

### Exercise WS11.1

1. Draw sketch graphs of  $y = 10^x$  and  $x = \log_{10} y$  for values of  $x$  from  $-3$  to  $+3$ .

Answer: see the book, fig. 11.2

Show algebraically how one graph may be derived from the other, using the definition of a logarithm.

Answer: given  $y = 10^x$ , then we can say immediately that  $x$  is, by definition, the log of  $y$ , to base 10. We write the statement " $x$  is the log of  $y$ , to base 10" in a more compact form as:  $x = \log_{10} y$  or  $x = \log y$ . (By convention, " $\log y$ " means  $\log_{10} y$ .)

Use the graphs to explain why (i) only positive numbers can have logarithms, and (ii) why the logarithms of numbers between 0 and 1 are negative.

Answer: (i) In fig. 11.2(a) in the book, we can see that the graph of  $y = 10^x$  always lies above the  $x$  axis; that is, there is no value of  $x$  that is associated with a negative value for  $y$ . Since by definition the  $x$  values are the logs of the  $y$  values, it follows that there is no log (value of  $x$ ) associated with a negative number (value of  $y$ ). This may also be seen more directly in fig. 11.2(b).

(ii) In fig. 11.2(a) we can see that when  $x$  is negative,  $y$  lies between 0 and 1. Again, since by definition the  $x$  values are the logs of the  $y$  values, it follows that for any number lying between 0 and 1, its log is negative. This may also be seen more directly in fig. 11.2(b).

2. (a) Sketch the graphs of each of the following functions.

(b) In each case, write down the inverse function.

(i)  $y = (25)10^x$

(ii)  $y = 10^{-0.5x}$

(iii)  $y = -\log 0.5x$

Answers: (i) The graph is identical to fig. 11.2(a) in the book, with all of the values of  $y$  on the vertical axis multiplied by 25. Thus for example at point P we now have  $x = 2$ ,  $y = 2500 (= 100 \times 25)$ . The inverse function is  $x = \log y - \log 25$ .

(ii) The graph is very similar to that of  $y = 10^{-x}$ , shown fig. 11.6 in the book. The difference is that  $x$  must now be twice as large to achieve a given value of  $y$ . Thus we will have  $y = 1000$  when  $x = -6$  (instead of  $-3$ );

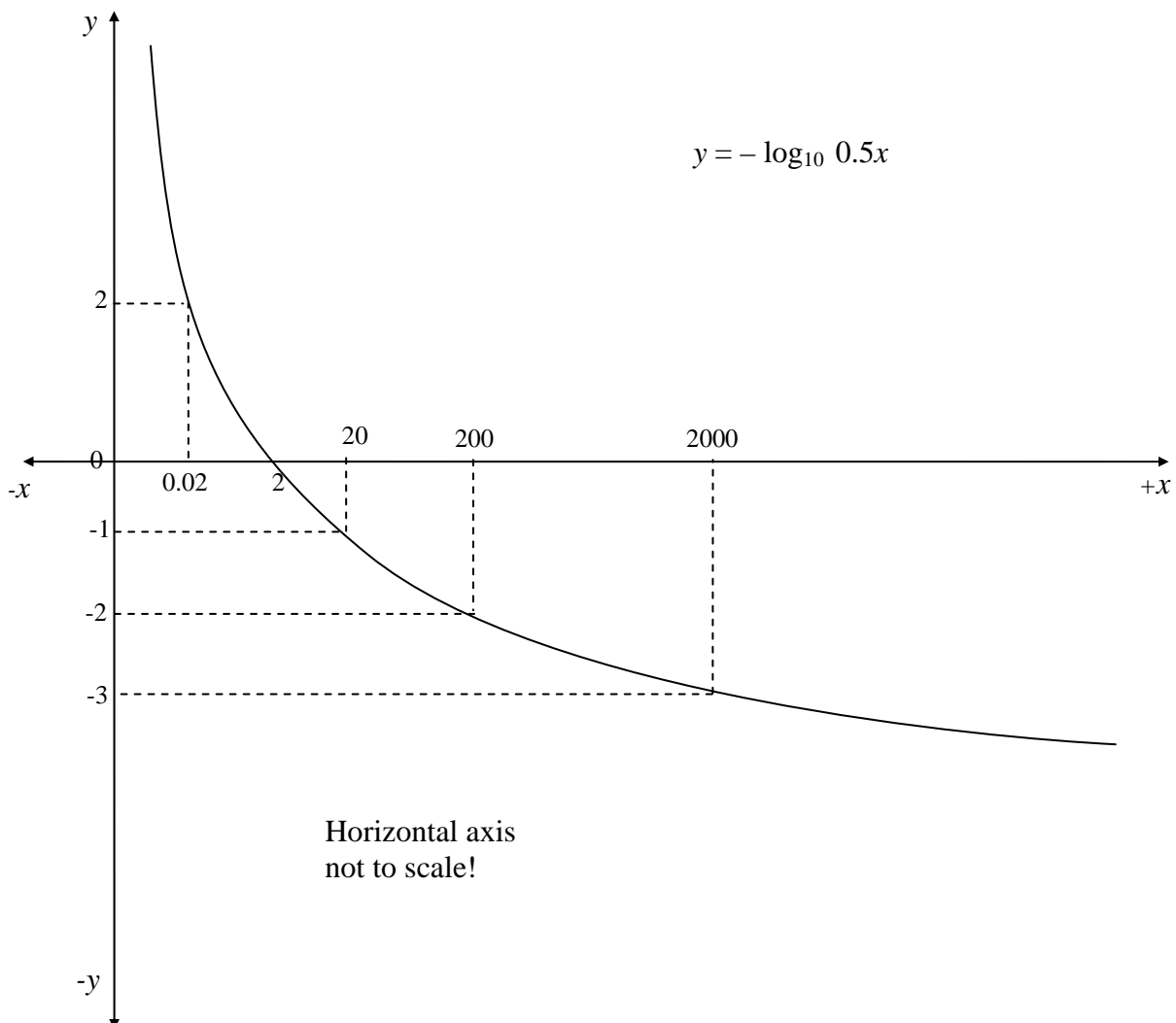
$y = 100$  when  $x = -4$  (instead of  $-2$ );  $y = 0.001$  when  $x = 6$  (instead of  $3$ ); and so on. The only exception is when  $x = 0$ . Since  $-0.5x = -x = 0$  when  $x = 0$ , this means that when  $x = 0$  we have  $y = 10^{-0.5x} = 10^0 = 1$  and  $y = 10^{-x} = 10^0 = 1$ . So the  $y$  intercept of both  $y = 10^{-0.5x}$  and  $y = 10^{-x}$  is at  $y = 1$ .

