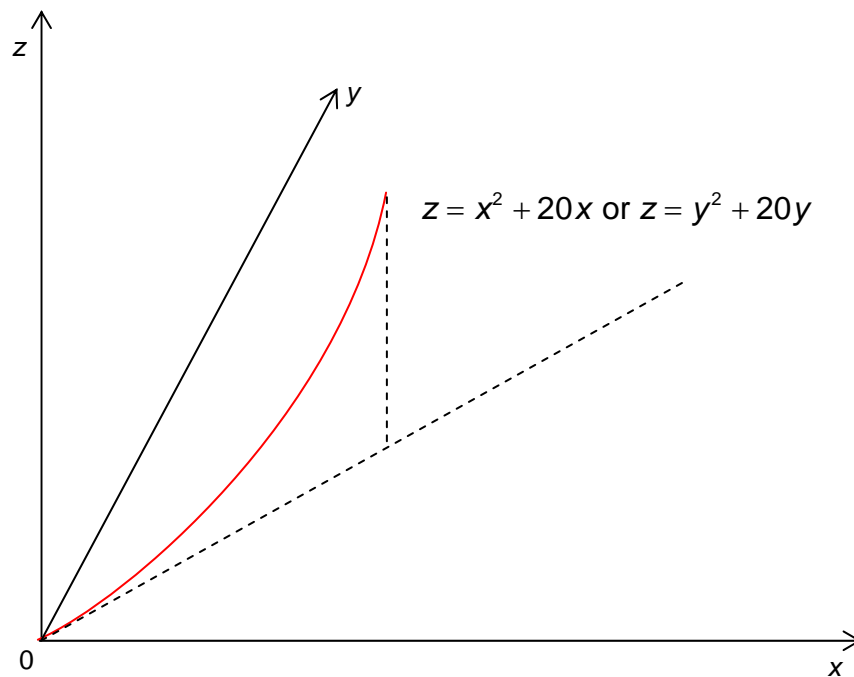
Because the function is symmetric in x and y , repeating the above with x variable and y held constant gives contour lines of exactly the same shape; see sketch B below.

(B)



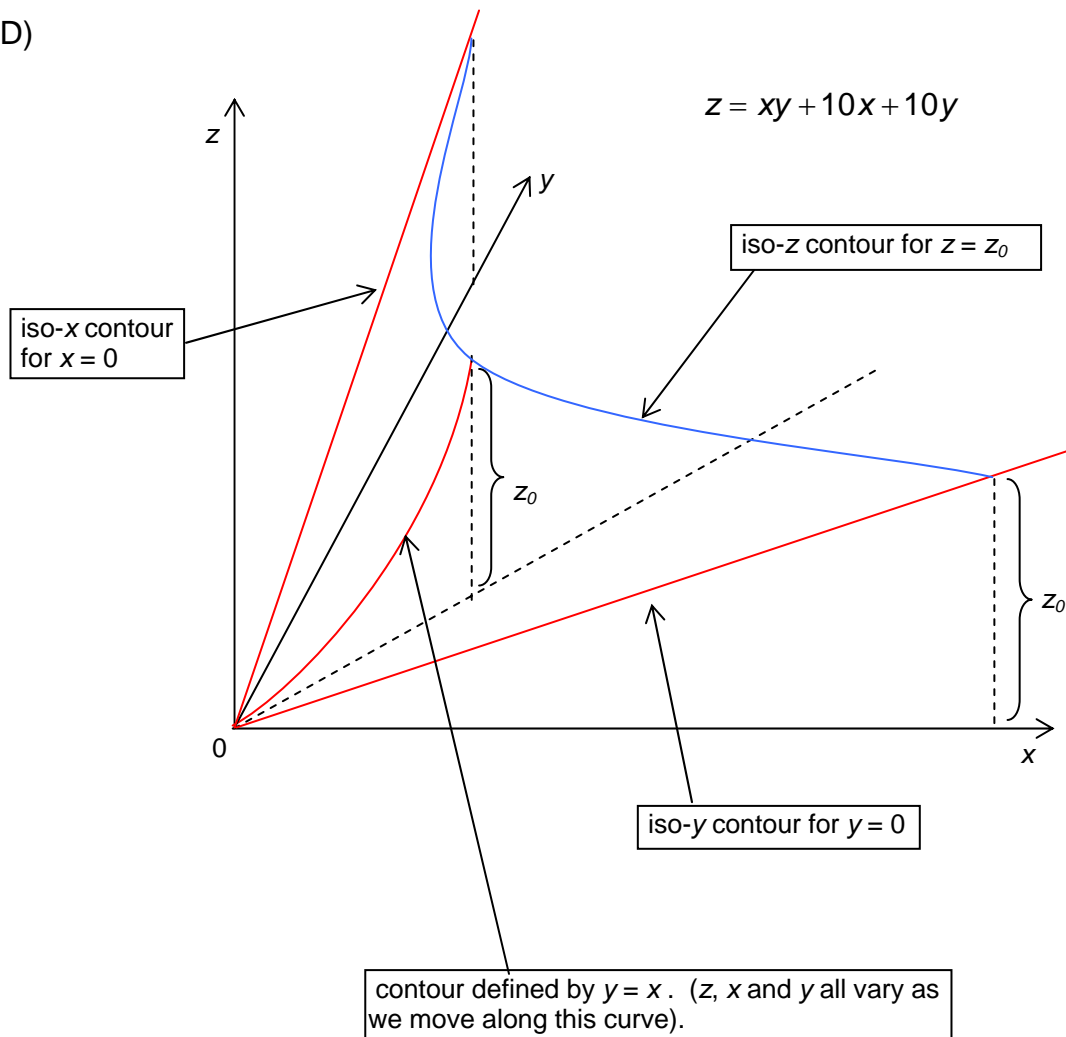
If we now let $x = y$ as suggested in the question, we find that the function can be written as $z = x^2 + 20x$ (or alternatively as $z = y^2 + 20y$). A three dimensional perspective drawing of this quadratic function is shown in sketch C. Note that there is a whole family of such curves whenever we fix the relative values of x and y . For example when y is fixed, relative to x , as $y = 3x$, we have $z = 3x^2 + 40x$.

(C)



Finally, we combine the information from the sketches above to produce a 3 dimensional perspective sketch of the surface (not easy).

(D)



Exercise WS14.2

For each of the following functions, find $\frac{\partial z}{\partial x}$ and $\frac{\partial z}{\partial y}$.

(Hint: All the rules of differentiation from chapters 6 and 13 for a function of one independent variable apply straightforwardly, with the other independent variable(s) treated as constants.)

1. (a) $z = 3x^2 + 2y^3 + 5$

Answer: To find $\frac{\partial z}{\partial x}$ we treat y as a constant, so $2y^3$ is an additive constant and will therefore not appear in the derivative; nor will the 5, by the same reasoning (see rule D3 in the book). So by the power rule of differentiation (rule D1 in the book) we have $\frac{\partial z}{\partial x} = 6x$. Similarly to find $\frac{\partial z}{\partial y}$ we treat x as a constant, so $3x^2$ is an additive constant and will therefore not appear in the derivative; nor will the 5. So by the power rule of differentiation we have $\frac{\partial z}{\partial y} = 6y$.

(b) $z = x^{\frac{1}{2}} + \frac{x^2}{y^3}$

Answer: To find $\frac{\partial z}{\partial x}$ it helps if we write this as $z = x^{\frac{1}{2}} + \left(\frac{1}{y^3}\right)x^2$. Then we see that $\frac{1}{y^3}$ is a multiplicative constant on x^2 and will therefore appear when we are differentiating with respect to x (Rule D3 in the book). So using the power rule of differentiation we get

$$\frac{\partial z}{\partial x} = \frac{1}{2}x^{-\frac{1}{2}} + \frac{1}{y^3}(2x) = \frac{1}{2}x^{-\frac{1}{2}} + \frac{2x}{y^3}$$

Similarly to find $\frac{\partial z}{\partial y}$ it helps if we write the given function as $z = x^{\frac{1}{2}} + x^2y^{-3}$.

Then we see that x^2 is a multiplicative constant on y^{-3} and will therefore appear when we are differentiating with respect to y . On the other hand, $x^{\frac{1}{2}}$ is an additive constant which therefore disappears. So using the power rule of differentiation we get

$$\frac{\partial z}{\partial y} = x^2(-3y^{-4}) = -\frac{3x^2}{y^4}$$

$$(c) \quad z = \frac{2x^3 + 3y^2}{2x + 4y^2}$$

Answer: (i) The partial derivative $\frac{\partial z}{\partial x}$. To find this, we need to use the quotient rule (see rule D7 in the book) because the given function is a quotient. We write the given function as $z = \frac{u}{v}$ where $u = 2x^3 + 3y^2$ and $v = 2x + 4y^2$. Then the quotient rule, slightly modified for partial differentiation, says that

$$\frac{\partial z}{\partial x} = \frac{v \frac{\partial u}{\partial x} - u \frac{\partial v}{\partial x}}{v^2} \quad (\text{equation A})$$

Next we need the partial derivatives $\frac{\partial u}{\partial x}$ and $\frac{\partial v}{\partial x}$. First, we will find $\frac{\partial u}{\partial x}$.

