

### Exercise WS7.1

1. A function is monotonic increasing if its derivative is positive for all values of  $x$ .  
Here,  $\frac{dy}{dx} = 3$ . This is always positive so the function is monotonic increasing.
2. Given  $y = x^2 + 4$ ,  $\frac{dy}{dx} = 2x$ . This is positive when  $x$  is positive and negative when  $x$  is negative. So the function is increasing (positively sloped) when  $x$  is positive and decreasing (negatively sloped) when  $x$  is negative.
3. Here  $\frac{dy}{dx} = 3x^2$ . This is positive when  $x$  is positive and also positive when  $x$  is negative. So the function is increasing (positively sloped) when  $x$  is either positive or negative. When  $x = 0$ ,  $\frac{dy}{dx} = 3x^2 = 0$ . At this point the function is neither increasing nor decreasing.

### Exercise WS7.2

1. Given  $y = -x^2 + 6x - 12$ ,  $\frac{dy}{dx} = -2x + 6$ . To find local max or min values, we set this equal to zero, giving  $-2x + 6 = 0 \Rightarrow x = 3$ . The value of  $y$  at this point is  $y = -(3)^2 + 6(3) - 12 = -3$

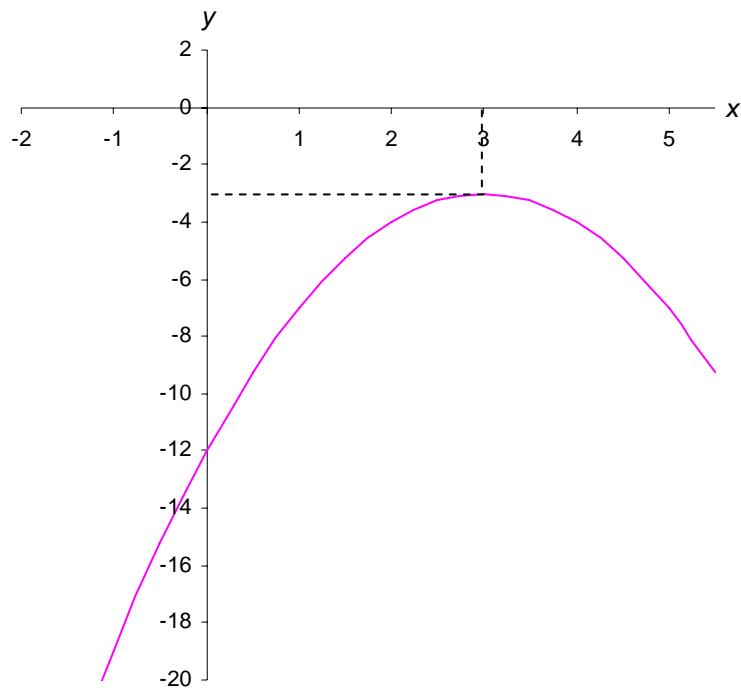
We also have  $\frac{d^2y}{dx^2} = -2$ , which is negative when  $x = 3$ , so this point is a local maximum of  $y$ .

To help us sketch the graph we now have the following information:

- (i) We know that  $y = -x^2 + 6x - 12$  is a quadratic function and since the coefficient of  $x^2$  is negative, the graph has the shape of an inverted (upside down) U.
- (ii) We know that it intercepts the  $y$  axis at  $y = -12$ .
- (iii) We know that its maximum is at  $x = 3$ ,  $y = -3$

Combining these pieces of information, we can sketch the graph below.

