Exercise WS12.1

1. (a) Without using a calculator, make a rough estimate of the value of each of the following. Then check your answers using a calculator.

(i) e^2

Answer: since *e* is a little less than 3 (see section 12.3 of the book), we expect e^2 to be a little less than 9; say, 7.5. To check this using our calculator, we key in "shift" or "inv" (depending on the calculator model), then "ln", then 1. The calculator will then find e^1, which of course equals *e*. (The calculator can of course only find an approximation to *e*.) We then key in " ^ " (depending on the calculator model) followed by 2, which gives $\left(e^1\right)^2$, or e^2 = 7.3891 to 4 d.p.

(ii) $e^{0.5}$

Answer: since $e^{0.5}$ is the positive square root of *e*, and *e* is a little less than 3, we expect e^2 to be less than 2 (because 2^2 = 4) but more than 1 (because 1^2 = 1). It is also more than 1.5, because $(1.5)^2 = \left(\frac{3}{2}\right)^2 = \frac{9}{4} = 2\frac{1}{4}$. So we guess, say, 1.7. To check this using our calculator, we find e^1 as in (i) above. We then key in "^" followed by 0.5, which gives $\left(e^1\right)^{0.5}$, or $e^{0.5}$ = 1.6488 (to 4 d.p.).

(iii) ln 0.5

Answer: We are looking for some number, *x*, such that *x* = ln 0.5. Then, from the definition of a natural log, we have $e^x = 0.5$. Since e^0 = 1, which is greater than 0.5, then *x* must be less than 0; that is, negative. In other words, *x* must be negative in order to make e^x smaller than 1.
Let us try *x* = -2. We then have $e^x = e^{-2} = \dfrac{1}{e^2} = \dfrac{1}{7.3891} = 0.1353$ (using our answer to (i) above). This is too small (0.1353 < 0.5) so let's try *x* = -0.5. We than have $e^x = e^{-0.5} = \dfrac{1}{e^{0.5}} = \dfrac{1}{1.6488} = 0.6065$, using our answer to (ii) above. This is too large (0.6065 > 0.5), so as a final guess let's try $x = -0.6$. Checking with our calculator we key " ln " followed by 0.5 and get –0.6931 on the display, not very far from our guess of –0.6.

(b) Write down the inverses of each of the above.

Answers: The functions $y = e^x$ and $x = \ln y$ are, by definition, inverse to one another.
In (i) we have 7.3891 = e^2, so we have y = 7.3891 and x = 2. So the inverse function is 2 = ln 7.3891.

In (ii) we have 1.6488 = $e^{0.5}$, so we have y = 1.6488 and x = 0.5. So the inverse function is 0.5 = ln 1.6488.

In (iii) we have –0.6931 = ln 0.5, so we have x = –0.6931 and y = 0.5. So the inverse function is 0.5 = $e^{-0.6931}$.
(Notice that your calculator makes small errors in these calculations. This is because the value of e that it uses is an approximation.)

2. (a) Sketch the graph of $y = e^X$ for values of x between –3 and +3. (Don't worry too much about the scale on the y axis).

Answer: see fig. 12.2 in the book. (Remember, we find e^2 on our calculator by keying in "inv" or "shift" (depending on calculator), followed by " ln " and 2.

(b) Use your graph to estimate the values of e^2, e^{-1}, $e^{0.5}$, $\ln 20$, and ln 0.5.

Answers: from fig. 12.2 we see that e^2 = 7.389 and e^{-1} = 0.368 (check these using your calculator). From fig. 12.2 we also see that $e^{0.5}$ lies somewhere between e^0 = 1 and e^1 = 2.71828, so we'll guess 1.5. Our calculator gives the true answer as 1.6487, so our guess was reasonably close.
In fig. 12.2, the x values are, by definition, the natural logs of the y values. Thus we see that ln 20.086 = 3, so ln 20 must be very slightly less than 3, say 2.98. Our calculator gives the true answer as 2.996.
In fig. 12.2 we see that the log of 0.368 is –1, so the log of 0.5 must be greater than –1; say, –0.7. Our calculator gives ln 0.5 = –0.6931 as the true answer.

