Exercise WS10.1

1. Find the 15th and 25th terms of the series: 3, 6, 9, 12,...,…

   This is an arithmetic series (= arithmetic progression, or AP) with first term A = 3 and common difference $d = 3$
   The formula for the nth term of an arithmetic series is $A + (n-1)d$. So the 15th term is $3 + (15-1)3 = 45$. Similarly the 25th term is $3 + (25-1)3 = 75$.

2. Find the sum of the first ten terms of the series: $\frac{1}{2}, \frac{3}{4}, 1, 1\frac{1}{4}, ...$

   This is an arithmetic series (= arithmetic progression, or AP) with first term $A = \frac{1}{2}$ and common difference $d = \frac{1}{4}$. The formula for the sum of the first n terms (denoted by $\sum_n$) of an arithmetic series is $\sum_n = \frac{n}{2}\left[2a + (n-1)d\right]$. So the sum of the first 10 terms is $\sum_{10} = \frac{10}{2}\left[2\left(\frac{1}{2}\right) + (10-1)\frac{1}{4}\right] = 16\frac{1}{4}$

3. Find the sum to infinity of the series: $1, \frac{1}{2}, \frac{1}{4}, \frac{1}{8}, ...,$

   This is a geometric series (= geometric progression, or GP) with first term A = 1. By dividing each term by the preceding term, we find R, the common ratio, as $R = \frac{1}{2}$. (For example, $\frac{1}{8} \div \frac{1}{4} = \frac{1}{8} \times \frac{4}{1} = \frac{1}{2}$). The general formula for the sum of the first n terms of a GP is $\sum_n = \frac{A(1-R^n)}{1-R}$. However, here we have the special case where R lies between 0 and 1, and the number of terms in the sum is infinite. Such a sum is called the sum to infinity. The formula for the sum to infinity is $\sum_\infty = \frac{A}{1-R}$ (see chapter 10, rule 10.3, in the book). So in this example, the required sum is: $\sum_\infty = \frac{1}{1-\frac{1}{2}} = 2$.

4. Given the following reduced form macroeconomic model:

$Y = C + I$

$C = 0.8Y_{-1} + 100$

where Y = national output (= national income); I = aggregate investment; C = aggregate consumption (all in the current time period); Y_{-1} = national income in the previous time period.

(a) What condition must be satisfied for this economy to be in equilibrium? (This condition, in the form of an equation, is known as an equilibrium condition.)

"Equilibrium" can have many definitions according to the context. Here we take the view that the economy cannot be in equilibrium if aggregate output and income, Y, is increasing or decreasing continuously. So for equilibrium we require $Y = Y_{-1}$ (that is, income in the current time period must be the same as income in the previous period). Thus to the two equations above we can add a third equation, the equilibrium condition $Y = Y_{-1}$.

(b) Find the equilibrium levels of Y and C, if $I = 900$.

Taking (a) above into account, our model is now

$Y = C + I$ (1)

$C = 0.8Y_{-1} + 100$ (2)

$Y = Y_{-1}$ (3)

$I = 900$ (4)

Substituting (4) into (1) gives (1a): $Y = C + 900$. Substituting (3) into (2) gives (2a): $C = 0.8Y + 100$. Using (2a) to substitute for C in (1a) gives

$Y = 0.8Y + 100 + 900$, with solution $Y = \frac{1000}{0.2} = 5000$. From (3), $Y_{-1} = Y = 5000$

Therefore, from (2), $C = 0.8Y_{-1} + 100 = 4100$, and $I = 900$ (given).

(c) If *I* then rises to 1000 and remains at this level, trace the paths of *C* and *Y* over the next 6 time periods.