Ex WS7.2 question 1

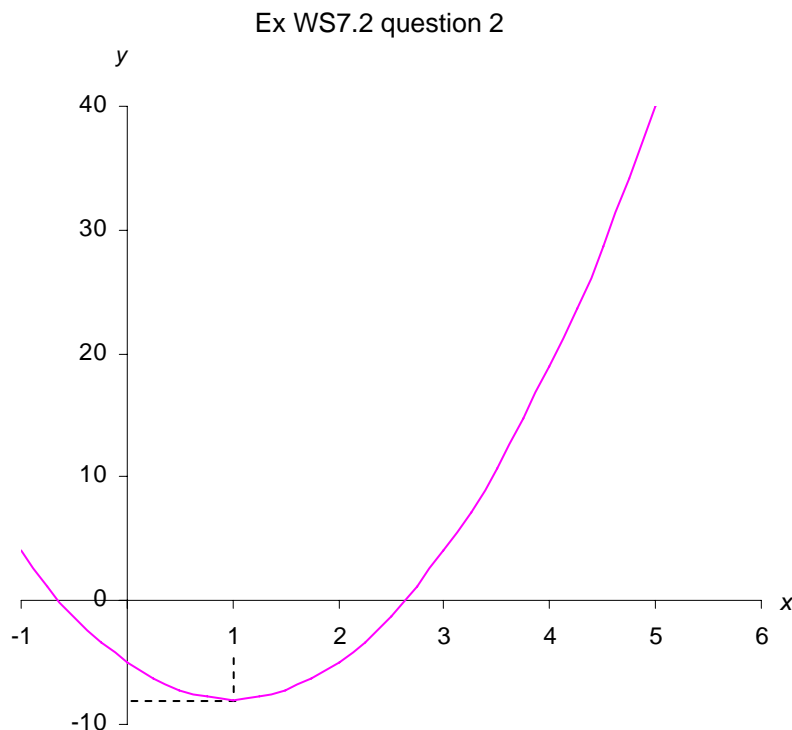


Note that we might try to make the sketch more accurate by solving the quadratic equation  $-x^2 + 6x - 12 = 0$ . The solutions would give us the intercepts of the curve on the  $x$  axis. However when we attempt to solve this equation using the formula, we find it has no real roots. This tells us that the graph of  $y = -x^2 + 6x - 12$  never cuts the  $x$ -axis, confirming our sketch graph above.

2. Given  $y = 3x^2 - 6x - 5$ , we have  $\frac{dy}{dx} = 6x - 6$ . To find local max or min values, we set this equal to zero and solve, giving  $x = 1$ . At this point,  $y = 3(1)^2 - 6(1) - 5 = -8$

We also have  $\frac{d^2y}{dx^2} = \frac{d}{dx}(6x - 6) = 6$ . This is positive when  $x = 1$ , so this point is a local minimum of  $y$ .

We can now sketch the graph, using the fact that the curve has a local minimum at  $x = 1, y = -8$ . We also know that the  $y$  intercept is at  $y = -5$ .



Note that we can make the sketch more accurate by solving the quadratic equation  $-3x^2 - 6x - 5 = 0$ . Using the formula, the solutions are  $x = 2.633$  and  $x = -0.633$  (to 3 decimal places). These are the intercepts of the curve on the  $x$  axis. The intercepts on the graph appear to have these values (though we can't be sure because the graph can never be drawn completely accurately). Thus we conclude that the graph is correctly drawn.

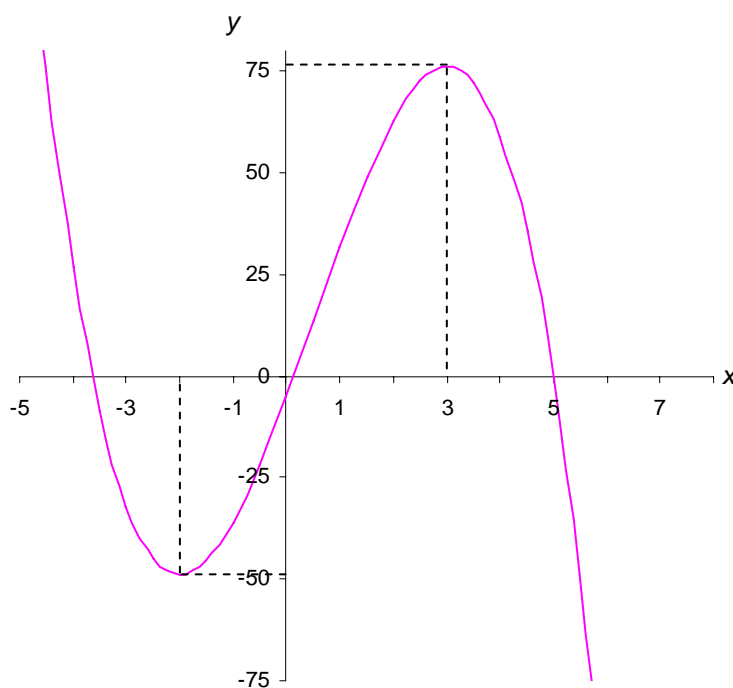
3. Given  $y = -2x^3 + 3x^2 + 36x - 5$ , we have

$\frac{dy}{dx} = -6x^2 + 6x + 36 = -6(x^2 - x - 6)$ . To find local max or min values, we set this

equal to zero and solve, giving  $x^2 - x - 6 = 0$ . Solving this quadratic by factorisation or by using the formula gives  $x = 3$  and  $x = -2$ . The corresponding values of  $y$  are  $y = 76$  and  $y = -49$ .

We also have  $\frac{d^2y}{dx^2} = -12x + 6 = -6(6x - 1)$ . This is negative when  $x = -2$ , so this point is a local maximum of  $y$ . It is positive when  $x = 3$ , so this point is a local minimum of  $y$ . This information, together with our knowledge that the  $y$  intercept is at  $y = -5$ , permits us to sketch the graph below.