3. Sketch the graph of $y = 100e^{0.05x}$.

Answer: the graph of $y = e^{0.5x}$ is given in fig. 12.4. The graph of $y = 100e^{0.05x}$ is exactly the same, except that every value on the y axis must be multiplied by 100. For example, when x = 1, $y = e^{0.5x}$ gives y = 1.649, while $y = 100e^{0.05x}$ gives y = 164.9 (check on your calculator).

Exercise WS12.2

1. Repeat question 3 of exercise WS11.2, assuming now that traffic grows continuously rather than in annual jumps. By how much are your answers changed by this change in assumption about the growth process? Which assumption about growth seems more appropriate in analysing these data?

(a) Calculate the average annual growth rate for each type of vehicle.

Answer: We now use the continuous compound growth formula $y = ae^{rx}$. To find the growth rate, *r*, we first divide both sides by *a* and take logs on both sides, giving: $\ln\left(\frac{y}{a}\right) = \ln\left[ae^{rx}\right]$

Using rule 12.4(b) on the right hand side, together with the fact that ln *e* ≡ 1 (see rule 12.4(e)) this becomes:

$\ln\left(\frac{y}{a}\right) = rx$, from which by elementary algebra

$$\frac{\ln\left(\frac{y}{a}\right)}{x} = r$$

For the "all vehicles" category, this gives

$$\frac{\ln\left(\frac{495}{277}\right)}{23} = r = 0.025240881$$

(Note that we do not have to take any anti-logs here, unlike in question 3 of Ex WS11.2).

To get the percentage rate we multiply by 100, giving 2.524% (to 3 d.p.). This is almost the same as the corresponding answer to Ex WS11.2 q.3, which was 2.556% to 3 d.p.

Using the same method, the % growth rates for the other categories are (with the answers to Ex WS11.2 q.3 in brackets):

Light vans 3.49 (3.55); Heavy goods vehicles 1.46 (1.47); Buses & coaches 1.89 (1.90). (All rounded to 2 d.p.) The two sets of answers are clearly very close.

(b) Between 1990 and 2003 the increase in traffic for all vehicles was 19%. If this growth rate continues, (i) calculate what the level of traffic will be in 2015; (ii) after how many years will traffic have increased to 50% above its 2003 level?

Answers (i) Using again the compound growth formula $y = ae^{rx}$, we first transform this into $\frac{\ln\left(\frac{y}{a}\right)}{x} = r$ as in (a) above.

We again assume abitrarily that the traffic level in 1990 was *a* = 100, we know that *y* = 119 (19% growth), and *x* = 13. Substituting these values into the equation above, we get

$\frac{\ln\left(\frac{119}{100}\right)}{13} = r = 0.01338$. The percentage rate is therefore 1.338% (compared to 1.347 in the corresponding part of Ex WS11.2 q.3).

If this growth rate continues from 2003 to 2015 (12 years), with a traffic level in 2003 of 495, we shall have a level in 2015 given by

$y = ae^{rx}$ with *a* = 495, *r* = 0.01338 and *x* = 12. This gives

$y = 495e^{0.16056} = 581.213$ = 581.22 to 2 d.p. (the same answer as in Ex WS11.2 q.3, to 2 d.p.)

(Don't forget that *r* must be entered in proportionate rather than percentage terms in the above formula; that is, 0.01338 not 1.338)

(ii) To find the number of years for 50% growth, we use $y = ae^{rx}$, giving *a* the arbitrary value of 100, so *y* has the value 150 (50% growth), and *r* = 0.01338 from (i) above. This gives

$150 = 100e^{0.01338x}$

We have to solve this for *x*, the unknown number of years required for 50% growth. Dividing both sides by 100 and taking logs on both sides, we get

$\ln 1.5 = 0.01338x$, from which $x = \frac{\ln 1.5}{0.01338} = 30.3038$ (years)

None of the answers is significantly changed by assuming continuous growth in traffic rather than growth in annual jumps. The assumption of continuous growth seems to reflect the real world more accurately, because traffic (measured here as kilometres per year) does in fact increase continuously from day to day. (Note that both formulae assume that traffic grows at a constant rate. In reality, the growth rate varies both within and between years, but any formula that tried to capture this variation would be very complex.)