Since $u = 2x^3 + 3y^2$, we see that $3y^2$ is an additive constant when we are differentiating with respect to x , and will therefore not appear in the derivative. So by the power rule, $\frac{\partial u}{\partial x} = 6x^2$. Next we find $\frac{\partial v}{\partial x}$. Since $v = 2x + 4y^2$, we see that $4y^2$ is an additive constant when we are differentiating with respect to x , and will therefore not appear in the derivative. So by the power rule, $\frac{\partial v}{\partial x} = 2$. Now we simply have to insert all these components into equation A. This gives:

$$\frac{\partial z}{\partial x} = \frac{v \frac{\partial u}{\partial x} - u \frac{\partial v}{\partial x}}{v^2} = \frac{(2x + 4y^2)6x^2 - (2x^3 + 3y^2)2}{(2x + 4y^2)^2}$$

Answer: (ii) The partial derivative $\frac{\partial z}{\partial y}$. Our derivation is the same as (i) above, except that we replace x with y and vice versa. As in (i), we use the quotient rule. As in (i) we write the given function as

$z = \frac{u}{v}$ where $u = 2x^3 + 3y^2$ and $v = 2x + 4y^2$. Then the quotient rule, slightly modified for partial differentiation, says that

$$\frac{\partial z}{\partial y} = \frac{v \frac{\partial u}{\partial y} - u \frac{\partial v}{\partial y}}{v^2} \quad (\text{equation B}) \quad (\text{Note that equation B is identical to equation A except that } x \text{ has been replaced by } y.)$$

Next we need the partial derivatives $\frac{\partial u}{\partial y}$ and $\frac{\partial v}{\partial y}$. First, we will find $\frac{\partial u}{\partial y}$.

Since $u = 2x^3 + 3y^2$, we see that $2x^3$ is an additive constant when we are differentiating with respect to y , and will therefore not appear in the derivative. So by the power rule, $\frac{\partial u}{\partial y} = 6y$. Next we find $\frac{\partial v}{\partial y}$. Since $v = 2x + 4y^2$, we see that $2x$ is an additive constant when we are differentiating with respect to y , and will therefore not appear in the derivative. So by the power rule, $\frac{\partial v}{\partial y} = 8y$. Now we simply have to insert all these components into equation B. This gives:

$$\frac{\partial z}{\partial y} = \frac{v \frac{\partial u}{\partial y} - u \frac{\partial v}{\partial y}}{v^2} = \frac{(2x + 4y^2)6y - (2x^3 + 3y^2)8y}{(2x + 4y^2)^2}$$

(d) $z = e^{x-y}$

Answer: here we have an exponential function, so we need rule 13.2 in the book, suitably modified to deal with partial derivatives. Rule 13.2 says:

if $y = e^{f(x)}$ where $f(x)$ is any function of x , then $\frac{dy}{dx} = e^{f(x)} \cdot f'(x)$, where $f'(x)$ denotes the derivative of $f(x)$.

The modified form says:

if $z = e^{f(x,y)}$ where $f(x, y)$ is any function of x and y , then

$$\frac{\partial z}{\partial x} = e^{f(x,y)} \cdot f_x \quad \text{and} \quad \frac{\partial z}{\partial y} = e^{f(x,y)} \cdot f_y$$

(where f_x and f_y denote the partial derivatives of $f(x, y)$).

In this example we have $e^{f(x,y)} = e^{x-y}$, so $f_x = 1$ (because we are differentiating with respect to x , so $-y$ is an additive constant which therefore disappears); and $f_y = -1$ (because now we are differentiating with respect to y so x is an additive constant which therefore disappears). So the two partial derivatives are

$$\frac{\partial z}{\partial x} = e^{f(x,y)} \cdot f_x = e^{x-y}(1) = e^{x-y}, \quad \text{and} \quad \frac{\partial z}{\partial y} = e^{f(x,y)} \cdot f_y = e^{x-y}(-1) = -e^{x-y}.$$

2. Find $\frac{\partial z}{\partial u}$, $\frac{\partial z}{\partial v}$ and $\frac{\partial z}{\partial w}$ for each of the following:

(a) $z = u^5 + v^2 + w^3 + uvw$

Answers: First, $\frac{\partial z}{\partial u}$. We must treat v and w as constants. This means that

v^2 and w^3 are additive constants which therefore will not appear in the derivative; while vw is a multiplicative constant on u , and therefore

reappears in the partial derivative. So we have $\frac{\partial z}{\partial u} = 5u^4 + vw$.

Next, $\frac{\partial z}{\partial v}$. By the same reasoning as above, we have $\frac{\partial z}{\partial v} = 2v + uw$.

Finally, by the same reasoning again, $\frac{\partial z}{\partial w} = 3w^2 + uv$.

(b) $z = u^{\frac{1}{2}}v^{\frac{1}{4}}w^{\frac{1}{3}}$

Answers: This is almost the same as (a) except that the relationship between the variables is entirely multiplicative. Beginning with $\frac{\partial z}{\partial u}$, we see that

$v^{\frac{1}{4}}$ and $w^{\frac{1}{3}}$ are multiplicative constants which therefore appear in the derivative.

Thus we have $\frac{\partial z}{\partial u} = \frac{1}{2}u^{-\frac{1}{2}}v^{\frac{1}{4}}w^{\frac{1}{3}}$.

Similarly $\frac{\partial z}{\partial v} = u^{\frac{1}{2}}\frac{1}{4}v^{-\frac{3}{4}}w^{\frac{1}{3}}$ and $\frac{\partial z}{\partial w} = u^{\frac{1}{2}}v^{\frac{1}{4}}\frac{1}{3}w^{-\frac{2}{3}}$.

Exercise WS14.3

1. (a) Given $z = 5x^4 + 3x^2y + y^2$, find the two first-order partial derivatives,

$\frac{\partial z}{\partial x}$ and $\frac{\partial z}{\partial y}$, and the four second-order partial derivatives,

$\frac{\partial^2 z}{\partial x^2}$, $\frac{\partial^2 z}{\partial y \partial x}$, $\frac{\partial^2 z}{\partial y^2}$, and $\frac{\partial^2 z}{\partial x \partial y}$.

Answers: $\frac{\partial z}{\partial x} = 20x^3 + 6xy$; $\frac{\partial^2 z}{\partial x^2} = 60x^2 + 6y$; $\frac{\partial^2 z}{\partial y \partial x} = 6x$;

$\frac{\partial z}{\partial y} = 3x^2 + 2y$; $\frac{\partial^2 z}{\partial y^2} = 2$; $\frac{\partial^2 z}{\partial x \partial y} = 6x$

- (b) Explain briefly what, in general, each of these partial derivatives measures.

Answers:

$\frac{\partial z}{\partial x}$ measures the slope of the surface in the x direction.

$\frac{\partial^2 z}{\partial x^2}$ measures how the slope of the surface in the x direction changes as x increases (with y constant).

$\frac{\partial^2 z}{\partial y \partial x}$ measures how the slope of the surface in the x direction changes as y increases (with x constant)

$\frac{\partial z}{\partial y}$ measures the slope of the surface in the y direction.

$\frac{\partial^2 z}{\partial y^2}$ measures how the slope of the surface in the y direction changes as y increases (with x constant).

$\frac{\partial^2 z}{\partial x \partial y}$ measures how the slope of the surface in the y direction changes as x increases (with y constant)

2. For each of the functions in exercise WS14.2, question 1, find

$$\frac{\partial^2 z}{\partial x^2}, \frac{\partial^2 z}{\partial y^2}, \frac{\partial^2 z}{\partial y \partial x}, \text{ and } \frac{\partial^2 z}{\partial x \partial y}.$$

Answers: (a) $z = 3x^2 + 2y^3 + 5$. From Ex WS14.2 question 1 we have $\frac{\partial z}{\partial x} = 6x$.

Therefore $\frac{\partial^2 z}{\partial x^2} = 6$; $\frac{\partial^2 z}{\partial y \partial x} = 0$ (this latter means the slope in the x direction is unaffected by changes in y)

From Ex WS14.2 question 1 we have $\frac{\partial z}{\partial y} = 6y$. Therefore

$\frac{\partial^2 z}{\partial y^2} = 6$; $\frac{\partial^2 z}{\partial x \partial y} = 0$ (this latter means the slope in the y direction is unaffected by changes in x)

(b) We have $\frac{\partial z}{\partial x} = \frac{1}{2}x^{-\frac{1}{2}} + \frac{2x}{y^3}$. Therefore $\frac{\partial^2 z}{\partial x^2} = -\frac{1}{4}x^{-\frac{3}{2}} + \frac{2}{y^3}$;

$$\frac{\partial^2 z}{\partial y \partial x} = 2x(-3y^{-4}) = -6xy^{-4}.$$

We have $\frac{\partial z}{\partial y} = -\frac{3x^2}{y^4}$. Therefore $\frac{\partial^2 z}{\partial y^2} = -3x^2(-4y^{-5}) = 12x^2y^{-5}$;

$$\frac{\partial z}{\partial x \partial y} = -\frac{1}{y^4}6x = -\frac{6x}{y^4}$$

(c) (Not easy!) We have from Ex WS14.2 question 1:

$$\frac{\partial z}{\partial x} = \frac{(2x+4y^2)6x^2 - (2x^3+3y^2)2}{(2x+4y^2)^2}.$$

Multiplying out and simplifying the

numerator, we get $\frac{\partial z}{\partial x} = \frac{8x^3 + 24x^2y^2 - 6y^2}{(2x+4y^2)^2}$. So, applying the quotient

rule to this we get

$$\frac{\partial^2 z}{\partial x^2} = \frac{\left[(2x+4y^2)^2 \right] (24x^2 + 48xy^2) - (8x^3 + 24x^2y^2 - 6y^2) 2(2x+4y^2)(2)}{(2x+4y^2)^4}$$