Ex WS7.2 question 3



**Exercise WS7.3**

1. Given  $y = 2x^3 - 27x^2 + 84x + 10$ , we have  $\frac{dy}{dx} = 6x^2 - 54x + 84 = 6(x^2 - 9x + 14)$ .

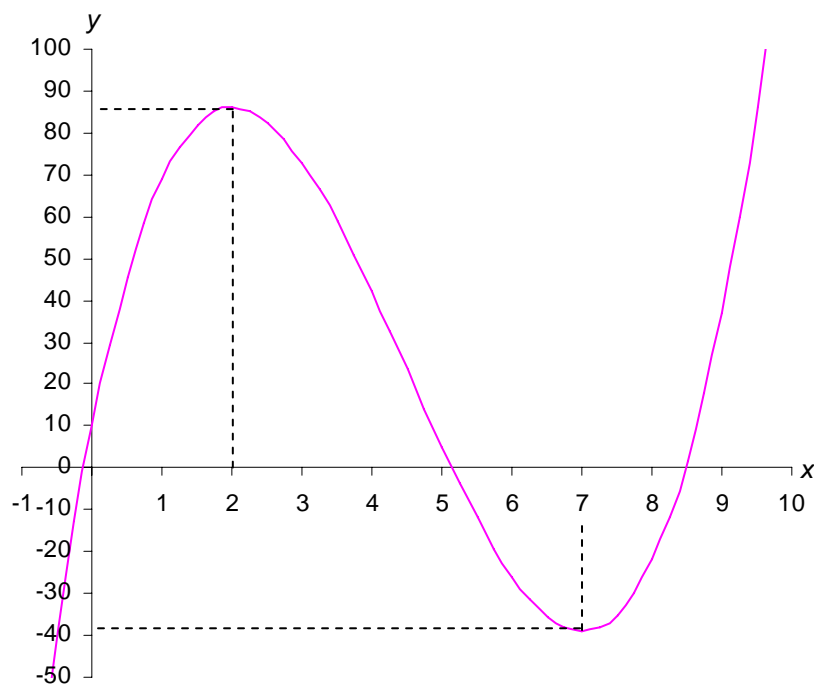
Setting equal to zero and solving (by factorisation or by using the formula) gives  $x = 2$  or  $7$ . Thus  $x = 2$  and  $x = 7$  are stationary points (SPs) of the function  $y = 2x^3 - 27x^2 + 84x + 10$ . At these points, the function is neither rising nor falling.

The associated values of  $y$  are, respectively,  $y = 2(2)^3 - 27(2)^2 + 84(2) + 10 = 86$  and  $y = 2(7)^3 - 27(7)^2 + 84(7) + 10 = -39$ .

The second derivative is  $\frac{d^2y}{dx^2} = 12x - 54 = 6(2x - 9)$ . This is negative when  $x = 2$ , so this is a local maximum. It is positive when  $x = 7$ , so this is a local minimum. (If at either point the second derivative had been equal to zero, this would have been a point of inflection, provided that the third derivative had not been zero at that point.)

Now that we know the maximum and minimum points of this function, together with the fact that the  $y$  intercept is at  $y = 10$ , we can sketch the graph below

Ex WS7.3 question 1



2. Given  $y = 2x^3 + 21x^2 + 60x + 30$ , we have

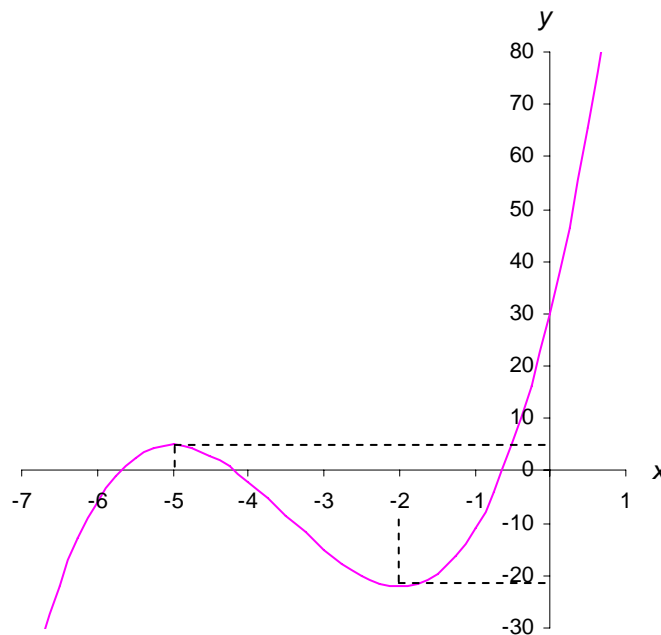
$$\frac{dy}{dx} = 6x^2 + 42x + 60 = 6(x^2 + 7x + 10) = 0. \text{ This last factorises as } 6(x+5)(x+2) = 0.$$

Thus  $x = -5$  or  $-2$ . At these points,  $y = 5$  and  $-22$  respectively.

