Exercise WS13.1

1. Find $\frac{dy}{dx}$ for each of the following functions, and in cases (a) and (b) draw a sketch graph of the function:

(a) $y = e^{3x}$

Answer: $\frac{dy}{dx} = 3e^{3x}$ (Rule 13.2).

The graph of $y = e^{3x}$ turns up very steeply and is therefore almost impossible to draw to scale. A sketch should be similar to fig. 12.2 in the book ($y = e^x$), but with the values of x shown all divided by 3. Thus for example when y = 7.389, x = $^2/_3$ instead of 2. The graph of $y = e^{3x}$ lies above the graph of $y = e^x$ when x is positive, and below it when x is negative. Both graphs have a y intercept of 1, because when x = 0 we have $e^x = e^0 = 1$ and $e^{3x} = e^0 = 1$.

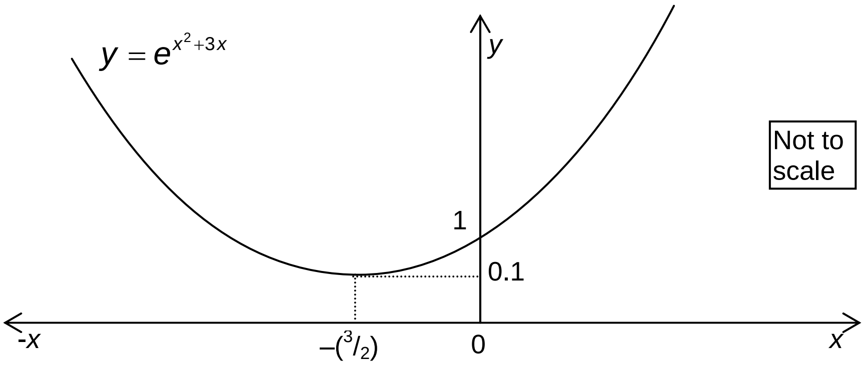
(b) $y = -\frac{1}{3}(\ln x)$

Answer: $\frac{dy}{dx} = -\frac{1}{3}\left(\frac{1}{x}\right) = -\frac{1}{3x}$ (Rule 13.3 plus the rule that a multiplicative constant reappears in the derivative).
To sketch the graph, we can start from fig. 12.9, the graph of $y = \ln x$. To get from this to the graph of $y = -\frac{1}{3}(\ln x)$ we must multiply each of the y values by $-\frac{1}{3}$. For example when x = 7.39 we will now have $y = -\frac{2}{3}$.
Both graphs have a x intercept of 1, because when x = 1 we have $\ln x = 0$ and $-\frac{1}{3}(\ln x) = 0$.

(c) $y = e^{x^2+3x}$

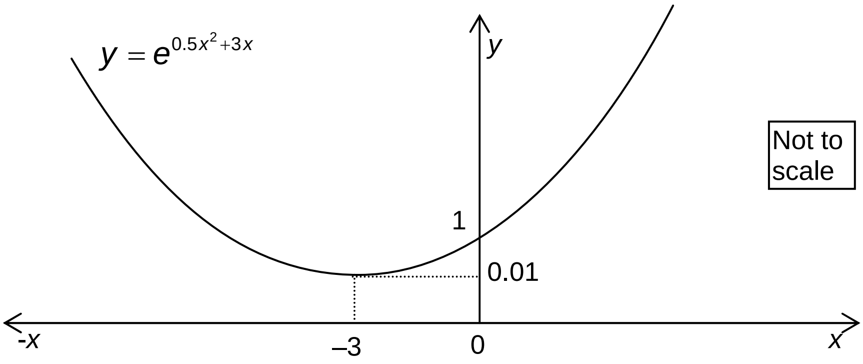
Answer: $\frac{dy}{dx} = e^{x^2+3x}\left(2x+3\right)$ (Rule 13.2).
To sketch the graph is not easy. We can start with the y intercept. This is easy because when x = 0 we have $y = e^0 = 1$. It is also easy to see that when x is positive, $x^2 + 3x$ is positive and increases very quickly as x increases. So the graph turns up steeply as x becomes increasingly positive. It is more difficult to work out the behaviour if y when x is negative. However the derivative gives us an important clue. Because $2x + 3 = 0$ when $x = -\frac{3}{2}$, the slope of the curve is zero at this point. And

because 2x + 3 < 0 when $x < -\frac{3}{2}$, the slope of the curve is negative when x is less than $-\frac{3}{2}$. Also, when $x = -\frac{3}{2}$, y = 0.1 approximately (using calculator).
From this information we can sketch the graph below.