(Note the use of the function of a function rule to find the partial derivative of $(2x+4y^2)^2$) Also:

$$\frac{\partial^2 z}{\partial y \partial x} = \frac{\left[(2x+4y^2)^2 \right] (48x^2y) - (8x^3 + 24x^2y^2 - 6y^2) 2(2x+4y^2)(8y)}{(2x+4y^2)^4}$$

We also have from Ex WS14.2 question 1:

$$\frac{\partial z}{\partial y} = \frac{(2x+4y^2)6y - (2x^3+3y^2)8y}{(2x+4y^2)^2}.$$

Multiplying out the numerator and

simplifying, we get $\frac{\partial z}{\partial y} = \frac{12xy - 16x^3y}{(2x+4y^2)^2}$. So, applying the quotient rule to this we

get: $\frac{\partial^2 z}{\partial y^2} = \frac{(2x+4y^2)^2 (12x - 16x^3) - (12xy - 16x^3y) 2(2x+4y^2)(8y)}{(2x+4y^2)^4}$. Similarly

$$\frac{\partial^2 z}{\partial x \partial y} = \frac{(2x+4y^2)^2 (12y - 48x^2y) - (12xy - 16x^3y) 2(2x+4y^2)(2)}{(2x+4y^2)^4}$$

3. For each of the following functions, find $\frac{\partial z}{\partial x}$, $\frac{\partial z}{\partial y}$, and $\frac{\partial^2 z}{\partial y^2}$

(a) $z = (x^2 + 2y)^{0.5}$

Here we use the function of a function rule (see rule D5 in the book). Let us be clear how this is modified for partial differentiation. Given the function $z = f(u)$ where $u = g(x, y)$ where f and g both denote functions, the partial derivatives are

$$\frac{\partial z}{\partial x} = f'(u) \frac{\partial u}{\partial x} \quad \text{and} \quad \frac{\partial z}{\partial y} = f'(u) \frac{\partial u}{\partial y} \quad [\text{where } f'(u) \text{ is the derivative of } f(u)]$$

In this example we have $f(u) = (x^2 + 2y)^{0.5}$ where $u = x^2 + 2y$. Therefore

$$f'(u) = 0.5(x^2 + 2y)^{-0.5}. \quad \text{Also } \frac{\partial u}{\partial x} = 2x \quad \text{and} \quad \frac{\partial u}{\partial y} = 2. \quad \text{Putting these pieces}$$

together, we get

$$\frac{\partial z}{\partial x} = f'(u) \frac{\partial u}{\partial x} = 0.5(x^2 + 2y)^{-0.5} (2x); \quad \text{and}$$

$$\frac{\partial z}{\partial y} = f'(u) \frac{\partial u}{\partial y} = 0.5(x^2 + 2y)^{-0.5} (2) = (x^2 + 2y)^{-0.5}$$

To find $\frac{\partial^2 z}{\partial y^2}$ we have to differentiate $\frac{\partial z}{\partial y} = (x^2 + 2y)^{-0.5}$ with respect to y .

Applying the function of a function rule to this, we get

$$\frac{\partial^2 z}{\partial y^2} = -0.5(x^2 + 2y)^{-1.5} (2) = -(x^2 + 2y)^{-1.5}$$

(b) $z = y^2 e^{x^2}$

$$\frac{\partial z}{\partial x} = y^2 e^{x^2} (2x) = 2xy^2 e^{x^2} \quad (\text{Note that } y^2 \text{ is a multiplicative constant});$$

$$\frac{\partial z}{\partial y} = 2ye^{x^2} \quad (e^{x^2} \text{ is a multiplicative constant});$$

$$\frac{\partial^2 z}{\partial y^2} = 2e^{x^2} \quad (2e^{x^2} \text{ is a multiplicative constant}).$$

4. Suppose you are given the following information about a function $z = f(x, y)$.

$$\text{At a point P, } \frac{\partial z}{\partial x} > 0; \frac{\partial^2 z}{\partial x^2} > 0; \frac{\partial^2 z}{\partial y \partial x} < 0; \frac{\partial z}{\partial y} < 0; \frac{\partial^2 z}{\partial y^2} > 0$$

Use this information to sketch the shapes of the iso-x and iso-y sections at P and thus indicate the shape of the surface in the vicinity of P. (Hint: Figures 14.12 and 14.13 in the book are relevant.)

Answer: First, the information that $\frac{\partial z}{\partial x}$ is positive at P tells us that the surface is positively sloped in the x direction at P, while the information that $\frac{\partial^2 z}{\partial x^2}$ is positive tells us that the slope in the x direction is increasing at P. Therefore the iso-y contour must have the shape shown by JK in fig. 1 below. (Note that the information about the signs of these partial derivatives only tells us about the shape of the contour line at, or in the immediate vicinity of, P. However for the sake of clarity in drawing the diagram we have assumed that the curve has this shape not only at P but at all points between J and K, even though these points are some distance away from P.)

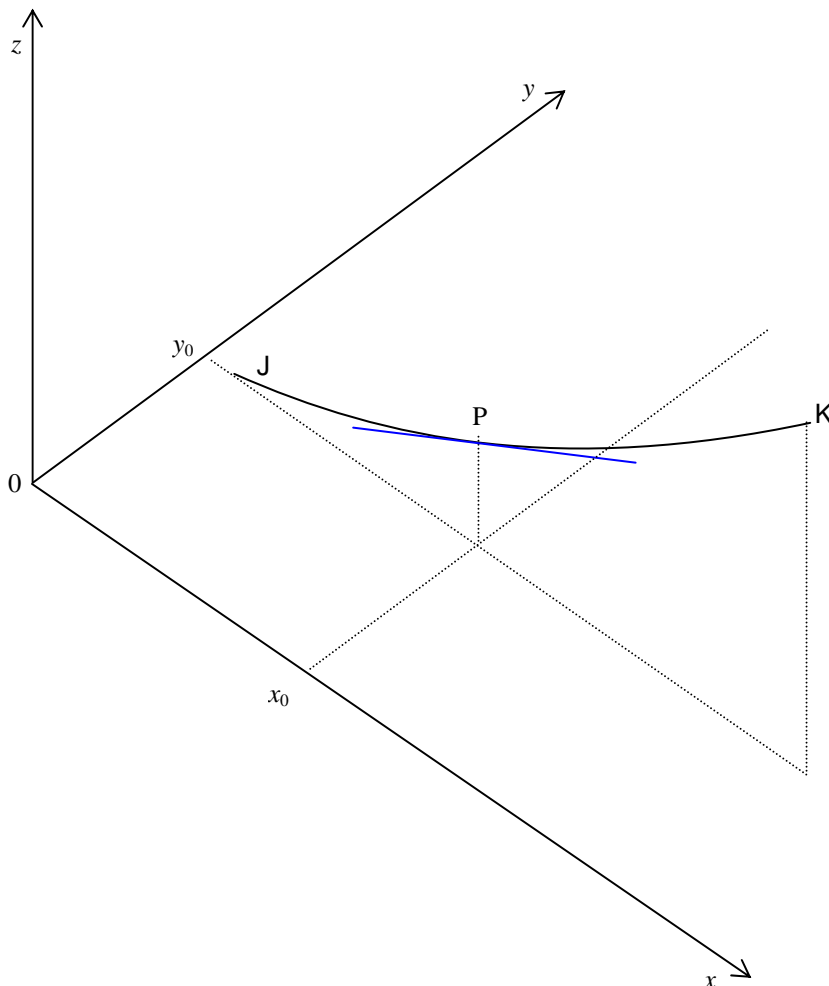
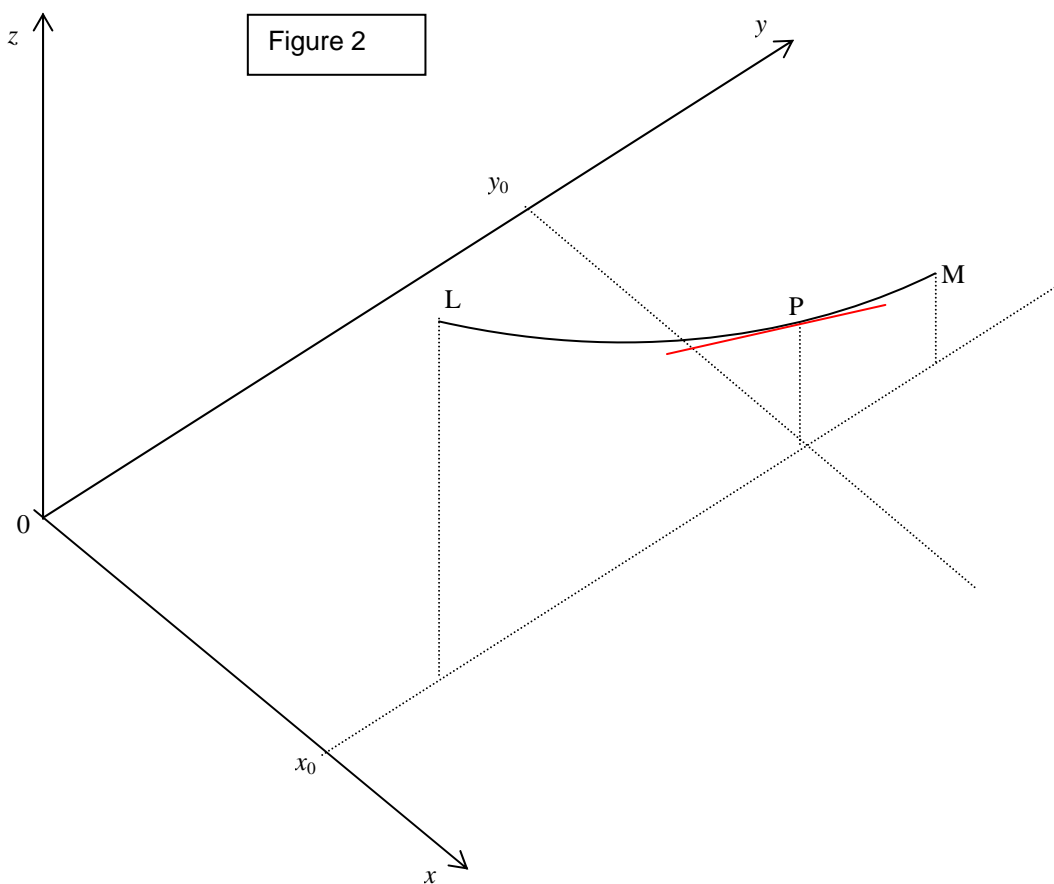