We also have  $\frac{d^2y}{dx^2} = 12x + 42 = 6(2x + 7)$ . When  $x = -5$  this expression is negative, so this point is a maximum. When  $x = -2$  this expression is positive, so this point is a minimum.

Now that we know the maximum and minimum points of this function, together with the fact that the  $y$  intercept is at  $y = 30$ , we can sketch the graph below.

Ex WS7.3 question 2



3. Given  $y = -x^3 + 9x^2 + 15x + 7$ , we have

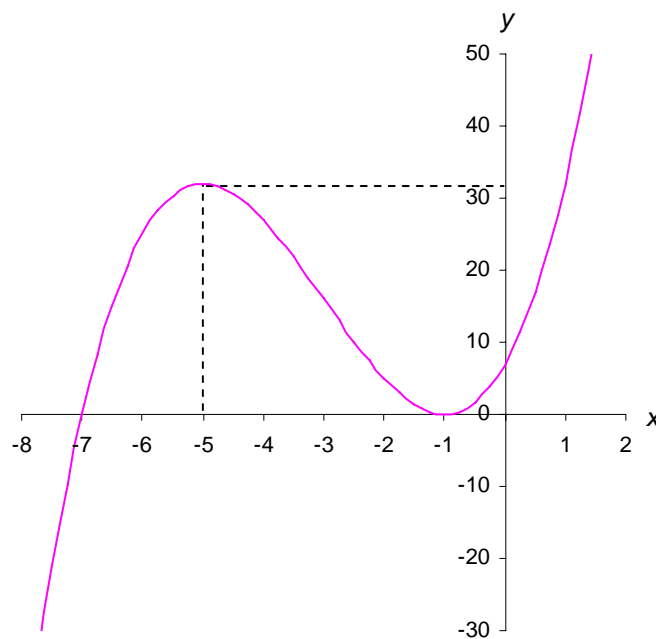
$$\frac{dy}{dx} = 3x^2 + 18x + 15 = 0 \Rightarrow 3(x^2 + 6x - 5) = 0 \Rightarrow 3(x+5)(x-1) = 0$$

Thus  $x = -5$  or  $-1$ . The values of  $y$  are  $y = 32$  and  $0$  respectively.

Also  $\frac{d^2y}{dx^2} = 6x + 18 = 6(x+3)$ . So when  $x = -5$ ,  $\frac{d^2y}{dx^2} < 0$  and this is therefore a local maximum. When  $x = -1$ ,  $\frac{d^2y}{dx^2} > 0$  and this is therefore a local minimum.

In the graph below, we see that because the value of  $y$  is zero at the local minimum, the curve is tangent to the  $x$  axis at this point.

Ex WS7.3 question 3



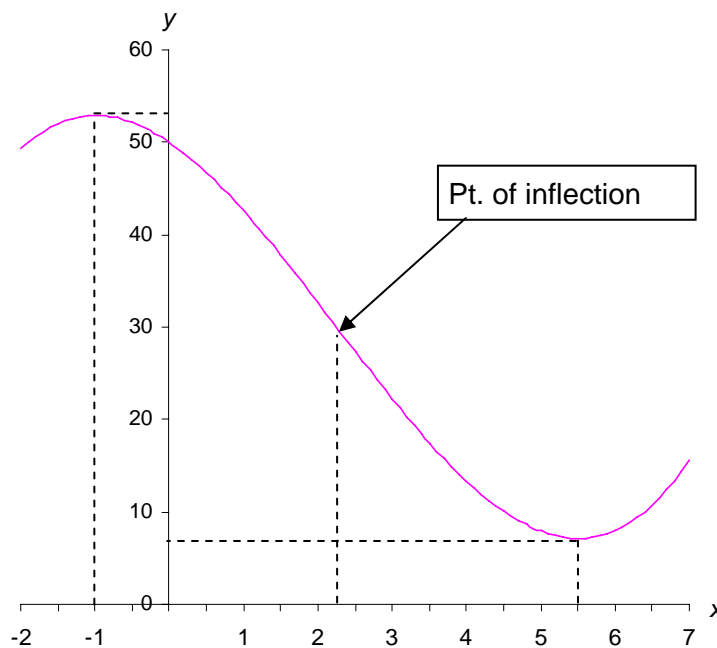
**Exercise WS7.4**

1. Given  $y = -\frac{1}{3}x^3 - 2\frac{1}{4}x^2 - 5\frac{1}{2}x + 50$ , setting the first derivative equal to zero and solving gives  $\frac{dy}{dx} = x^2 - \frac{9}{2}x - \frac{11}{2} = 0 \Rightarrow 2x^2 - 9x - 11 = 0$ . Solving this quadratic gives  $x = 5\frac{1}{2}$  or  $-1$ . The values of  $y$  are  $y = 7.15$  and  $52.92$  respectively (to 2 dp). To determine whether each of these two stationary points is a maximum, minimum or point of inflection we examine sign of second derivative, which is