The inverse function is  $x = -2\log y$ .

(iii) See sketch below for graph of  $y = -\log 0.5x$ . This is obtained from the graph of  $y = \log x$  in fig. 11.3 of the book, with two changes.

First, we "flip" the curve vertically so that, for example, when  $x = 1000$  we now have  $y = -3$  instead of  $+3$ . Second, we now need  $x$  to be twice as large to achieve a given value of  $y$ , so the values  $1, 10, 100, 1000$  on the  $x$  axis are renumbered as  $2, 20, 200, 2000$  and so on.

The inverse function is  $x = (2)10^{-y}$ .



3. Using your graphs drawn in questions 1 and 2 above, estimate the values of the following. Then check your estimates using a calculator.
- (a)  $10^{2.5}$  Answer: from fig. 11.2(a) it is not easy to estimate the value of  $y$  when  $x = 2.5$ , because the vertical axis is not drawn to scale. However it is clear that, if the graph were in fact drawn to scale, the point  $x = 3$ ,  $y = 1000$  would be well off the top of the page. This implies that the value of  $y$  when  $x = 2.5$  is nearer to  $y = 100$  than to  $y = 1000$ . We guess at  $y = 300$ . Our calculator gives  $10^{2.5}$  as 316.22, so our guess wasn't wildly wrong.
- (b)  $10^{0.5}$  Answer: using the same reasoning as in (a) above, in fig. 11.2(a) the value of  $y$  when  $x = 0.5$  must be nearer to  $y = 1$  than to  $y = 10$ . We guess at  $y = 3$ . Our calculator gives  $10^{0.5}$  as 3.1622, so our guess was close. (Another way of guessing this one is to note that  $10^{0.5}$  is the (positive) square root of 10. Since  $3^2 = 9$ , the number we are looking for must be slightly greater than 3, since we want it to equal 10, not 9, when squared).
- (c)  $\log 300$  Answer: Fig. 11.3, the graph of  $y = \log x$ , is relevant here. As in (a) above, we must make some correction for the fact that the horizontal axis is not drawn to scale (for if it were, the point  $x = 1000$ ,  $y = 3$  would be off the page to the right). Allowing for this we see that when  $x = 300$ ,  $y$  will be closer to  $y = 2$  than to  $y = 3$ . We guess at 2.3. Our calculator gives  $\log 300$  as 2.477, so our guess was reasonably close.
- (d)  $\log 15$  Answer: Again using fig. 11.3, the graph of  $y = \log x$ , and allowing for the fact that the horizontal axis is not drawn to scale, we see that when  $x = 15$ ,  $y$  will be closer to  $y = 1$  than to  $y = 2$ . We guess at 1.3. Our calculator gives  $\log 15$  as 1.176, so our guess was too high but reasonably close.