Time period	$C = 0.8Y_{-1}+100$	*I*	*Y=C+I*	change in *C*	change in *I*	change in *Y*	sum of changes in *Y*
0	4100	900	5000	0.00	0	0.00	0.00
1	4100	1000	5100	0.00	100	100.00	100.00
2	4180	1000	5180	80.00	0	80.00	180.00
3	4244	1000	5244	64.00	0	64.00	244.00
4	4295	1000	5295	51.20	0	51.20	295.20
5	4336	1000	5336	40.96	0	40.96	336.16
6	4369	1000	5369	32.77	0	32.77	368.93
7	4395	1000	5395	26.21	0	26.21	395.14

Notes to table: The values for time period 0 are from (b) above. We assume the levels of *Y*, *C*, and *I* in period zero are the same as in the previous (period –1). In period 1, investment is assumed to rise exogenously by 100 and thereafter remain at its new level. This raises *Y* by 100 in period 1. This rise in *Y* in turn raises *C* by 80 (= 0.8 × 100) in period 2. Since *I* is unchanges, this rise in *C* raises *Y* by 80 too. This rise in *Y* of 80 in turn raises *C* by 64 (= 0.8 × 80) in period 3. And so on. The key point is to see that consumption rises in each period, but by decreasing amounts (100, 80, 64,..,..)

(d) What are the levels of *C* and *Y* in the new equilibrium? How long does it take to reach this new equilibrium? How near to equilibrium is the economy after 12 time periods have passed?

The changes in *Y* (column 7 of the table) form the series

100, 100 × 0.8, 100 × 0.8 × 0.8, 100 × 0.8 × 0.8 × 0.8,…, This is a GP with 1st term A = 100 and common ratio R = 0.8. As R is lies between 0 and 1, the sum to infinity is $\sum_{\infty} = \frac{A}{1-R} = \frac{100}{1-0.8} = \frac{100}{0.2} = 500$. As the initial equilibrium level of Y was 5000, the new equilibrium level is 5000 + 500 = 5500. This is also the value of $Y_{-1,}$ since otherwise income would be changing from one period to the next and the economy would therefore not be in equilibrium. The new equilibrium level of C is $C = 0.8Y_{-1} + 100$ = 4500, with I = 1000.

But, this level of Y is reached only after an infinite number of time periods have passed; that it, it is never reached. The reason for this is that although the changes in Y in column 7 of the table are getting smaller as time passes, the change is always positive. So we never actually reach equilibrium as this requires $Y = Y_{-1}$ as discussed above. However, after enough time periods we will reach a point where the change in Y from one period to the next is so small that we are willing to ignore it. Then we can say that the new equilibrium has been reached.

After 12 periods, the sum of the changes in Y is given by

$$\sum_{n} = \frac{A(1-R^n)}{1-R} = \frac{100(1-0.8^{12})}{1-0.8} = 465.64$$

Thus after 12 periods the increase in Y is already about 93% of the final increase (because 465.64 is about 93% of 500).

(e) Do you regard this model as reasonable in its assumptions? Give reasons for your answer.

The key feature of this macreconomic model is its assumption that consumption is determined by the previous period's income. For households with no assets and no ability to borrow this seems not unreasonable.

Exercise WS10.2

1. A firm wishes to increase its output from 1000 to 2000 units per week. Due to labour shortages and other supply side constraints, it can increase output by a maximum of 100 units per week. After 7 weeks, how near to its target will the level of output be?

The firm's weekly output series will be:

1000, 1100, 1200, 1300,…,…

This is an arithmetic progression (AP) with first term *A* = 1000 and common difference *d* = 100. The formula for the *n*th term of an arithmetic series is $A+(n-1)d$. After 7 weeks it will have reached the 8th term of the series, given by 1000 + (8 – 1)100 = 1700.

2. Suppose in question 1 that the supply side constraints limited the growth of output to 10% per week. After 7 weeks, how near to its target will the level of output be? Explain the difference between your answers to questions 1 and 2.

The output levels now form the series: 1000, 1100, 1210, 1331, …, …

This is a geometric series (= geometric progression, or GP) with first term *A* = 1000 and common ratio, *R* = 1.1. (*R* is found by dividing any term by the preceding term.) The formula for the *n*th term of such a series is AR^{n-1}. So the 8th term is $1000(1.1^7) = 1000(1.9487) = 1948.7$

In question 2 the weekly increase is a constant proportion of the previous week's output, the absolute increase gets larger as output gets larger. So the total increase after 7 weeks is much larger in question 2 than in question 1.