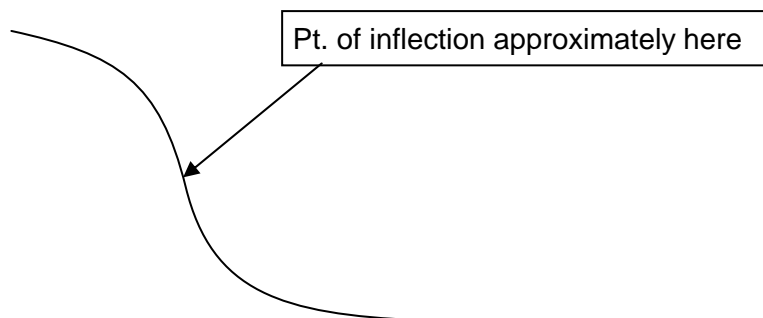
$\frac{d^2y}{dx^2} = 2x - \frac{9}{2} = 2(x - \frac{9}{4})$ . When  $x = -1$ ,  $\frac{d^2y}{dx^2} < 0$  so this is a local maximum. When  $x = 5\frac{1}{2}$ ,  $\frac{d^2y}{dx^2} > 0$  so this is a local minimum.

To find any points of inflection we set the second derivative equal to zero, giving  $\frac{d^2y}{dx^2} = 2(x - \frac{9}{4}) = 0$  with solution  $x = \frac{9}{4}$ . To confirm that this is a point of inflection, we must check that the third derivative is not zero. Here  $\frac{d^3y}{dx^3} = 2 > 0$  as required. (The fact that the third derivative is positive tells us that this point is a minimum of the slope, and therefore the slope is flatter (more positive or less negative) on each side of the point; see graph below.)

Ex WS7.4 question 1



In case you are finding it difficult to see why there is a point of inflection in the graph above, below is an exaggerated sketch of the curve, hopefully showing the point of inflection more clearly. By definition, a point of inflection occurs where, as  $x$  increases, the curvature of the function's graph changes from concave to convex, as below, or from convex to concave.

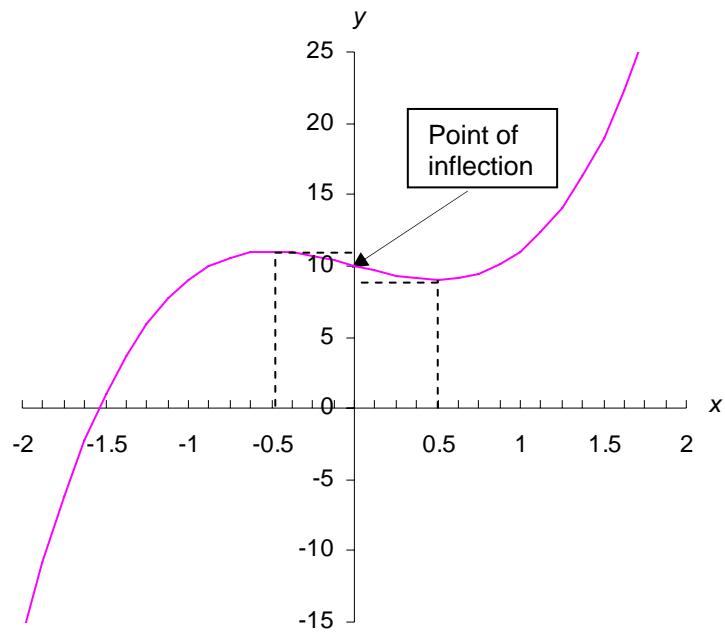


2. We are given  $y = 4x^3 - 3x + 10$  (note there is no  $x^2$  term in this function). Setting the first derivative equal to zero gives  $\frac{dy}{dx} = 12x^2 - 3 = 3(4x^2 - 1) = 0$ . This quadratic has roots  $x = \frac{1}{2}$  and  $x = -\frac{1}{2}$ . So these are stationary points (SPs). The  $y$  values are 9 and 11 respectively.

Next,  $\frac{d^2y}{dx^2} = 24x$ . This is negative when  $x = -\frac{1}{2}$  so this is a max. It is positive when  $x = \frac{1}{2}$  so this is a minimum.

To find points of inflection which are not SPs, we set the 2nd derivative equal to zero, giving  $\frac{d^2y}{dx^2} = 24x = 0$ , with solution  $x = 0$ . To confirm this is a point of inflection we check that the third derivative is non-zero at this point. The third derivative is  $\frac{d^3y}{dx^3} = 24$ , which is non-zero when  $x = 0$  so this point is confirmed as a point of inflection. (Note that the inflection lies between the max and min values, as expected. Also, the fact that the third derivative is positive tells us that this point is a minimum of the slope, and therefore the slope is greater (more positive or less negative) on each side of the point.)