Figure 1

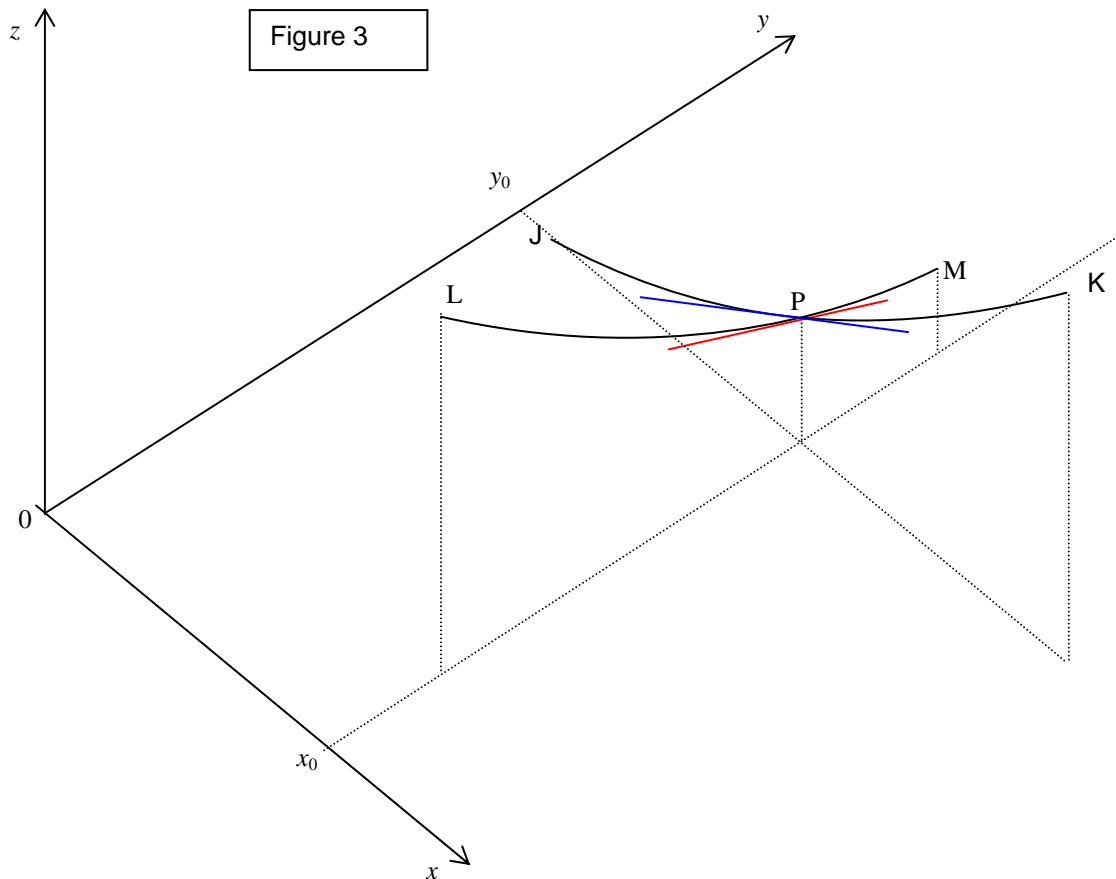
Second, the information that $\frac{\partial z}{\partial y}$ is negative at P tells us that the surface is

negatively sloped in the y direction at P, while the information that $\frac{\partial^2 z}{\partial y^2}$ is positive

tells us that the slope in the y direction is increasing at P; that is, becoming increasingly positive or decreasingly negative. Therefore the iso-x contour must have the shape shown by LM in fig. 2 below. (Note that again for the sake of clarity in drawing the diagram we have assumed that the curve has this shape not only at P but at all points between L and M, even though these points are some distance away from P.)



Third, we can combine figs. 1 and 2 to produce fig. 3. There we see that, starting from P, we move downhill (lower values of z) if either y is increased (movement towards M) or x decreased (movement towards J). Conversely, we move uphill (higher values of z) if either y is decreased (movement towards L) or x increased (movement towards K).



Finally, the information that $\frac{\partial^2 z}{\partial y \partial x}$ is positive tells us that the slope in the x direction increases when we move slightly in the y direction; see fig. 14.12 in the book.

5. Repeat question 4 for a function $z = h(x, y)$, where at a point R,

$$\frac{\partial z}{\partial x} < 0; \quad \frac{\partial^2 z}{\partial x^2} < 0; \quad \frac{\partial^2 z}{\partial y \partial x} < 0; \quad \frac{\partial z}{\partial y} > 0; \quad \frac{\partial^2 z}{\partial y^2} > 0$$

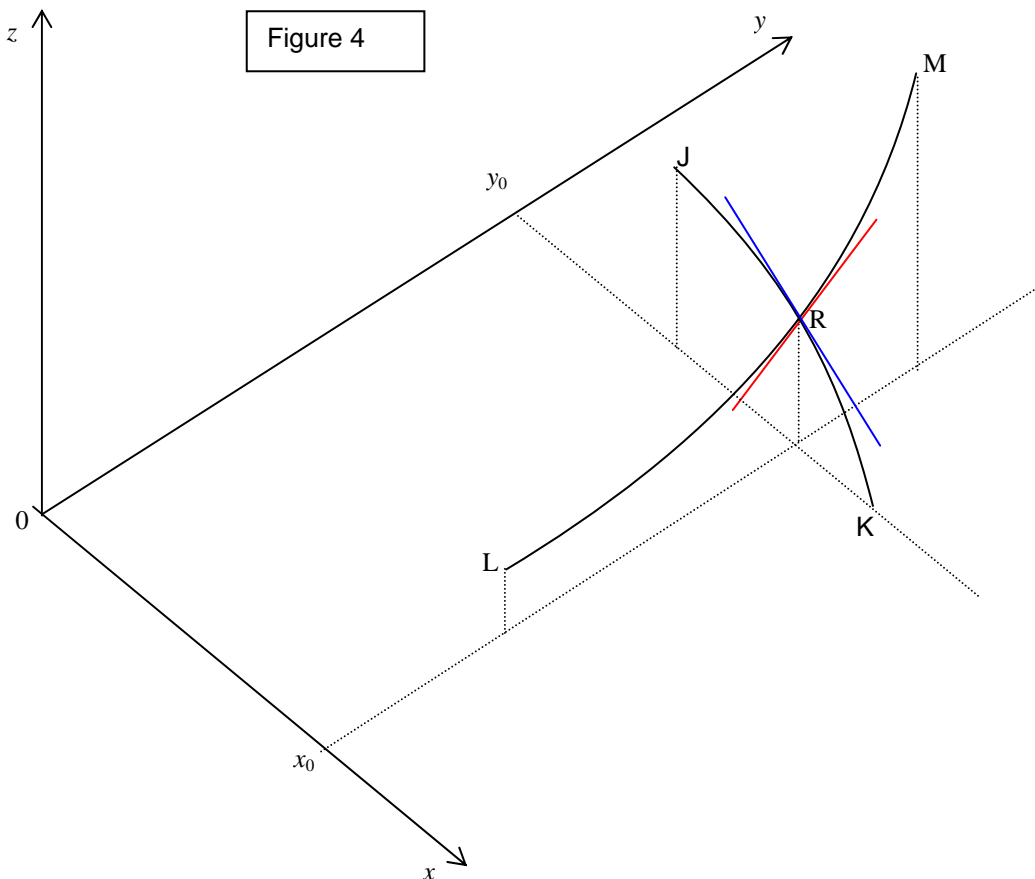
Answer: See fig. 4 below. The information that $\frac{\partial z}{\partial x} < 0$ tells us that the surface is

negatively sloped in the x direction at R; while the information that $\frac{\partial^2 z}{\partial x^2} < 0$ tells us that the slope is decreasing (that is, becoming increasingly negative). These two pieces of information give the contour line JK its shape.