Exercise WS12.3

1. Between 1955 and 2003, the volume of UK exports of goods (in index number form, with 2001 = 100) grew from 13 to 97.8. (The source of these trade data is the ONS Time Series Data set.)

(a) Calculate the average annual growth rate, assuming (i) continuous growth; (ii) growth in annual jumps.

Answers: (i) Using again the compound growth formula $y = ae^{rx}$, we first transform this into $\frac{\ln\left(\frac{y}{a}\right)}{x} = r$ as in Ex WS12.2 question 1(a) above. With *a* = 13, *y* = 97.8 and *x* = 48, this gives $\frac{\ln\left(\frac{97.8}{13}\right)}{48} = r = 0.04204$; that is, 4.204% per year.

(ii) We use the compound growth formula $y = a(1+r)^x$. Following the steps of Ex WS11.2 question 3(a), we transform this into

$$\frac{\log\left(\frac{y}{a}\right)}{x} = \log(1+r)$$

With *a* = 13, *y* = 97.8 and *x* = 48, this gives

$$\frac{\log\left(\frac{97.8}{13}\right)}{48} = \log(1+r) = 0.018258239$$

We then take anti-logs (see Ex WS11.2 question 3(a)) and get

$1+r = 10^{0.01826} = 1.04294$, so r = 0.04294 or 4.294%

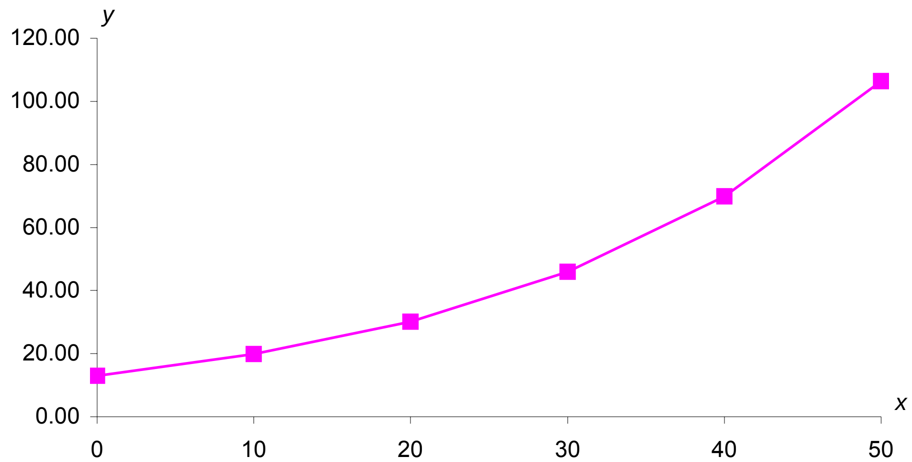
(b) Assuming continuous growth, sketch the graph of the average export volume, *X*, (also known as the trend value) as a function of time. Also sketch the graph of ln *X* against (= as a function of) time.

Note: The wording of this question is potentially confusing because export volume is the dependent variable, which we have labelled as *y* in (a) above, rather than *X*. To avoid this confusion we will continue this labelling below: that is, we will write *y* rather than *X* as export volume.