(d) $y = e^{0.5x^2+3x}$

Answer: $\dfrac{dy}{dx} = e^{0.5x^2+3x}\left(x+3\right)$ (Rule 13.2).
The shape of this curve is almost the same as in (c) above. Again the y intercept is at y = 1, because when x = 0 we have $y = e^0 = 1$. Again, when x is positive, $0.5x^2 + 3x$ is positive and increases very quickly as x increases. So the graph turns up steeply as x becomes increasingly positive, though not quite as steeply as in (c) above because of the 0.5 coefficient on x^2.
Looking at the derivative, we see that x + 3 = 0 when $x = -3$, the slope of the curve is zero at this point, and the slope is negative when x is less than -3. Also, when $x = -\frac{3}{2}$, y = 0.01 approximately (using calculator).
From this information we can sketch the graph below.



2. The table below gives an index of UK labour productivity (defined as output per worker for the whole economy) for 1980 to 2003, with 2001 = 100.

(a) Calculate the year-to-year growth of labour productivity for the years 1980-90 and 1997-2003.

Answer: the method is exactly the same as that of Ex WS12.3, question 2(e). For example, the percentage growth between 1980 and 1981 is $\left(\frac{65.8-64.4}{64.4}\right)100 = 2.1739$ to 4 d.p. The complete list is:

Year	Productivity index	Year-on-year growth
1980	64.4	
1981	65.8	2.17
1982	68.6	4.26
1983	71.3	3.94
1984	71.3	0.00
1985	73.1	2.52
1986	75.6	3.42
1987	77.3	2.25
1988	78.2	1.16
1989	77.6	-0.77
1990	77.9	0.39
1991	79.5	2.05
1992	82.4	3.65
1993	85.2	3.40
1994	88.4	3.76
1995	89.7	1.47
1996	91.4	1.90
1997	92.7	1.42
1998	95	2.48
1999	96.1	1.16
2000	98.8	2.81
2001	100	1.21
2002	100.7	0.70
2003	101.9	1.19

(b) Calculate the average annual growth rate of labour productivity for the periods (i) 1980-90; (ii) 1990-97; (iii) 1997-2003.

Answer: We can answer this either by using the formula $y = a(1+r)^x$ which assumes growth occurs in annual jumps, or by using the formula $y = ae^{rx}$ which assumes growth is continuous. As we have seen in Ex WS12, the two formulae give very similar answers. We will use $y = a(1+r)^x$. Here we are solving for *r*, so we transform the formula into $\dfrac{\log\left(\dfrac{y}{a}\right)}{x} = \log(1+r)$ (See the answer to Ex WS11.2 question 3).

For (i) 1980-90 we have a = 64.4, y = 77.9 and x = 10. Thus in $\dfrac{\log\left(\dfrac{y}{a}\right)}{x} = \log(1+r)$, we get $\dfrac{\log\left(\dfrac{77.9}{64.4}\right)}{10} = \log(1+r) = 0.008265159$.

We then find the anti-log of this (see again the answer to Ex WS11.2 question 3), which is 1.0192. This is 1+r, so r = 0.0192 or 1.92%.

The remaining answers, obtained in the same way, are: (ii) 2.52%; (iii) 1.36**%**

Exercise WS13.2

1. (a) A variable with an initial value of 200 is growing through time at a constant nominal rate of 5% per year. Write down the equation that describes this growth, assuming (i) continuous growth; (ii) growth in annual jumps.

   Answers: (i) $y = 200e^{0.05x}$ (ii) $y = 200(1.05)^x$
   (where y is the value of the variable after x years)

   (b) If in (a) the actual growth of the variable is continuous, but we choose to model the growth as occurring in annual jumps, what is the resulting error in (i) our calculation of the effective annual growth rate; (ii) the level of the variable after 10 years?

   Answer: (b)(i) The effective annual growth rate is the proportionate or percentage increase in the variable in any one year. With continuous growth, the value of the variable after 1 year (x = 1) is $y = 200e^{0.05} = 200(1.05127)$. So the proportionate increase is $\dfrac{200(1.05127) - 200}{200} = \dfrac{1.05127 - 1}{1} = 0.05127$. The percentage increase is therefore 5.127%. (See p.436 of the book, "The effective interest rate", for a slightly different explanation of this). With growth in annual jumps, the value of the variable after 1 year (x = 1) is $y = 200(1.05)$. So the proportionate increase is $\dfrac{200(1.05) - 200}{200} = \dfrac{1.05 - 1}{1} = 0.05$. The percentage increase is therefore 5%. (That is, with growth in annual jumps, the nominal and the effective growth rates are the same.)
   Thus if we choose to model the growth as occurring in annual jumps, when

in fact it occurs continuously, we will calculate the effective growth rate as 5% when in fact it is 5.127%, and error of 0.127 percentage points.