### Exercise WS11.2

1. Attempt the following without using a calculator. Then use a calculator to check your answers.
- (a) Given  $\log 100 = 2$  and  $\log 3 = 0.4771$ , what is  $\log 300$ ?

Answer: We know that  $300 \equiv 100 \times 3$ , so  $\log 300 \equiv \log (100 \times 3)$ . From rule 11.2 in the book,  $\log (100 \times 3) \equiv \log 100 + \log 3$ . We are given these two logs in the question, so we have

$$\log 300 \equiv \log 100 + \log 3 = 2 + 0.4771 = 2.4771.$$

Our calculator confirms this.

- (b) Given  $\log\left(\frac{100}{x}\right) = -1$ , what is  $x$ ?

Answer: from rule 11.2(c) in the book, we have  $\log\left(\frac{100}{x}\right) \equiv \log 100 - \log x$ .

We know that  $\log 100 = 2$  (because  $10^2 = 100$ ), so we have

$\log\left(\frac{100}{x}\right) \equiv \log 100 - \log x \equiv 2 - \log x$ . Using this in the equation we are given, we get

$$2 - \log x = -1$$

From this, by elementary algebra,  $\log x = 3$ . Therefore  $x = 10^3 = 1000$ .

- (c) Given  $\log 3 = 0.4771$ , what is (i)  $\log 0.3$ , and (ii)  $\log 30$ ?

Answers: (i) Since  $0.3 \equiv \frac{3}{10}$ ,  $\log 0.3 \equiv \log\left(\frac{3}{10}\right) \equiv \log 3 - \log 10$  (using rule 11.2(c)). Since  $\log 10 \equiv 1$  (because  $10^1 = 10$ ), and we are given  $\log 3$  in the question, we have

$$\log 0.3 \equiv \log 3 - \log 10 = 0.4771 - 1 = -0.5229.$$

(Our calculator confirms this).

(ii) Since  $30 \equiv 3 \times 10$ ,  $\log 30 \equiv \log(3 \times 10) \equiv \log 3 + \log 10$  (using rule 11.2(a)). Since  $\log 10 \equiv 1$  (because  $10^1 = 10$ ), and we are given  $\log 3$  in the question, we have

$$\log 30 \equiv \log 3 + \log 10 = 0.4771 + 1 = 1.4771.$$

(Our calculator confirms this).

2. Solve the equations:

- (a)  $105^{0.5x} = 200$

Answer: (See example 11.2 in the book). Taking logs on both sides gives

$$\log(105^{0.5x}) = \log(200).$$

$$0.5x \log(105) = \log(200)$$

After dividing both sides by  $0.5 \log(105)$  this becomes

$$x = \frac{\log(200)}{0.5\log(105)}. \text{ Using our calculator, we get}$$

$$x = \frac{\log(200)}{0.5\log(105)} = \frac{2.3010}{1.0106} = 2.2769.$$

Our calculator confirms this, giving  $105^{1.1385} = 200$

(b)  $200 = 100(1+r)^{20}$

Answer: (See example 11.3 in the book, though this is not identical).  
Dividing both sides by 100, and then taking logs on both sides, we get

$$\log 2 = \log \left[ (1+r)^{20} \right]. \text{ Using rule 11.2(b) this becomes}$$

$$\log 2 = 20\log(1+r). \text{ Dividing both sides by 20 gives}$$

$$\frac{\log 2}{20} = \log(1+r). \text{ Since } \log 2 = 0.3010 \text{ from our calculator, we have}$$

$$\frac{0.3010}{20} = 0.01505 = \log(1+r)$$

We now have to take anti-logs on both sides; that is, reverse the process of taking logs. The right hand side becomes  $(1+r)$  and the left hand side becomes  $10^{0.01505} = 1.03526$  (using calculator). So we have

$$1.03526 = 1+r. \text{ Subtracting 1 from both sides gives } r = 0.03526.$$

Note that this is an example of the compound growth formula. We have found that if a variable has an initial value of 100 and grows to 200 over 20 years, the annual proportionate growth rate is 0.03526. To find the annual percentage growth rate we multiply this by 100, giving 3.5626%.

Note also that there is a simpler method of solving this type of problem without using logs. Dividing both sides by 100 and then raising both sides to the power  $\frac{1}{20}$ , we get

$$(2)^{\frac{1}{20}} = \left[ (1+r)^{20} \right]^{\frac{1}{20}}. \text{ Using rule 2.4 (chapter 2) of the book, the right hand}$$

$$\text{side becomes } \left[ (1+r)^{20} \right]^{\frac{1}{20}} = (1+r)^{20 \times \frac{1}{20}} = (1+r)^1 = 1+r. \text{ Also, using our}$$

$$\text{calculator, the left hand side is } (2)^{\frac{1}{20}} = 2^{0.05} = 1.03526$$

So we have  $1.03526 = 1+r$ , as before.