Ex WS7.4 question 2



3. Given  $y = x^3 + 15x^2 - 225x$ . We have

$$\frac{dy}{dx} = 3x^2 + 30x - 225 = 0. \text{ This quadratic has roots}$$

$x = -15$  with  $y = 3375$ , and  $x = 5$  with  $y = -625$ .

Next,  $\frac{d^2y}{dx^2} = 6x + 30 = 6(x + 5)$ . This is negative when  $x = -15$  so this is a max. It

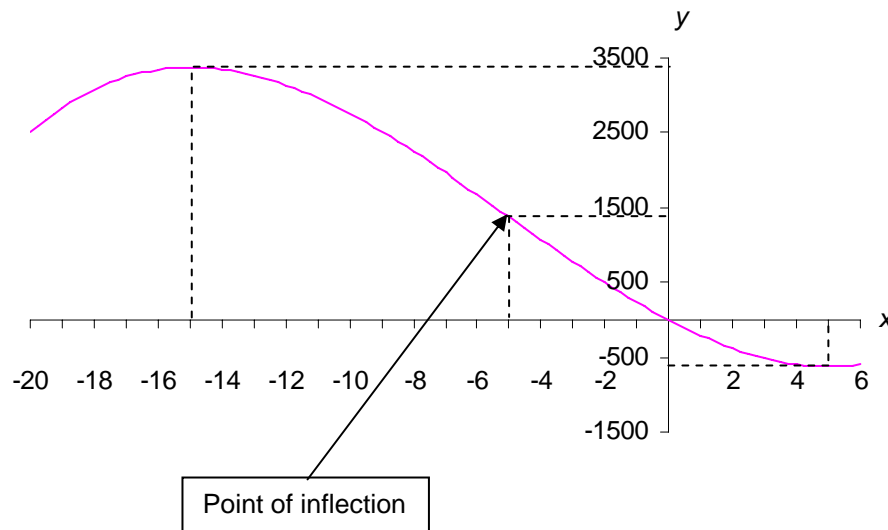
is positive when  $x = 5$  so this is a minimum.

To find any point of inflection which is not an SP, we set the 2nd derivative equal to zero, giving  $(x + 5) = 0$  with solution  $x = -5$  (and  $y = 1375$ ). To confirm

this is a point of inflection we check that the third derivative is non-zero at this point. The third derivative is  $\frac{d^3y}{dx^3} = 6$ , which is non-zero when  $x = -5$  so this

point is confirmed as a point of inflection. Because the third derivative is positive, this is a minimum of the slope, that is, the (locally) most negatively sloped point on the curve; see graph below.

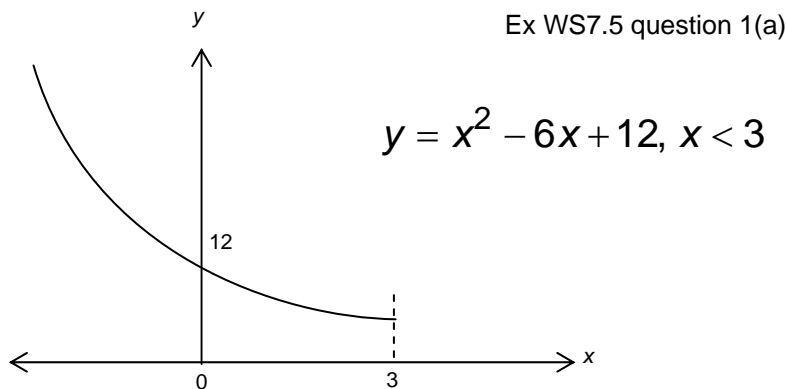
Ex WS7.4 question 3



**Exercise WS7.5**

1. (a) Given  $y = x^2 - 6x + 12$ , we have  $\frac{dy}{dx} = 2x - 6 = 2(x - 3)$ . This is negative when  $x < 3$ , so the function is downward sloping. Also,  $\frac{d^2y}{dx^2} = 2$ . This is positive when  $x < 3$ , so the function is convex when viewed from below. Another way of putting this is to say that because the second derivative is positive, the slope increases as  $x$  increases.

This information, together with the fact that the  $y$  intercept is 12, is used to draw the very rough sketch below. (To help with the sketch, we also attempted to solve the equation  $y = x^2 - 6x + 12 = 0$ , as the solutions would give us the  $x$  intercepts of the function. We quickly found that the equation has no real roots, so the curve never cuts the  $x$  axis.) Note that the curve rises continuously as  $x$  decreases; it never turns down again.