Answer: (b)(ii) After 10 years, the value of the variable, assuming continuous growth, will be $y = 200e^{0.05(10)} = 329.7443$. Assuming growth in annual jumps, the value of the variable will be $y = 200(1.05)^{10} = 325.7789$. So the difference is 3.9654.

2. (a) Demonstrate algebraically that if a variable, *Y*, is growing continuously at a constant (proportionate) rate, the graph of ln *Y* as a function of time will be linear, and with a slope equal to the growth rate of *Y*.

Answer: The function $Y = ae^{rx}$ gives the value of any variable, *Y*, that has been growing continuously at a constant (proportionate) rate, *r*, for *x* years. Taking natural logs on both sides, we get:

$$\ln Y = \ln a + rx$$

If this function is plotted with *Y* on the vertical axis and *x* on the horizontal (*a* and *r* being parameters), we see that this will be a linear function because *x* is not raised to any power other than 1 (see section 3.4). As *a* and *r* are constants, we see that the intercept on the *Y* axis is ln *a* (a constant since *a* is a constant) and the slope is *r*.

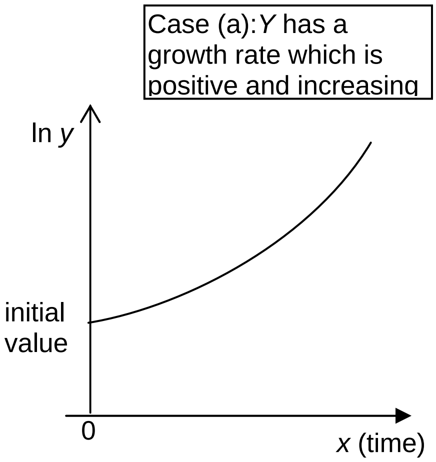
We can show that the slope, *r*, is also the growth rate as follows. Given $Y = ae^{rx}$, the derivative is $\frac{dY}{dx} = ae^{rx}(r)$ (see rule 13.2). So the (instantaneous) growth rate is $\frac{1}{Y}\frac{dY}{dx} = \frac{1}{Y}\left[ae^{rx}(r)\right] = r$ (since $Y = ae^{rx}$). (See rule 13.8).

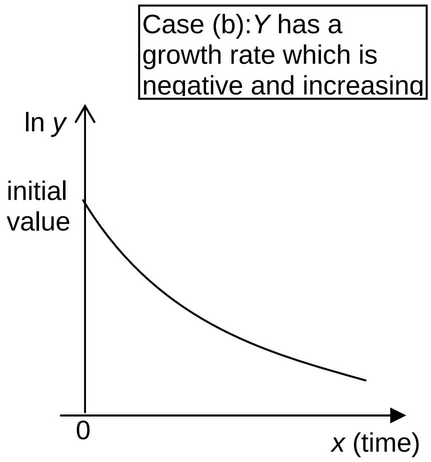
(b) Illustrate (a) when *Y* has an initial value of 100 and a nominal growth rate of 2.5% per year.

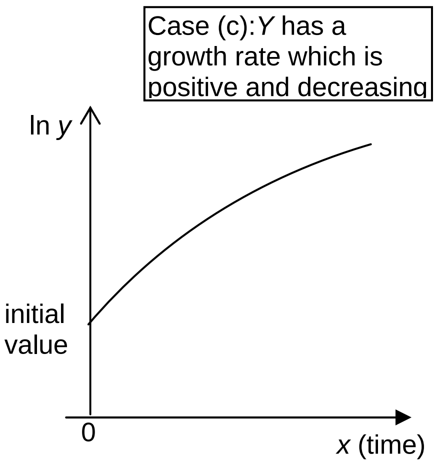
Answer: here we have $Y = 100e^{0.025x}$ so taking natural logs gives $\ln Y = \ln 100 + 0.025x$. The graph of this has an intercept on the *Y* axis of ln100 (= 4.605) and a slope of 0.025.