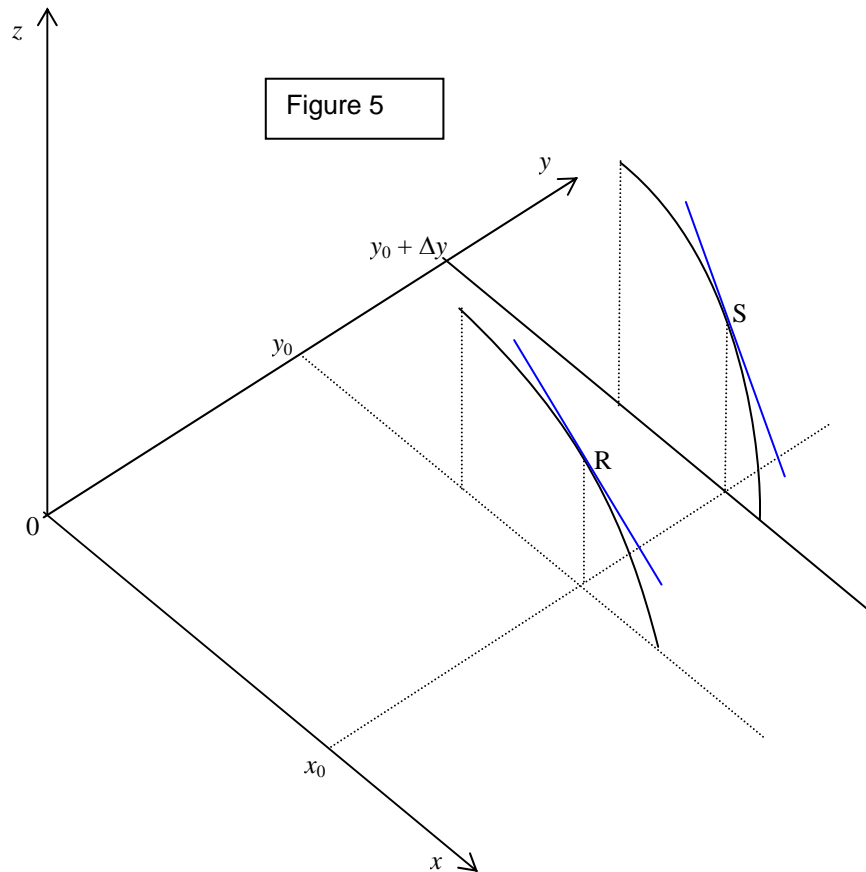
The information that $\frac{\partial z}{\partial y} > 0$ tells us that the surface is positively sloped in the y

direction at R; while the information that $\frac{\partial^2 z}{\partial y^2} > 0$ tells us that the slope is increasing

(that is, becoming increasingly positive). These two pieces of information give the contour line LM its shape.



Finally, the information that $\frac{\partial^2 z}{\partial y \partial x}$ is negative tells us that the slope in the x direction decreases when we move slightly in the y direction; that is, from R to S in fig. 5 below. In this example the slope in the x direction decreases in the sense of becoming more negative; that is, the tangent at S is more negatively sloped than the tangent at R. In another example it might decrease in the sense of becoming less positive.



Exercise WS14.4

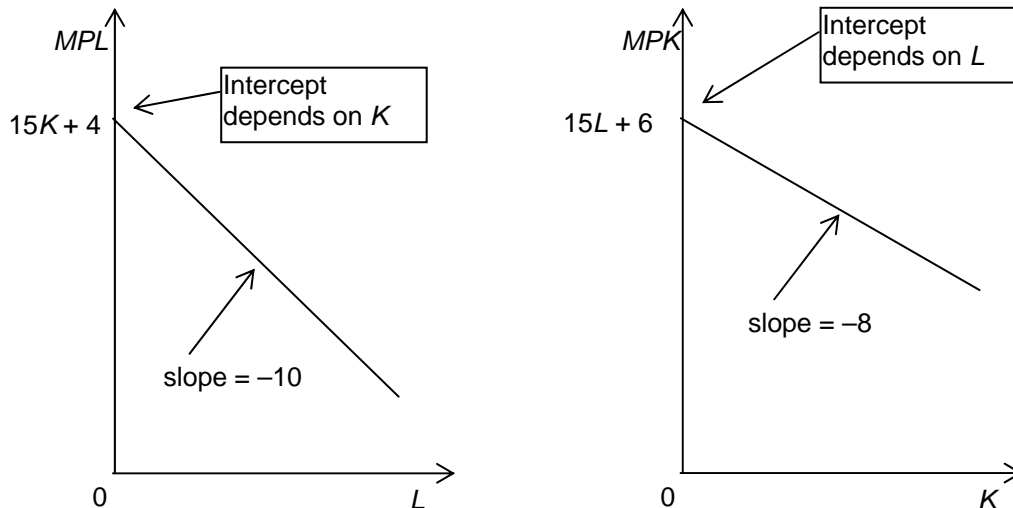
1. A firm finds that its production function is

$$Q = 15KL - 4K^2 - 5L^2 + 6K + 4L$$

where Q is weekly output (in thousands of units) and K and L are weekly inputs of machine-hours and worker-hours respectively (measured in thousands).

(a) By differentiation, find the marginal products of labour and capital, and sketch their graphs.

Answer: $MPL \equiv \frac{\partial Q}{\partial L} = 15K - 10L + 4$; $MPK \equiv \frac{\partial Q}{\partial K} = 15L - 8K + 6$



(b) Confirm that your graphs in (a) are correctly drawn by examining the signs of the direct second derivatives $\frac{\partial^2 Q}{\partial L^2}$ and $\frac{\partial^2 Q}{\partial K^2}$. Is this production function characterised by diminishing marginal productivity?

Answer: $\frac{\partial^2 Q}{\partial L^2} = -10$ and $\frac{\partial^2 Q}{\partial K^2} = -8$. This confirms that the slopes of the two sketch

graphs above are -10 and -8 respectively. As $\frac{\partial^2 Q}{\partial L^2}$ is negative, this tells us that MPL decreases as L increases with K constant; that is, there is diminishing marginal productivity of labour. Similarly, because $\frac{\partial^2 Q}{\partial K^2}$ is negative, this tells us that MPK decreases as K increases with L constant; that is, there is diminishing marginal productivity of capital.

- (c) By examining the signs of the cross partial derivatives $\frac{\partial^2 Q}{\partial L \partial K}$ and $\frac{\partial^2 Q}{\partial K \partial L}$ determine the effect on the MPL of an increase in the capital input and the effect on the MPK of an increase in the labour input. Give an economic interpretation to this finding. Would you expect it to be generally true of all or most production functions?

Answer: $\frac{\partial^2 Q}{\partial L \partial K} = \frac{\partial}{\partial L}$ of $\frac{\partial Q}{\partial K} = 15$ and $\frac{\partial^2 Q}{\partial K \partial L} = \frac{\partial}{\partial K}$ of $\frac{\partial Q}{\partial L} = 15$. (They are equal due

to Young's theorem.) Because $\frac{\partial^2 Q}{\partial K \partial L}$ is positive, at any given level of labour input *MPL* is increased by a small increase in the capital input. This means that a small increase in the capital input increases the *additional* output obtained from a small increase in the labour input. In the sketch above, the *MPL* function shifts upwards when *K* increases.

Similarly because $\frac{\partial^2 Q}{\partial L \partial K}$ is positive, at any given level of capital input *MPK* is increased by a small increase in the labour input. This means that a small increase in the labour input increases the *additional* output obtained from a small increase in the capital input. In the sketch above, the *MPK* function shifts upwards when *L* increases. Although there is no iron law of logic or economics that says that this must be true, it seems reasonable to expect that it will be found to be true in many cases.

- (d) Find the equation of the average product of labour (APL). By differentiation, show that APL reaches a maximum when $L = \left(\frac{2}{5} K(2K - 3)\right)^{0.5}$. Show that, when $K = 10$, the value of L that maximises the APL is $L = 8.246$. Find the maximised value of the APL. Show that, in the usual way, $MPL = APL$ at this point. Hence sketch the graphs of the APL and MPL on the same diagram.

Answer: $APL \equiv \frac{Q}{L} = 15K - 4\frac{K^2}{L} - 5L + 6\frac{K}{L} + 4$. We can write this as:

$APL = 15K - 4K^2L^{-1} - 5L + 6KL^{-1} + 4$ so taking the partial derivative of this with respect to L and setting it equal to zero, we get $\frac{\partial APL}{\partial L} = 4K^2L^{-2} - 5 - 6KL^{-2} = 0$.

Solving this equation (treating L as the variable and K as a constant, of course) we get $4K^2L^{-2} - 6KL^{-2} = 5$, from which $L^{-2}(4K^2 - 6K) = 5$, and

hence $L = \left(\frac{2K(2K - 3)}{5}\right)^{0.5}$, as we were asked to show. Substituting $K = 10$ into this gives $L = 8.246$. Substituting $K = 10$, $L = 8.246$ into the equation for *APL* above gives $APL = 15(10) - 4\frac{100}{8.246} - 5(8.246) + 6\frac{10}{8.246} + 4 = 71.54$ to 2 d.p.

- (e) Repeat (d) for the average product of capital, *APK*.

Answer: $APK = 15L - 4K - 5L^2K^{-1} + 6 + 4LK^{-1}$ so taking the partial derivative of this with respect to K we get $\frac{\partial APK}{\partial K} = -4 + 5L^2K^{-2} - 4LK^{-2} = 0$. Solving this equation (treating K as the variable and L as a constant) we get

$K = \left(\frac{L(5L - 4)}{4} \right)^{0.5}$. Substituting $L = 10$ into this gives $K = 10.72$. Substituting $K = 10.72$, $L = 10$ into the equation for APK above gives the maximised value of APL as 70.21.

- (f) Comment on the general plausibility of this functional form as a production function.