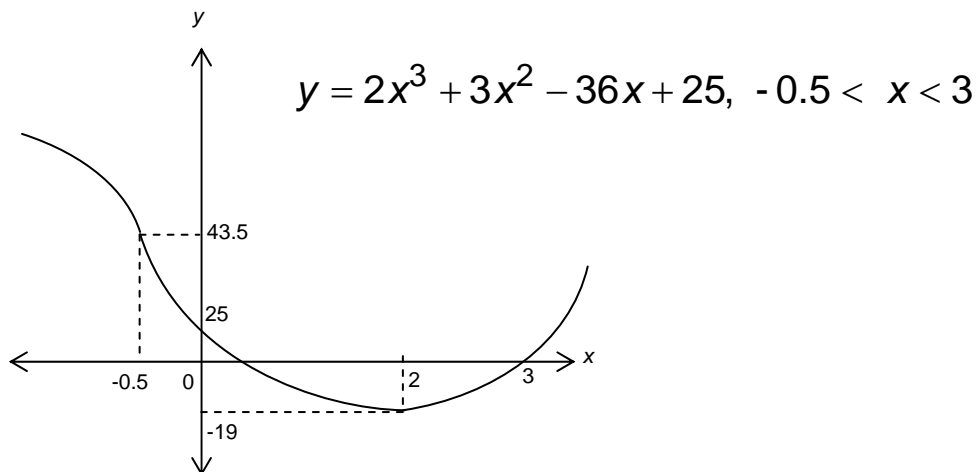
- 1 (b) Given  $y = 2x^3 + 3x^2 - 36x + 25$ , we have  $\frac{dy}{dx} = 6x^2 + 6x - 36 = 6(x^2 + x + 6)$ . This factorises as  $6(x^2 + x + 6) = 6(x + 3)(x - 2)$  so  $\frac{dy}{dx} = 0$  when  $x = -3$  or  $2$ . The second of these two points,  $x = 2$ , interests us because it lies within the interval  $-0.5 < x < 3$ , which we are asked to look at. We have  $\frac{d^2y}{dx^2} = 12x + 6 = 6(2x + 1)$ . This is positive when  $x = 2$ , so this point is a minimum of  $y$ . (At this point,  $y = 2(2)^3 + 3(2)^2 - 36(2) + 25 = -19$ .) So we can divide the interval  $-0.5 < x < 3$  into two sub-intervals, one to the left of  $x = 2$  and the other to the right of  $x = 2$ . To the left (that is, where  $-0.5 < x < 2$ ) we have  $\frac{dy}{dx}$

negative and the graph is negatively sloped. To the right (that is, where  $2 < x < 3$ ) we have  $\frac{dy}{dx}$  positive and the graph is positively sloped. The boundary between these two sub-intervals is  $x = 2$  where the slope is zero and the function is at a local minimum.

Looking now at curvature, we have  $\frac{d^2y}{dx^2} = 6(2x+1)$ . This is positive when  $x > -0.5$ , so is positive in our interval  $-0.5 < x < 3$  and the function is therefore convex from below in this interval.

Using all of the above information we can draw the rough sketch graph below.

Ex WS7.5 question 1(b)



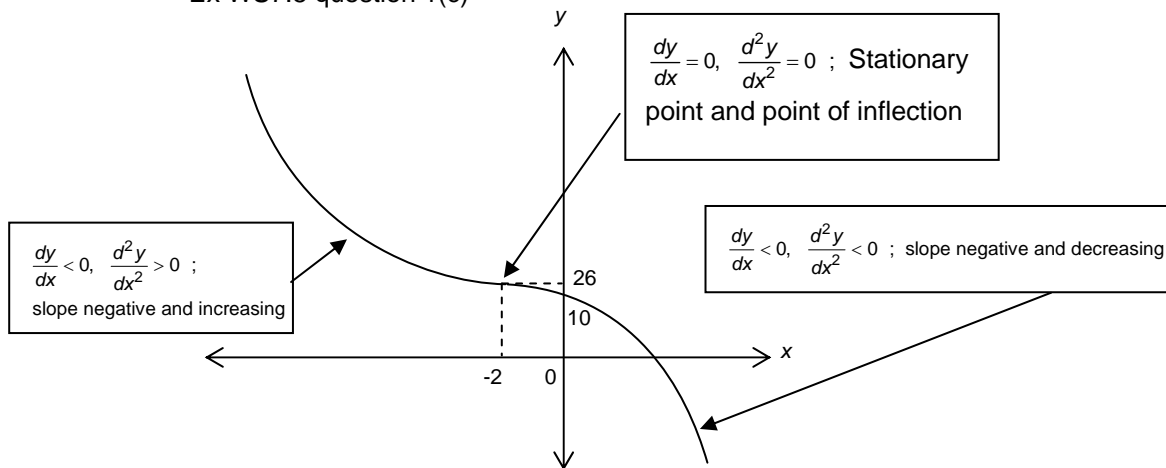
1.(c) We have  $\frac{dy}{dx} = -6x^2 - 24x - 24 = -6(x^2 + 4x + 4) = -6(x+2)(x+2)$ . So  $\frac{dy}{dx} = 0$  when  $x = -2$ . When  $x < -2$ ,  $\frac{dy}{dx} < 0$  since both expressions in brackets are negative. When  $x > -2$ ,  $\frac{dy}{dx} < 0$  since both expressions in brackets are positive.