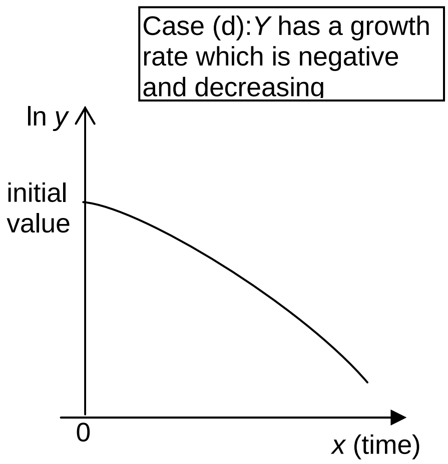
(c) Sketch the graph of ln *Y* as a function of time, if the growth rate is (i) increasing; (ii) decreasing.

Answer: If the growth rate is not constant, the variable can no longer be described by the function $Y = ae^{rx}$ as this assumes a constant growth rate, *r*. The appropriate functional form could be very complex. However, in general it follows from the analysis above that if the growth rate is increasing the graph of ln *Y* will have an increasing slope (cases (a) and (b) below), while if the growth rate is decreasing the graph of ln *Y* will have an decreasing slope (cases (c) and (d)). (See also Ex WS12.3 question 2(c))





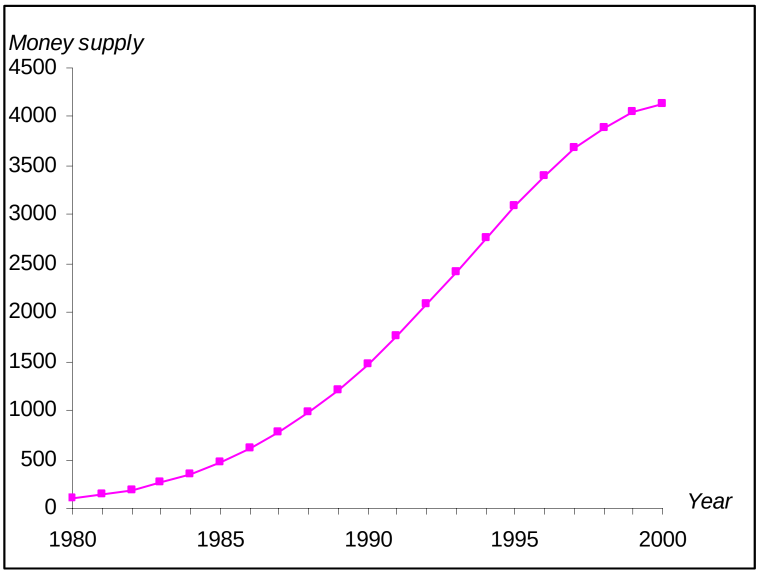




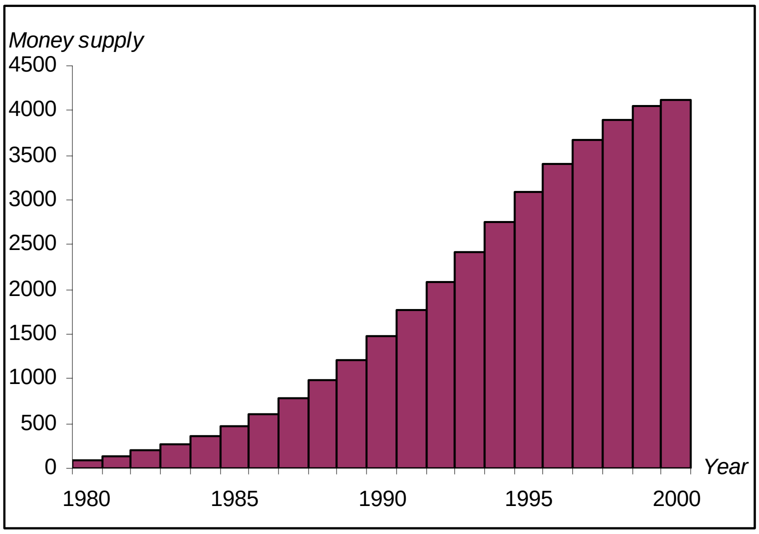
3. The table below gives money supply data for the Democratic Republic of Agraria for 1980-2000, with 1980 = 100.