Answer: Looking at the production function $Q = 15KL - 4K^2 - 5L^2 + 6K + 4L$ we see that, because both K^2 and L^2 terms appear, with negative coefficients, then for many positive values of K and L , Q will be negative. For example, when $K = 1$ and $L = 5$, $Q = -28$.

We can explore this further by looking at a typical short-run production function; that is, when K is fixed at, say, $K = 10$. The production function is then

$$Q = 150L - 400 - 5L^2 + 60 + 4L = -340 + 154L - 5L^2$$

This is a quadratic function in L . Setting it equal to zero and solving for L gives $L = 2.39$ or 28.4 . At these labour inputs, output is zero.

We can also find the maximum level of output by differentiating

$$Q = -340 + 154L - 5L^2, \text{ giving } \frac{dQ}{dL} = 154 - 10L. \text{ Setting this equal to zero and}$$

solving gives $L = 15.4$. This is a maximum of Q because $\frac{d^2Q}{dL^2} = -10$, which is negative when $L = 15.4$. Thus the short run production function for $K = 10$ has the shape of the sketch below.



Whatever fixed value is assigned to K , the resulting short run production function will always have this quadratic form. Thus this production function (unlike those considered in the book) has the following properties: (i) when K is fixed, output is zero when the labour input is low (< 2.39 in the example above). This is not implausible, if the technology of production is such that a certain minimum labour input is required before the production process will work; (ii) when K is fixed, as labour input increases output increases at first, then reaches a maximum, beyond which further increases in labour input actually reduce output. This is also not implausible, if the technology of production is such that there is an optimum labour input which gives maximum output, and further if further workers are employed they simply get in the way, causing output to fall.

Moreover, the same is true of the relationship between Q and K , with L fixed.

Another feature of this production function is that some of its isoquants cut the K or L axis, but we will not explore this here.

2. A firm's production function has the Cobb-Douglas form $Q = 100K^{0.5}L^{0.75}$.

(a) Find the marginal products of capital and labour and show that they can be written respectively as

$$\frac{\partial Q}{\partial K} = 0.5 \frac{Q}{K} \quad \text{and} \quad \frac{\partial Q}{\partial L} = 0.75 \frac{Q}{L}$$

(Hint: There is a little trick involved here; see the appendix to chapter 14.)

Answer: $\frac{\partial Q}{\partial K} = (0.5)100K^{-0.5}L^{0.75} = (0.5)100K^{-0.5}L^{0.75} \frac{K}{K} = (0.5)100K^{0.5}L^{0.75} \frac{1}{K} = 0.5 \frac{Q}{K}$

Thus the little trick is to multiply and divide by K (see Appendix 14.1 in the book)

We show that $\frac{\partial Q}{\partial L} = 0.75 \frac{Q}{L}$ in the same way, this time multiplying and dividing by L .

(b) Are the marginal products of capital and labour ever negative? Explain how you reached your answer. Give an economic interpretation to your answer.

Answer: If we restrict Q and K to being non-negative (that is, equal to or greater than zero), then $\frac{\partial Q}{\partial K} = 0.5 \frac{Q}{K}$ is always positive; and similarly for $\frac{\partial Q}{\partial L} = 0.75 \frac{Q}{L}$.

The economic interpretation is that if the MPL is always positive, then adding more workers, with the capital input fixed, will always increase output. That is, the possibility that extra workers might get in the way of existing workers, reducing total output, is ruled out (unlike in question 1 above). And similarly for the MPK .

- (c) How are the marginal products of capital and labour related to the respective average products? Explain how you reached your answer.

Answer: From (a) above we have $MPK = \frac{\partial Q}{\partial K} = 0.5 \frac{Q}{K}$. Since by definition $\frac{Q}{K}$ is the APK , we see that the MPK is always half of the APK , for this production function. Similarly from (a) the MPL is always three-quarters of the APL .

- (d) Sketch the graphs of the marginal and average product functions.

Answer: From (a) we have $\frac{\partial Q}{\partial K} = 50K^{-0.5}L^{0.75}$. If we fix L at, say, $L = 16$, we have

$L^{0.75} = 8$, so $\frac{\partial Q}{\partial K} = 400K^{-0.5} = \frac{400}{K^{0.5}}$. As K increases, $K^{0.5}$ increases, and so

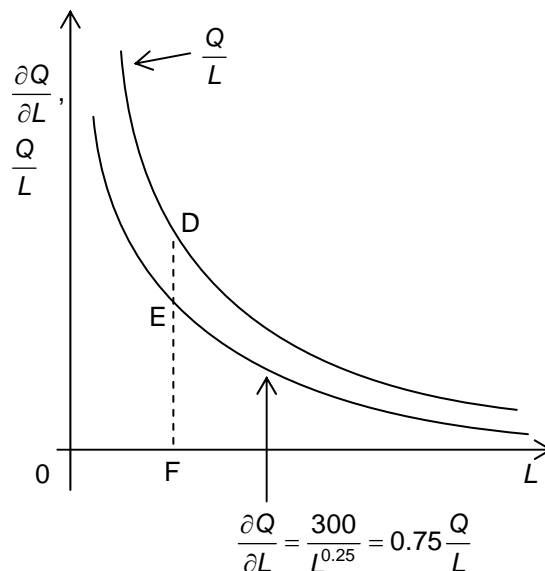
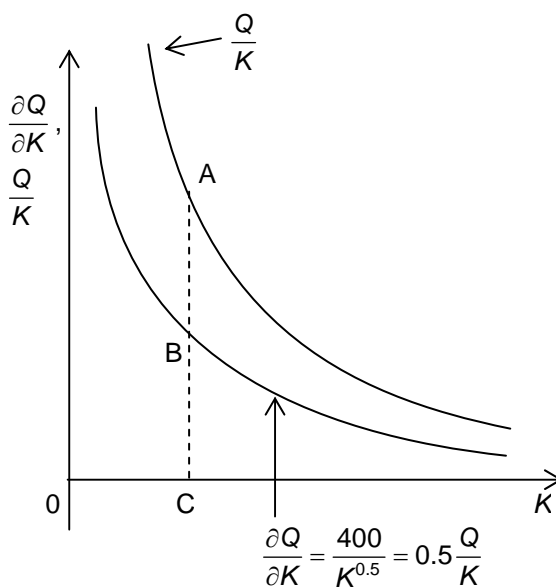
$\frac{\partial Q}{\partial K} = \frac{400}{K^{0.5}}$ approaches zero. Thus the graph of $\frac{\partial Q}{\partial K}$ approaches the horizontal axis asymptotically. At the other extreme, as K approaches zero, $K^{0.5}$ approaches zero, hence $\frac{\partial Q}{\partial K} = \frac{400}{K^{0.5}}$ approaches infinity. Thus the graph of $\frac{\partial Q}{\partial K}$ approaches the vertical axis asymptotically too. See left-hand graph below.

Similarly, we have $\frac{\partial Q}{\partial L} = 75K^{0.5}L^{-0.25}$, so if we fix K at, say, 16, we have

$\frac{\partial Q}{\partial L} = 300L^{-0.25} = \frac{300}{L^{0.25}}$. By the same reasoning as above, the graph of $\frac{\partial Q}{\partial L}$

approaches both the horizontal and vertical axis asymptotically. See right hand graph below.

(By taking first and second derivatives and examining their signs, we can confirm that the slope and curvature of these graphs is correct; see (f) below.)



Note that BC is half of AC, while EF is three-quarters of DF.

(e) Show that the four second-order partial derivatives can be written as

$$\frac{\partial^2 Q}{\partial K^2} = -0.25 \frac{Q}{K^2} ; \quad \frac{\partial^2 Q}{\partial L^2} = -\frac{3}{16} \frac{Q}{L^2} ; \quad \frac{\partial^2 Q}{\partial L \partial K} = \frac{3}{8} \frac{Q}{KL} ; \quad \text{and} \quad \frac{\partial^2 Q}{\partial K \partial L} = \frac{3}{8} \frac{Q}{KL}$$

(Hint: use the same trick as in (a) above.)

Answer: From (a) we have $\frac{\partial Q}{\partial K} = (0.5)100K^{-0.5}L^{0.75}$, so

$$\frac{\partial^2 Q}{\partial K^2} = (-0.5)(0.5)100K^{-1.5}L^{0.75} = (-0.25)100K^{-1.5}L^{0.75} \frac{K^2}{K^2} = (-0.25) \frac{Q}{K^2}; \text{ and}$$

$$\frac{\partial^2 Q}{\partial L \partial K} = (0.75)(0.5)100K^{-0.5}L^{-0.25} = (0.375)100K^{-0.5}L^{-0.25} \frac{KL}{KL} = (0.375)Q \frac{1}{KL}$$

The other second order derivatives can be found in the same way.

(f) Is diminishing marginal productivity a characteristic of this production function? If so, does it operate at some, or all, levels of input and output? Explain how you reached your answer.

Answer: from (e) we have $\frac{\partial^2 Q}{\partial K^2} = (-0.25) \frac{Q}{K^2}$. As K^2 is always positive, and we

assume Q positive, $\frac{\partial^2 Q}{\partial K^2}$ is always negative. Therefore the graph of $\frac{\partial Q}{\partial K}$ is

always negatively sloped, so $\frac{\partial Q}{\partial K} \equiv MPK$ decreases as K increase (with L

constant). Similarly we can show $\frac{\partial Q}{\partial L} \equiv MPL$ decreases as L increase (with K

constant). So diminishing marginal productivity operates on both inputs at all levels of output. (See (d) above).

(g) What is the effect on the marginal product of one input if the other input is increased to a new fixed value? Explain how you reached your answer.