Answer: The first graph asked for. Given that we have calculated in (a) above the average annual continuous growth rate as 0.04204, the equation giving the trend value of exports is $y = ae^{rx}$, with *a* = 13 and *r* = 0.04204; that is: $y = 13e^{0.04204x}$. (For the given values of *a* and *r*, this equation gives the volume of exports, *y*, as a function of time, *x*). For $y = 13e^{0.04204x}$ we can draw up the following table of values (using a calculator):

Table of values for $y = 13e^{0.04204x}$						
x (years)	0	10	20	30	40	50
y (exports)	13.00	19.79	30.14	45.89	69.86	106.37

From this table we can sketch the graph below:



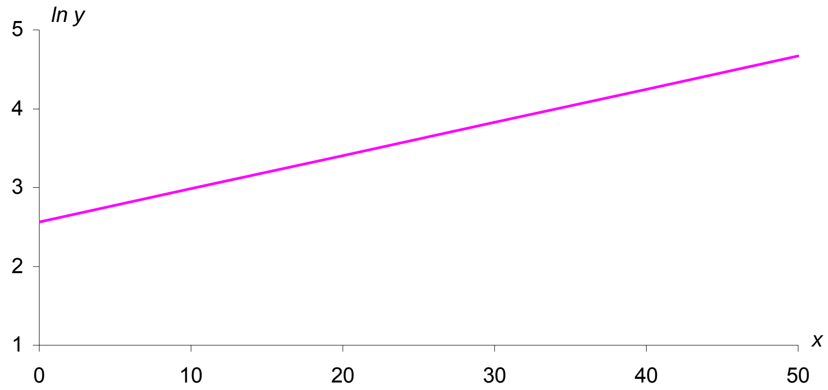
Note that our graph is not a smooth curve (as it is in theory), because the table of values gives us only a small number of points lying on the curve, which we have joined together by straight lines. By increasing the number of points, we could make the graph smoother.

Answer: The second graph asked for. Given $y = 13e^{0.04204x}$, taking natural logs on both sides (and using rules 12.4a and 12.4b) gives

$$\ln y = \ln 13 + 0.04204x$$

In this equation, the dependent variable is the natural log of export volume and the independent variable is time, x. We then construct the following table of values and graph:

x (years)	0	10	20	30	40	50
ln y (log of exports)	2.56	2.99	3.41	3.83	4.25	4.67



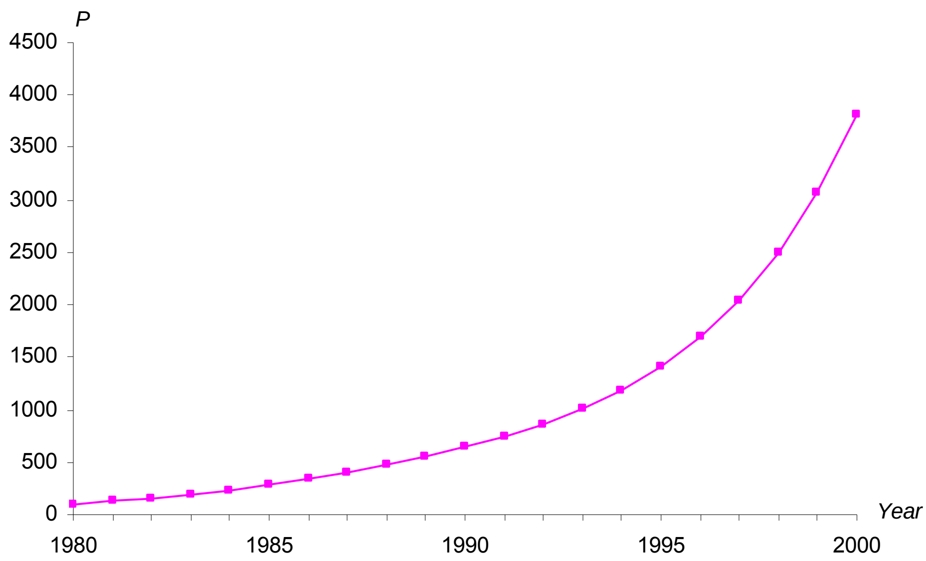
The key point about the graph above is that it is linear, reflecting the assumption that the dependent variable, export volume, grows continuously at a constant rate (see section 12.9 of the book). Moreover, looking at the equation $\ln y = \ln 13 + 0.04204x$ we see that the slope is 0.04204, the proportionate growth rate. (The intercept is ln 13, but this is of no real significance.)