We also have  $\frac{d^2y}{dx^2} = -12x - 24 = -12(x+2)$ . This equals zero when  $x = -2$ , so this is probably a point of inflection. This is confirmed when we check the sign of the third

derivative, which is  $\frac{d^3y}{dx^3} = -12$  (that is, not zero). When  $x < -2$ ,  $\frac{d^2y}{dx^2}$  is positive, and when  $x > -2$ ,  $\frac{d^2y}{dx^2}$  is negative.

Collecting results, to the left of  $x = -2$  we have  $x < -2$  with  $\frac{dy}{dx} < 0$  and  $\frac{d^2y}{dx^2} > 0$ . The curve is therefore downward sloping and convex (the slope is increasing as  $x$  increases). To the right of  $x = -2$  we have  $x > -2$  with again  $\frac{dy}{dx} < 0$  but  $\frac{d^2y}{dx^2} < 0$ . The curve is therefore downward sloping and concave (the slope is decreasing as  $x$  increases). At  $x = -2$  there is a point of inflection. See sketch below.

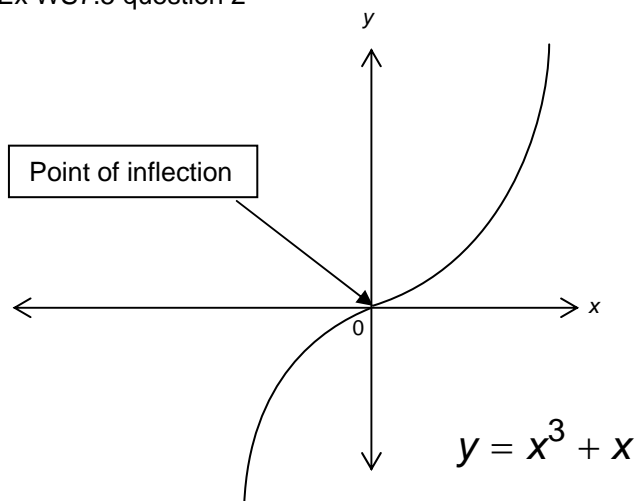
Ex WS7.5 question 1(c)



$$y = -2x^3 - 12x^2 - 24x + 10, x < -2 \text{ and } x > -2$$

2. Given  $y = x^3 + x$ , we have  $\frac{dy}{dx} = 3x^2 + 1$ . This is positive for all  $x$ , because  $x^2$  is never negative. So the function is always positively sloped. Also,  $\frac{d^2y}{dx^2} = 6x$ . This is positive when  $x$  is positive, so the function is then convex (it becomes increasingly positively sloped as  $x$  increases). It is negative when  $x$  is negative, so the function is then concave (it becomes decreasingly positively sloped as  $x$  increases). At  $x = 0$  we have  $\frac{d^2y}{dx^2} = 0$  so this is probably a point of inflection, confirmed when we see that  $\frac{d^3y}{dx^3} = 6$  which is non-zero when  $x = 0$ .
- Though the question does not ask for a sketch, we give one below (noting also that the  $y$  intercept is at  $y = 0$ ).

Ex WS7.5 question 2



2. Given  $y = 2x^2 - 6x + 15$ , and given  $x_0 = -10$ ,  $x_1 = -9$ , so  
 $dx \equiv x_1 - x_0 = -9 - (-10) = +1$ .

We have  $\frac{dy}{dx} = 4x - 6 = -46$  when  $x = x_0 = -10$ . Given  $dx = 1$ , the differential formula gives the approximate change as  $dy = \frac{dy}{dx} dx = -46(1) = -46$ .

To find the true change, we have  $y_0 = 2(-10)^2 - 6(-10) + 15 = 275$  (when  $x = x_0 = -10$ ).  
And  $y_1 = 2(-9)^2 - 6(-9) + 15 = 231$  (when  $x = x_1 = -9$ ).

So true change in  $y$  is  $y_1 - y_0 = 231 - 275 = -44$ .

We can define the absolute error as:

absolute error = approximated change minus true change =  $-46 - (-44) = -2$ . (That is, the approximated change under-stated the true change by 2 units.)

We can define the percentage error as:

$$\text{Percentage error} = \frac{\text{approximated change} - \text{true change}}{\text{true change}} \times 100 = \frac{2}{-46} \times 100 = -4.3 \text{ percent.}$$

(That is, the approximated change under-stated the true change by 4.3 per cent.)

(We could also define the error as true change minus approximated change.)