3. The price level in the UK is currently increasing by about 2.5% per year. Show that if this inflation rate continues for 10 years, the price level will have risen by 28%. (Hint: Give the initial price level an arbitrary value, such as 100.)

This is an example of a constant compound growth rate, so the formula required is $y = a(1+r)^x$. Here we have *a* = initial value of the variable =100 (say), *r* its annual growth rate = 2.5% or 0.025 in proportionate terms, and *x* =10, the number of years. So solution is $y = 100(1+0.025)^{10} = 128.008$ (to 3 dp). So the proportionate increase is $\frac{128-100}{100} = 0.28$ and the percentage increase is $\frac{128-100}{100} \times 100 = 28$.

4.(a) If I deposit 500 euros in a bank account paying interest of 6% per year, how much will I get back after 10 years, including compounded interest? (Assume interest is added once per year.)

When the constant compound growth occurs in a single annual jump, the formula required is $y = a(1+r)^x$. Here we have *a* = 500, *r* = 6% or 0.06 in proportionate terms, and *x* =10, the number of years. So solution is $y = 500(1+0.06)^{10} = 895.42$ (to 2 dp).

(b) Repeat (a) above with the assumption that interest is added quarterly (that is, every 3 months). What is the effective annual interest rate (EAR) in this case?

The formula required is $y = a(1+\frac{r}{n})^{nx}$ where *n* is the number of times per year that growth jumps occur. So solution is
$y = 500(1+\frac{0.06}{4})^{4(10)} = 500(1.814) = 907$

Using rule 10.6, the *EAR* is $EAR = (1+\frac{r}{n})^n - 1 = (1+\frac{0.06}{4})^4 = 6.136\%$ (to 3 dp).

(c) Repeat (b) above, assuming that interest is added monthly.

Formula is same as (b) but with *n* = 12. So solution is
$y = 500(1+\frac{0.06}{12})^{12(10)} = 500(1.8194) = 909.70$.

EAR = 6.168% to 3 dp.

Note that difference between answers to (a) and (b) is much larger then difference between answers to (b) and (c)

5. I deposit 10,000 euros in a bank account that pays interest of 5% per year (credited annually in arrears; that is, at the end of each year). At the end of each year I withdraw and spend x percent of my bank balance at that time. After 10 years I find that I have only 100 euros left in the bank. What is x?

To get the *overall* growth rate, we use the formula $y = a(1+r)^x$ with *a* =10,000, *y* = 100, *x* = 10. This gives $100 = 10000(1+r)^{10}$. This can be rearranged as $\frac{100}{10000} = (1+r)^{10}$, from which $\left(\frac{100}{10000}\right)^{\frac{1}{10}} = 1+r$ = 0.63905. So $r = 0.63905 - 1 = -0.36095$ = – 36.095%. That is, my bank balance declined by 36.095% per year. But this is the *overall* (negative) growth rate. As the bank also paid 5% per year interest, I must have withdraw 36.095% + 5% = 41.095% per year. (Note that this assumes that, at the end of each year, two things happened: (i) the bank paid interest equal to 5% of the capital at the beginning of the year, and (ii) that I withdrew 41.095% of the capital at the beginning of the year.)

Exercise WS10.3

1. Assuming that I can borrow or lend any amount at 10% per annum, with interest paid or received annually in arrears, what is the present discounted value (PV) to me of £10 000 due 20 years from now?

   The present value formula for a single future payment or receipt is
   $y = \frac{a}{(1+r)^x}$, where y is the present value (PV). This gives
   $y = \frac{10000}{(1+0.1)^{20}} = 1486.44$

2. In the context of a British government policy entitled the Private Finance Initiative, the National Health Service can acquire a new hospital either by buying it now for a single payment of £100 million, or by instalments of £7 million per year for the next 30 years. If the government can borrow or lend at 6% per year, and all risks and uncertainty are ignored, which is the cheaper method of acquiring the hospital?