2. The table below gives the price level, *P*, of the Democratic Republic of Ruritania, in index number form with 1980 = 100.

Year	Price index, *P*	Year	Price index, *P*
1980	100.00	1990	643.63
1981	125.00	1991	740.17
1982	155.00	1992	858.60
1983	190.65	1993	1004.56
1984	232.59	1994	1185.39
1985	281.44	1995	1410.61
1986	337.73	1996	1692.73
1987	401.89	1997	2048.20
1988	474.23	1998	2498.81
1989	554.85	1999	3073.54
		2000	3811.18

(a) Draw a reasonably accurate graph of *P* as a function of time, for 1980-2000.

Answer:

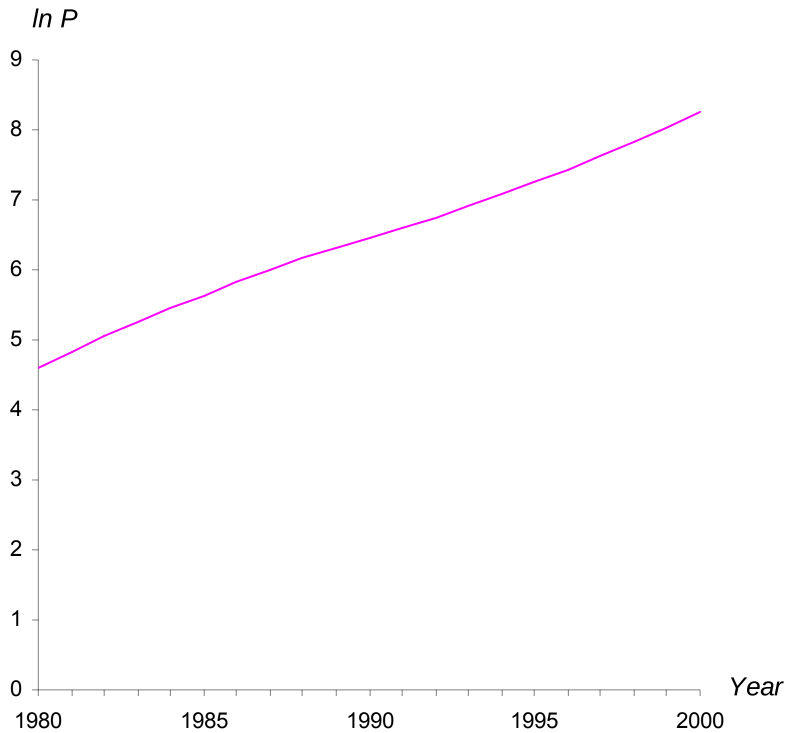


(b) From inspection of the graph, what can you infer about the inflation rate; that is, the growth rate of *P*? Does it appear to be constant, increasing, or declining?

Answer: We can't make a confident inference about the growth rate. The slope of the graph, which measures the *absolute* change in *P* from one year to the next, is clearly increasing. But this is compatible with a *proportionate* change (growth rate) which is increasing, decreasing or constant.

(c) Draw another reasonably accurate graph, this time taking the natural log of *P* as the dependent variable. (You can instead take the common log if you wish.)

Answer:



(d) From inspection of this graph, what can you infer about the growth rate of *P*? Does it appear to be constant, increasing, or declining?

Answer: We know that, with a log scale on the vertical axis, the slope measures the growth rate of the dependent variable (see section 12.11 of book). From inspection of the graph, the curve looks slightly S-shaped; that is, the slope seems to be decreasing in the late 1980s but increasing again in the early 1990s. This means that the growth rate of the price level (= the inflation rate) declined in the late 1980s, then increased again. Overall, though, the curve is roughly linear, indicating that the inflation rate has been fairly constant.

(e) Use your calculator to check the actual year-to-year inflation rate, from the table. (See chapter 9.2 of the book if you need to review how to do this.)