Answer: from (a) we have $\frac{\partial Q}{\partial K} = 50K^{-0.5}L^{0.75}$. Since $L^{0.75}$ is a multiplicative term with

positive sign, and $L^{0.75}$ increases as L increases, we can say that an increase in the labour input always increases MPK . More rigorously, if we differentiate $\frac{\partial Q}{\partial K}$

with respect to L we get (from (e) above) the cross-partial derivative

$$\frac{\partial^2 Q}{\partial L \partial K} = (0.375)Q \frac{1}{KL}, \text{ which is positive when } Q, K \text{ and } L \text{ are positive. This}$$

cross-partial, of course, measures the effect on MPK of a small increase in L . In terms of the graph in part (d) above, an increase in L shifts the MPK curve bodily upwards.

In the same way, we can show that an increase in K always increases MPL

3. Continuing with the production function in the previous question, find the equation of a typical isoquant (say, $Q = 100$) with K as the dependent variable. Show that the isoquants are downward sloping and convex in the K, L plane. Explain the relationship between the marginal products of capital and labour and the slope of an isoquant.

Answer: given $Q = 100K^{0.5}L^{0.75}$, with $Q = 100$ we have $1 = K^{0.5}L^{0.75}$, from which

$$K = \left(\frac{1}{L^{0.75}} \right)^{\frac{1}{0.5}} = \frac{1}{L^{1.5}}. \text{ From this, we can see that } K \text{ goes to infinity as } L \text{ approaches}$$

zero, so the isoquant is asymptotic to the vertical (K) axis. Also, K goes to zero as L approaches infinity, so the isoquant is also asymptotic to the horizontal (L) axis.

More rigorously, we can differentiate $K = \frac{1}{L^{1.5}} \equiv L^{-1.5}$ to find the slope of the isoquant.

This gives $\frac{dK}{dL} = (-1.5)L^{-2.5} \equiv \frac{-1.5}{L^{2.5}}$. As $L^{2.5}$ is always positive, the slope is always

negative. Differentiating again, we get $\frac{d^2K}{dL^2} = (3.75)L^{-3.5} \equiv \frac{3.75}{L^{3.5}}$. As $L^{3.5}$ is always

positive, the second derivative is always positive, telling us that the isoquant is convex from below.

Regarding the relationship between the marginal products of capital and labour and the slope of an isoquant, this part of the question is a little premature as this relationship is not fully explained until chapter 15. However in general terms we can see that, as we move down an isoquant (with K on the vertical axis) the capital input is falling and the labour input rising, in such a way as to maintain output constant. Therefore: (i) the higher is MPK , the greater is the loss of output when the K input falls by one unit, and therefore the greater is the increase in L input required to keep output constant, and therefore the flatter is the indifference curve; (ii) the higher is MPL , the smaller is the increase in L input required to keep output constant when the K input falls by one unit, and therefore the steeper is the indifference curve.

4. An individual's utility function has the Cobb-Douglas form

$$U = X^{0.5}Y^2$$

where U is an index of her utility and X and Y are the weekly quantities of the two goods consumed.

- (a) Find the marginal utilities of the two goods. Does the consumer ever experience satiation with respect to either good?

Answer: $MU_x \equiv \frac{\partial U}{\partial X} = 0.5X^{-0.5}Y^2$; $MU_y \equiv \frac{\partial U}{\partial Y} = 2X^{0.5}Y$. Satiation with X occurs

when, with the amount of Y consumed held constant, consumption of X increases to the point where its MU becomes zero or negative. As $0.5X^{-0.5}Y^2$ is positive provided X and Y are positive, MU_x is always positive, so satiation with X does not occur. By the same reasoning, satiation with Y does not occur, because $2X^{0.5}Y$ is always positive when X and Y are positive.

- (b) How does the marginal utility of each good vary as consumption of it increases with consumption of the other good held constant? What is the economic significance of the signs of these partial derivatives?

Answer: The second derivative $\frac{\partial^2 U}{\partial X^2}$ tells us how $\frac{\partial U}{\partial X}$ changes as consumption of X

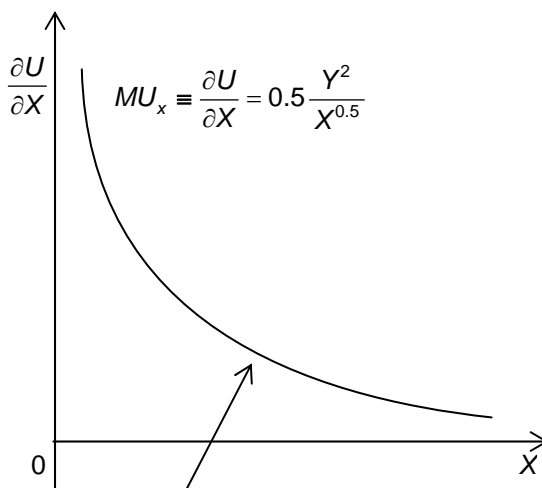
increases with consumption of Y held constant. Here $\frac{\partial U}{\partial X} = 0.5X^{-0.5}Y^2$, so

$$\frac{\partial^2 U}{\partial X^2} = (-0.5)(0.5)X^{-1.5}Y^2 = -0.25\frac{Y^2}{X^{1.5}}.$$

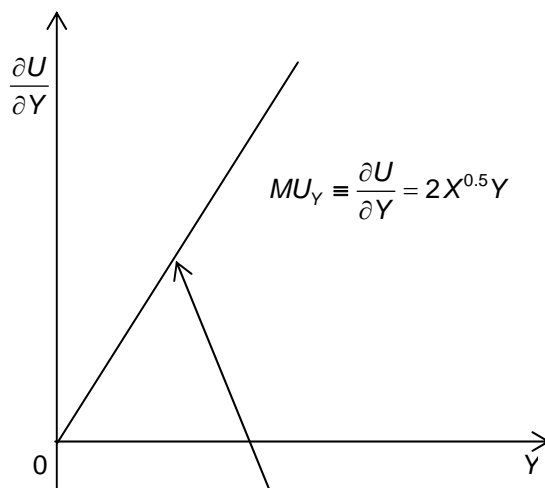
As $\frac{Y^2}{X^{1.5}}$ is always positive when X and Y are positive, the second derivative is always negative, meaning that MU_X decreases as consumption of X increases with consumption of Y held constant. The graph of MU_X as a function of X is therefore negatively sloped. The third derivative $\frac{\partial^3 U}{\partial X^3} = 0.375\frac{Y^2}{X^{2.5}}$ is positive, telling us that the slope of the graph of MU_X increases (becomes less negative) as X increases.

Applying the same reasoning to $\frac{\partial U}{\partial Y}$ we get $\frac{\partial^2 U}{\partial Y^2} = 2X^{0.5}$. This is positive when X is

positive, but is independent of Y , meaning that the graph of MU_Y is upward sloping. MU_Y increases as consumption of Y increases with consumption of X held constant. The third derivative $\frac{\partial^3 U}{\partial Y^3} = \frac{\partial}{\partial Y}$ of $2X^{0.5} = 0$, telling us that the slope of the graph of MU_X remains constant as Y increases.

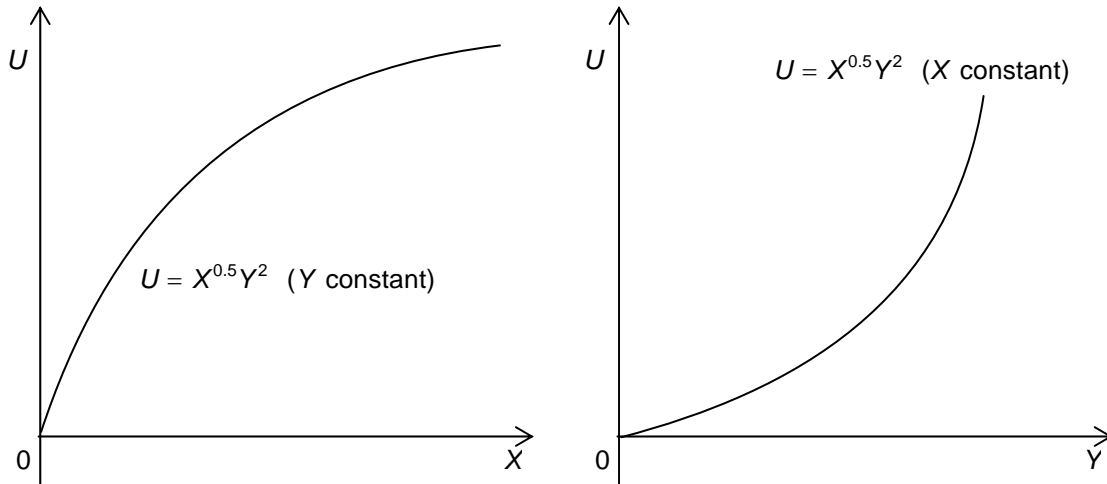


slope $\left[= \frac{\partial^2 U}{\partial X^2} = -0.25\frac{Y^2}{X^{1.5}} \right]$
 is negative but increasing



slope $\left[= \frac{\partial^2 U}{\partial Y^2} = 2X^{0.5} \right]$ is
 positive and constant

- (c) Use the information obtained above to sketch the graph of a typical iso- Y section; that is, the curve showing how U varies as consumption of X increases with consumption of Y held constant. Similarly, sketch the graph of an iso- X section.



- (d) By examining the signs of the cross-partial derivatives, determine whether an increase in consumption of X increases the marginal utility of Y , and vice versa. What is the economic interpretation of the signs of these cross partial derivatives?