   We answer this by comparing the PVs of the two options. Option (i): immediate outright purchase. The PV of £100m paid or received now (that is, zero years from now) is $y = \frac{100m}{(1+r)^0} = \frac{100m}{1} = 100m$. (Note that we don't need to know the interest rate, as the denominator is always equal to 1.)

   Option (ii): payment by instalments. The PV of £7m a year for 30 years, discounted at 6% per year, measured in millions of pounds (= £million), is

   $$PV = \frac{7}{(1.06)^1} + \frac{7}{(1.06)^2} + \frac{7}{(1.06)^3} + \ldots + \frac{7}{(1.06)^{30}}$$

   This is a GP with first term $A = \frac{7}{1.06}$, common ratio $R = \frac{1}{1.06}$, and $n = 30$.

   Using the formula for this sum, we get

   $$\sum_n = \frac{\frac{7}{1.06}\left(1-\left(\frac{1}{1.06}\right)^{30}\right)}{1-\frac{1}{1.06}} = \frac{6.0368(1-(0.9434)^{30})}{1-0.9434} = 96.36 \text{ (£million)}$$

   As the PV of the instalment method is less than the PV of the outright purchase, we conclude that the former is the better buy.

3. How is your answer to question 2 changed if the relevant discount rate falls from 6% to 5%? Explain your answer.

Recalculating option (ii) above gives the new PV as 107.61. The PV of the immediate outright purchase is, of course, left unchanged. As the PV of the instalment method is now greater than the PV of the outright purchase, we conclude that the former is no longer the better buy. Because the interest rate appears in the denominator in the PV formula, the PV of all future payments is increased when the interest rate falls.

4. Calculate the present value (PV) of the following income series, discounted once-yearly at 4% per year.

Years fom now	1	2	3	4	5	6	7	8
Income series	200	200	150	150	100	100	50	50

The formula is: $PV = \frac{200}{(1.04)^1} + \frac{200}{(1.04)^2} + \frac{150}{(1.04)^3} + \frac{150}{(1.04)^4} + \frac{100}{(1.04)^5} + \frac{100}{(1.04)^6}$

$$+ \frac{50}{(1.04)^7} + \frac{50}{(1.04)^8} = 874.54$$

5. (a) A 'perpetual' bond issued by the government pays interest of 10 euros per year, forever, the first payment due one year from now. Assuming that there is no risk that the government will default on these payments, what will be the market value of the bond if the market rate of interest on riskless loans is 5% per year?

As this flow of future receipts continues forever, the *PV* is

$$PV = \frac{a}{(1+r)^1} + \frac{a}{(1+r)^2} + \frac{a}{(1+r)^3} + \ldots + \ldots$$

where *a* = 10, *r* = 0.05.

As we saw in Ex WS10.1 question 4, this is an infinite GP with 1st term $A = \frac{a}{1+r}$ and common ratio $R = \frac{1}{1+r}$. The formula for the infinite sum is

$$\sum_{\infty} = \frac{A}{1-R} = \frac{\frac{a}{1+r}}{1-\frac{1}{1+r}} = \frac{\frac{10}{1.05}}{1-\frac{1}{1.05}} = 200$$

(b) If the market rate of interest doubled or halved, calculate the resulting percentage change in the market value of the bond. Derive the equation giving the relationship between the market rate of interest and the market value of the bond, and sketch its graph.

As shown in the book, ch. 10, the formula above can be rearranged as

$$\sum_{\infty} = \frac{A}{1-R} = \frac{\frac{a}{1+r}}{1-\frac{1}{1+r}} = \frac{\frac{a}{1+r}}{\frac{1+r-1}{1+r}} = \frac{a}{r}$$

From this we can see that if initially we had $a = a_0$ and $r = r_0$, then the initial *PV* would be be $PV_0 = \frac{a_0}{r_0}$. If the interest rate then rises to $r_1 = 2r_0$, the new *PV* is $PV_1 = \frac{a_0}{2r_0} = \frac{1}{2}\frac{a_0}{r_0}$. Thus doubling the interest rate halves the *PV* of a perpetual income stream. Similarly halving the interest rate doubles the *PV*.