   (a) Plot a reasonably accurate graph of the money supply time series. (Hint: You can format your graph in either of two ways: (i) by joining up the data points to form a continuous curve, which means you are treating the money supply as a continuous variable; or (ii) by creating a bar chart in which the height of each bar gives the money supply for the year in question, which means you are treating the money supply as a discrete variable that varies in annual jumps.) From inspection of the graph, try to assess whether the annual growth rate of the money supply is constant, increasing or decreasing through time.

   Answer: (i) continuous variable

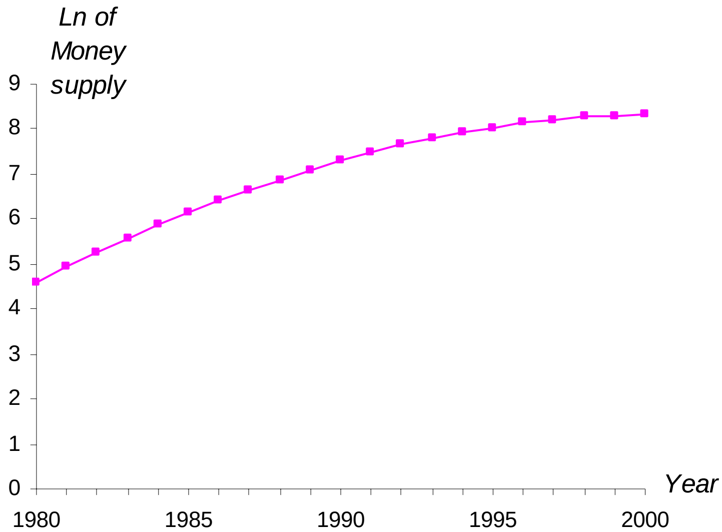


   Answer: (ii) annual jumps

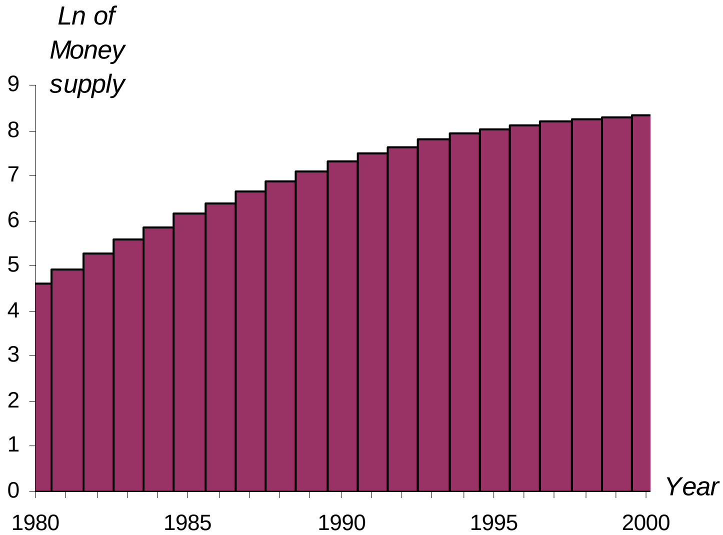


(b) Repeat (a) above, but this time plotting the natural log of the money supply.

Answer (i) continuous growth



Answer (ii) annual growth



(c) Explain in words (preferably supported by the relevant maths) why assessing the growth rate is easier in (b) than in (a).

Answer: In part (a), with the money supply on the vertical axis, the graphs turn up very steeply and it appears that the money supply is increasing very rapidly. However in part (b), with the natural log of the money supply on the vertical axis, we see that the slope of the graph (which measures the growth rate of the money supply) is decreasing continuously throughout the period. The maths to support this is simply a repetition of the answer to question 2 above.

(You can check that the growth rate declines continuously by calculating the year-to-year growth rates of the money supply, as in question 2(a) above. You will find that the growth rate falls from 40% in 1980-81 to 2% in 1999-2000.

4. Given the demand function $q = 100p^{-1.2}$

(a) Find an expression for the price elasticity of demand.

Answer: the derivative is $\frac{dq}{dp} = -1.2(100p^{-0.2})$ so the demand elasticity is

$$E^D \equiv \frac{p}{q}\frac{dq}{dp} = \frac{p}{q}\left[-1.2(100p^{-0.2})\right] = \frac{1}{q}\left[-1.2(100p^{-0.2+1})\right]$$

$$= \frac{-1.2(100p^{-1.2})}{q} = -1.2 \qquad \text{(since } q = 100p^{-1.2}\text{)}$$

(b) Take natural logs on both sides and hence show that the function is "log-log linear".