Answer: the growth rate 1980 – 1981 is given by $\frac{125-100}{100}=0.25$. As usual we can multiply this by 100 to get the percentage growth rate, 25%. The remaining year-to-year growth rates, calculated in the same way, are:

Year	Growth rate (%) of *P*
1980	-
1981	25
1982	24
1983	23
1984	22
1985	21
1986	20
1987	19
1988	18
1989	17
1990	16
1991	15
1992	16
1993	17
1994	18
1995	19
1996	20
1997	21
1998	22
1999	23
2000	24
Growth rates are rounded to nearest integer.	

From the table we can see that the growth rate of the price level (the inflation rate) declined continuously through the 1980s, reaching a minimum in 1991, but thereafter accelerated again. This confirms our inference from the graph of ln *P*. (Note that this table, of growth rates, could also be graphed; but the question does not ask us to do this.)

(f) Hence explain briefly why we often use a logarithmic scale to graph a variable growing through time.

Answer: we do this because the slope of the graph then gives the (proportionate) growth rate of the variable in question.

3. Find the present value of 1000 euros due in 10 years' time, discounted continuously at the nominal rate of 5% per year.

Answer: the present value (PV) of *a* due in *x* years time, discounted continuously at a nominal interest rate of *r*, is given by:

$y = \frac{a}{e^{rx}}$ or $y = ae^{-rx}$ where *y* is the PV. So the answer is

$$y = \frac{1000}{e^{0.05(10)}} = \frac{1000}{e^{0.5}} = 606.53$$

Compare this with the present value if discounting occurs (a) once per year; (b) four times per year.

Answer: the present value (PV) of *a* due in *x* years time, discounted *n* times per year at a nominal annual interest rate of *r*, is given by:

$y = \frac{a}{(1+\frac{r}{n})^{nx}}$ or $y = a(1+\frac{r}{n})^{-nx}$ So the answers to (a) and (b) are:

(a) $y = \frac{1000}{(1+\frac{0.05}{1})^{10}} = \frac{1000}{(1.05)^{10}} = 613.91$

(b) $y = \frac{1000}{(1+\frac{0.05}{4})^{40}} = \frac{1000}{(1.0125)^{40}} = 608.41$

4. In the Kingdom of Valetudinaria, expenditure on government provision of health services is currently 5% of GDP. The government wishes to increase this ratio to 8%, the regional average, and accordingly plans to increase health spending by 6% per year until this target is achieved. If GDP grows at 2.5% per year, after how many years will the government's target be reached?

Answer: Let *H* denote health expenditure, and H_0 its initial level (in year zero, or now). Similarly let *Y* denote GDP, and Y_0 its initial level (in year zero, or now). We are given the information that:

(1) $H_0 = 0.05Y_0$ (that is, health spending now is 5% of GDP)

(2) *H* is growing at 6% per year, so assuming continuous growth its level, *H*, *x* years from now will be given by $H = H_0 e^{0.06x}$.

(3) Similarly *Y* is growing at 2.5% per year, so assuming continuous growth its level, *Y*, *x* years from now will be given by $Y = Y_0 e^{0.025x}$.

(4) The government's target is to have: $H = 0.08Y$.

When this target is achieved, we will have (by substituting (2) and (3) into (4)):

(5) $H_0 e^{0.06x} = 0.08 Y_0 e^{0.025x}$

Finally, by substituting (1) into (5) we get:

$0.05 Y_0 e^{0.06x} = 0.08 Y_0 e^{0.025x}$ Dividing both sides by Y_0 gives

(6) $0.05 e^{0.06x} = 0.08 e^{0.025x}$

Equation (6) contains only one unknown, *x*, and we can solve for this. If we divide both sides by $e^{0.025x}$ and also by 0.05 we get

$\frac{e^{0.06x}}{e^{0.025x}} = \frac{8}{5}$ which simplifies to $e^{0.06x-0.025x} = e^{0.035x} = 1.6$

Taking logs on both sides and then dividing by 0.035 we get

$$x = \frac{\ln 1.6}{0.035} = 13.43 \text{ (to 2 d.p.)}$$

that is, between 13 and 14 years from now.