(c)  $50^2 = 25^x$

Answer: taking logs on both sides, we get  $\log(50^2) = \log(25^x)$ . Applying rule 11.2(b), this becomes  $2\log(50) = x\log(25)$ . Dividing both sides by  $\log(25)$  and using calculator gives

$$x = \frac{2\log(50)}{\log(25)} = \frac{3.3979}{1.3979} = 2.4306.$$

Our calculator confirms that  $25^{2.4306} = 2500$  and  $50^2 = 2500$ .

3. The levels of traffic of various types on UK roads (defined as total distance travelled, in billions of kilometres, by all vehicles in the chosen type) was as follows:

Vehicle type	1980	2003
All vehicles	277	495
Light vans	26	58
Heavy goods vehicles	20	28
Buses and coaches	3.5	5.4
Source: www.dft.gov.uk		

- (a) Calculate the average annual growth rate for each type of vehicle.

Answer: We use the compound growth formula  $y = a(1+r)^x$ . To find the growth rate,  $r$ , we first divide both sides by  $a$  and take logs on both sides, giving:  $\log\left(\frac{y}{a}\right) = \log\left[(1+r)^x\right]$

Using rule 11.2(b) on the right hand side, this becomes:

$$\log\left(\frac{y}{a}\right) = x\log(1+r), \text{ from which by elementary algebra}$$

$$\frac{\log\left(\frac{y}{a}\right)}{x} = \log(1+r)$$

For the "all vehicles" category, this gives

$$\frac{\log\left(\frac{495}{277}\right)}{23} = \log(1+r) = 0.01096$$

We then take anti-logs (that is, reverse the process of taking logs) on both sides. The anti-log of  $\log(1+r)$  is, of course, simply  $(1+r)$ , while the anti-log of 0.01096 is  $10^{0.01096} = 1.02556$ . So we have

$1+r = 1.02556$ , from which  $r = 0.02556$ . This is the average proportionate yearly growth rate; to get the percentage rate we multiply by 100, giving 2.556%.

Note that on our calculator we can find the anti-log of 0.01096 in two ways: (i) by keying in 10, then the power sign  $\wedge$ , then 0.01096, then equals; (ii) by keying in "shift" (or "inv" depending on calculator model), then log, then 0.01096, then equals. "Shift" followed by "log" reverses the process of taking a log; that is, it finds the anti-log of the number you key in.

Using the same method, the growth rates for the other categories are:

Light vans 3.55%; Heavy goods vehicles 1.47%; Buses & coaches 1.90%.

- (b) Between 1990 and 2003 the increase in traffic for all vehicles was 19%. If this growth rate continues, (i) calculate what the level of traffic will be in 2015; (ii) after how many years will traffic have increased to 50% above its 2003 level?

Answers (i) Using again the compound growth formula  $y = a(1+r)^x$ , we

first transform this into  $\frac{\log\left(\frac{y}{a}\right)}{x} = \log(1+r)$  as in (a) above.

If we assume arbitrarily that the traffic level in 1990 was  $a = 100$ , we know that  $y = 119$  (19% growth), and  $x = 13$ . Substituting these values into the equation above, we get

$$\frac{\log\left(\frac{119}{100}\right)}{13} = \log(1+r) = 0.005811305. \text{ By the method of (a) above,}$$

$$r = 0.01347 \text{ or } 1.347\%.$$

If this growth rate continues from 2003 to 2015 (12 years), with a traffic level in 2003 of 495, we shall have a level in 2015 given by

$$y = a(1+r)^x \text{ with } a = 495, r = 1.347 \text{ and } x = 12. \text{ This gives}$$

$$y = 495(1+0.01347)^{12} = 581.2$$

(Don't forget that  $r$  must be entered in proportionate rather than percentage terms in the above formula; that is, 0.01347 not 1.347)

(ii) To find the number of years for 50% growth, we use  $y = a(1+r)^x$  with  $a$  given the arbitrary value of 100, so  $y$  has the value 150 (50% growth), and  $r = 0.01347$  from (i) above. This gives

$$150 = 100(1+0.01347)^x$$

We have to solve this for  $x$ , the unknown number of years required for 50% growth. Dividing both sides by 100 and taking logs on both sides, we get

$$\log 1.5 = x \log 1.01347, \text{ from which } x = \frac{\log 1.5}{\log 1.01347} = 30.3 \text{ (years)}$$