Answer: taking natural logs on both sides of $q = 100p^{-1.2}$ gives

$\ln q = \ln 100 - 1.2(\ln p)$. Here the variables are ln q and ln p, with constants ln 100 and –1.2. Since ln p is not raised to any power except 1, this function is linear. A function such as this, which has the property that when plotted with log scales on both axes the result is a linear function, is said to be "log-log linear".

(c) Differentiate your expression in (b) above to find $\frac{d(\ln q)}{d(\ln p)}$

(Hint: Just apply the normal rules of differentiation.)
Why does it matter whether the exponent of p is greater or less than 1 in absolute value?

Hence verify that $\frac{d(\ln q)}{d(\ln p)} = \frac{p}{q}\frac{dq}{dp}$ and explain this result.

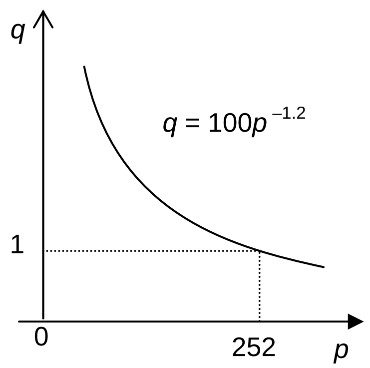
Answer: the derivative of this function is $\frac{d\ln q}{d\ln p} = -1.2$. Since the derivative measures the slope, it follows that when the function is plotted with logarithmic scales on both axes, the slope measures the elasticity.

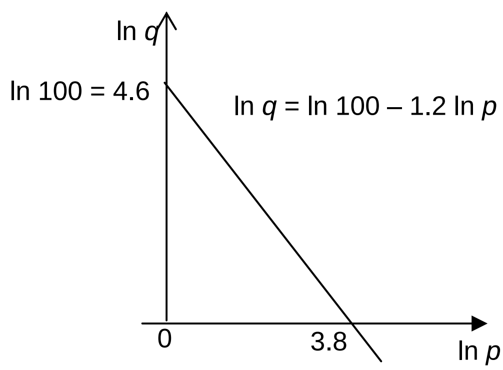
Combining answers to (a) and (b) we see that $\frac{\mathrm{d}(\ln q)}{\mathrm{d}(\ln p)} = \frac{p}{q}\frac{\mathrm{d}q}{\mathrm{d}p} = -1.2$. (See section 13.10 of the book). In general, we see that in the demand function $q = 100p^{-\alpha}$, the exponent of p (that is, α) is the elasticity. So if $\alpha > 1$, demand is always elastic whatever the price, while if $\alpha < 1$ demand is always inelastic whatever the price.

(In this example, the elasticity is constant, but that is purely to simplify the analysis. If the elasticity varies along the demand curve, it remains true that the slope at any point, with log scales on both axes, measures the elasticity at that point. And the result is valid for any function, no matter what p and q stand for.)

(d) Sketch the graph of the demand function, (i) with q and p on the axes, and (ii) with ln q and ln p on the axes. What presentational advantage does (ii) have over (i)?

Answer:





To sketch the graph of $q = 100p^{-1.2}$, it helps to write $q = 100p^{-1.2}$ as $q = \frac{100}{p^{1.2}}$.

Next, it helps if we can find one point on the curve. Here, we can find the point where $q = 1$ by setting $p^{1.2} = 100$. Solving this equation gives $p = 100^{\frac{1}{1.2}} = 252$ approximately.

The, we can see that as p increases, so $p^{1.2}$ increases, so $q = \frac{100}{p^{1.2}}$ decreases but is always positive. So the curve is asymptotic to the p axis.

Similarly, as p decreases, so $p^{1.2}$ decreases, so $q = \frac{100}{p^{1.2}}$ increases, and approaches infinity as p approaches zero. So the curve is also asymptotic to the q axis.

To sketch the graph of $\ln q = \ln 100 - 1.2 \ln p$, we see that the intercept on the vertical axis is where ln p = 0; that is, where p = 1. At this point, ln q = ln 100 = 4.605 (using calculator). The intercept on the ln p axis is where ln q = 0, which implies $\ln 100 - 1.2 \ln p = 0$. Solving this equation gives $\ln p = \frac{\ln 100}{1.2} = 3.84$ approximately (using calculator).
The presentational advantage of the graph of $\ln q = \ln 100 - 1.2 \ln p$ over the graph of $q = 100p^{-1.2}$ is that we can immediately see what the elasticity of demand is, simply by inspecting the slope.