(c) Repeat (a) with the assumption that the first payment of interest is due tomorrow. Compare and explain your two answers.

The payment due tomorrow (effectively, now) has a *PV* of

$$PV = \frac{a}{(1+r)^0} = \frac{10}{(1.05)^0} = 10$$ (see also question 2 above)

The *PV* of the other receipts is unchanged, so the new *PV* is the answer to (a) + 10; that is, 210.

Exercise WS10.4

1. I borrow 10 000 euros to buy a car, the loan to be repaid in 60 equal monthly instalments consisting of interest and capital repayment. Interest will be charged every month at 1.5% of the outstanding debt. Calculate (a) the amount of each monthly instalment; (b) the total amount of interest that I will pay; (c) the effective annual interest rate (EAR).

(a) We first calculate the capital repayment at the end of the first month (P_1) using rule 10.11 from the book; that is $P_1 = K\left[\frac{r}{(1+r)^x - 1}\right]$. Here *K* = 10000, *r* = 0.015 (15%), *x* = 60. (Note we are working here with time measured in months). This gives $P_1 = 10000\left[\frac{0.015}{(1.015)^{60} - 1}\right] = 103.93$.

Then we calculate the interest paid at the end of the first month, which is simply 1.5% of 10000, which equals 150. So the total payment at the end of the first month is 103.93 + 150 = 253.93. As the formula was developed to make each monthly payment equal, we know that 253.93 is the payment in each of the 60 months.

(b) The total amount paid is 253.93 × 60 = 15236.06. But this includes the capital repayment of 10000. So total interest paid is 15236.06 – 10000 = 5236.06.

(c) Using rule 10.6 from the book, the EAR formula is $EAR = (1+\frac{r}{n})^n - 1$, where r is the nominal annual interest rate and n is the number of times per year that interest is added or calculated. Here we know that the monthly interest rate is 0.015 (1.5%), so r = 0.015 × 12 = 0.18 (18%), and n =12. So $EAR = (1+\frac{0.18}{12})^{12} - 1 = 0.19562\ (19.562\%)$.

2.(a) In question 1, by how much would my total interest payments be reduced if I increased my initial deposit by 1 000 euros and thus borrowed only 9 000 instead of 10 000 euros?

Re-working question 1 with an initial debt of 9000, the first month capital repayment is 93.54, the first month interest repayment is 135, total 228.54. Total repayments are 228.54 × 60 = 13712.4. Of this, 9000 is repayment of capital, so total interest payment is 4712.4.

(b) In question 1, what would be the EAR if the monthly rate rose to 2%? Does the EAR rise in the same proportion as the rise in the monthly rate?

New $EAR = (1+\frac{0.24}{12})^{12} - 1 = 0.2682\ (26.82\%)$

3. Repeat question 5(a) from exercise WS10.3 assuming that the bond, instead of being perpetual, will be redeemed (= repurchased) at its nominal or face value of 100 euros by the government in 10 years' time. What would the bond be worth if it were due to be redeemed tomorrow?

As the sum is now finite, to calculate the PV of the interest payments over the 10 years we use rule 10.2b from the book: $\sum_n = \frac{A(1-R^n)}{1-R}$, with $A = \frac{a}{1+r} = \frac{10}{1.05}$, common ratio $R = \frac{1}{1+r} = \frac{1}{1.05}$, and n = 10. This gives:

$$\sum_n = \frac{A(1-R^n)}{1-R} = \frac{9.5238(1-(0.9524)^{10})}{1-0.9524} = 77.22$$

To this we must add the PV of the capital redemption in 10 years' time. This PV is $\frac{100}{(1.05)^{10}} = 61.39$.

So the market value of the bond is the sum of these two, 138.61.

If we treat tomorrow as effectively now, and there were no more interest payments to be made, the PV would be simply 100. If we can borrow and lend for 1 day at a time, at a nominal interest rate of 5% per year, the PV of

100 due in one day's time is: $\frac{100}{(1+\frac{0.05}{365})^1} = 99.986$. (This calculation takes the time unit as a day, so uses the daily interest rate.)