Answer: $\frac{\partial^2 U}{\partial Y \partial X} = X^{-0.5}Y$. As this is positive for all positive values of X and Y , an increase in the amount of Y consumed increases the marginal utility of X . Since $\frac{\partial^2 U}{\partial X \partial Y} = X^{-0.5}Y$ also, an increase in the amount of X consumed increases the marginal utility of Y . The economic interpretation is that these goods are mutually complementary in the individual's consumption, such as toast and marmalade.

- (e) (i) Find the equation of a typical indifference curve, with either K or L as the dependent variable. (ii) Do any indifference curves ever cut either of the axes? (iii) How does the answer to this question relate to the issue of satiation in (a) above?

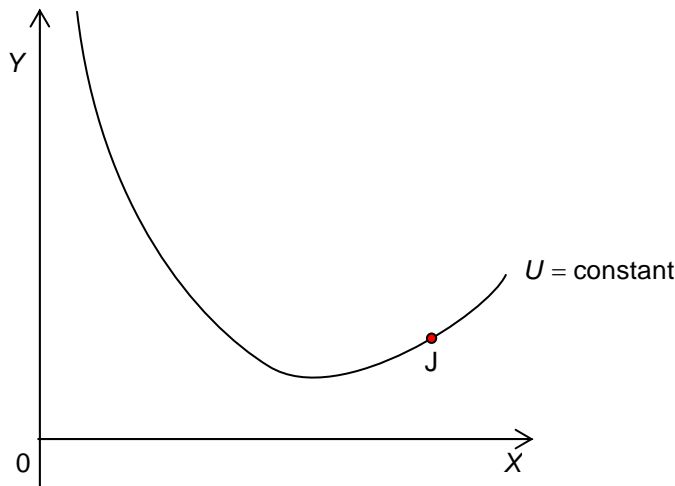
Answer: (i) Given the utility function $U = X^{0.5}Y^2$, if we fix the utility level at, say, $U = 100$, we derive the equation of the indifference curve for $U = 100$ as

$$100 = X^{0.5}Y^2, \text{ which we can rearrange as } \frac{100}{X^{0.5}} = Y^2 \text{ and hence } Y = \frac{10}{X^{0.25}}.$$

(ii) As X increases, $X^{0.25}$ increases and hence $Y = \frac{10}{X^{0.25}}$ decreases, but is always positive, so the indifference curve is asymptotic to the X axis. But it never cuts the axis, because $X^{0.25}$ is always positive and so therefore is

$Y = \frac{10}{X^{0.25}}$. At the other extreme, as X approaches zero $Y = \frac{10}{X^{0.25}}$ approaches infinity, so the indifference curve is also asymptotic to the Y axis. These conclusions are not dependent on the assumption that $U = 100$, but will be true whatever fixed value we assign to U , and therefore none of the indifference curves cut the axes.

(iii) If the consumer is satiated with, say, good X , this means that any further increase in X consumption (with Y constant) will reduce the individual's utility and an increase in Y consumption is required to hold utility constant. If consumption of both goods is increasing while utility is constant, the indifference curve must be positively sloped, as at J is sketch below. In the utility function we are considering, we showed in (ii) above that the indifference curves are asymptotic to the axes. Therefore they never turn up, as in the sketch below, so satiation does not occur. (Actually this is not completely rigorous, because the curve could turn up, then down again to become asymptotic. The rigorous demonstration is in (f) below.)



- (f) (i) By differentiation, show that the indifference curve is negatively sloped and convex in the K, L plane. (ii) Give an economic interpretation to the slope and curvature of the indifference curve. (iii) Sketch the graph of the indifference curve.

Answer: (i) Using the indifference curve from (e) above, $Y = \frac{10}{X^{0.25}} \equiv 10X^{-0.25}$, by

differentiation its slope is $\frac{dY}{dX} = -2.5X^{-1.25} = -\frac{2.5}{X^{1.25}}$. This is negative, since $X^{1.25}$ is always positive; that is, the indifference curve is negatively sloped.

Differentiating again, we get $\frac{d^2Y}{dX^2} = 3.125X^{-2.25} = \frac{3.125}{X^{2.25}}$. As this is always positive, the slope of the indifference curve is increasing; that is, it is convex from below. As noted above, these conclusions are not dependent on the

assumption that $U = 100$, therefore all of the indifference curves are negatively sloped and convex from below. This convexity proves rigorously that satiation does not occur (see (e)(iii) above).

5. A person's utility function is $U = (X + 2)(Y + 1)$.

(a) Find the marginal utilities of goods X and Y .

Answer: $\frac{\partial U}{\partial X} = Y + 1$; $\frac{\partial U}{\partial Y} = X + 2$

(b) Find the direct second derivatives and the cross partial derivatives.

Answer: $\frac{\partial^2 U}{\partial X^2} = 0$; $\frac{\partial^2 U}{\partial Y \partial X} = 1$; $\frac{\partial^2 U}{\partial Y^2} = 0$; $\frac{\partial^2 U}{\partial X \partial Y} = 1$

What do these four derivatives tell us about how:

(i) the marginal utility of X changes when consumption of X increases with Y constant;

Answer: The fact that $\frac{\partial^2 U}{\partial X^2} = 0$ tells us that there is no change in the marginal utility of X when consumption of X increases with Y constant.

(ii) how the marginal utility of X changes when consumption of Y increases with X constant;

Answer: The fact that $\frac{\partial^2 U}{\partial Y \partial X}$ is positive tells us that the marginal utility of X increases when consumption of Y increases with X constant.

(iii) how the marginal utility of Y changes when consumption of Y increases with X constant; and

Answer: The fact that $\frac{\partial^2 U}{\partial Y^2} = 0$ tells us that there is no change in the marginal utility of Y when consumption of Y increases with X constant.

(iv) how the marginal utility of Y changes when consumption of X increases with Y constant?

Answer: The fact that $\frac{\partial^2 U}{\partial X \partial Y}$ is positive tells us that the marginal utility of Y increases when consumption of X increases with Y constant.

- (c) Find the equation of a typical indifference curve, with either Y or X as the dependent variable. Do the indifference curves cut either or both axes?

Answer: Given $U = (X + 2)(Y + 1)$, if we set $U = 100$ we get $100 = (X + 2)(Y + 1)$ which rearranges as $Y = \frac{100}{X+2} - 1$. This is a rectangular hyperbola. To find its intercept on the Y axis we set $X = 0$, giving $Y = 49$. To find its intercept on the X axis we set $Y = 0$, giving $\frac{100}{X+2} = 1$, from which $X = 98$

- (d) By differentiation, determine the slope and convexity or concavity of the indifference curve. Sketch some indifference curves.

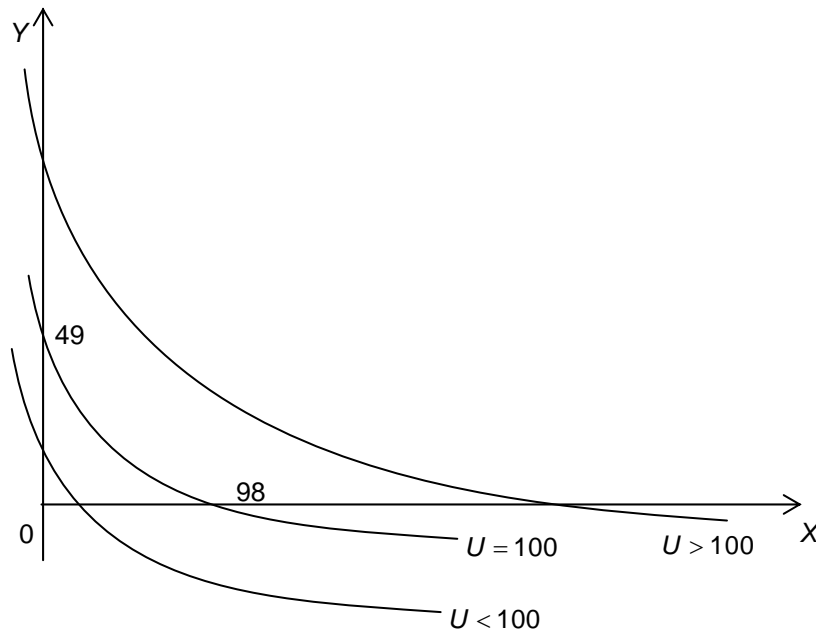
Answer: From (c) we have $Y = \frac{100}{X+2} - 1$ as the indifference curve for $U = 100$. Hence

$$\frac{\partial Y}{\partial X} = -\frac{100}{(X+2)^2},$$

which is negative for all X , so the indifference curve is

negatively sloped. Differentiating again, we get $\frac{\partial^2 Y}{\partial X^2} = \frac{200}{(X+2)^3}$, which is

positive when X is positive, so the indifference curve is convex from below. Changing the fixed value of U would not change the signs of the first or second derivatives, so we conclude that all indifference curves are negatively sloped and convex. See sketch below.



- (e) In the light of your answers to (a) and (d), what can we infer about the relationship between diminishing marginal utility and the convexity of indifference curves?

Answer: Although it is tempting to suppose that diminishing marginal utilities of both goods is necessary to make the indifference curves convex from below, this example shows that this is not correct; for in this example we have

$\frac{d^2U}{dX^2} = 0$ and $\frac{d^2U}{dY^2} = 0$, so marginal utilities do not diminish, but nevertheless the indifference curves are convex from below.