Exercise WS4.1

1. (a) $x^2 + 3x - 4$

(b) $x^2 - 25$

(c) $\frac{3}{4}x^2 - \frac{7}{8}x - \frac{5}{8}$

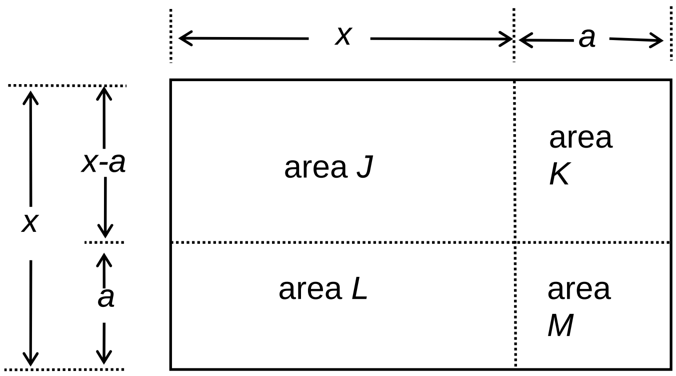
2. From the diagram below we see that $(x+a)(x-a)$ = area J + area K.

But area K = a(x – a) = ax – a^2 = area L – area M.

So $(x+a)(x-a)$ = area J + area L – area M.

But area J + area L = x^2. And area M = a^2.

So $(x+a)(x-a)$ = area J + area L – area M = $x^2 - a^2$, as we were asked to show.



Exercise WS4.2

(a) From the book we know that $(x+a)(x+b) = x^2 + (a+b)x + ab$. Given $x^2 + 5x + 6$, we have $(a+b) = 5$ and $ab = 6$. By trial and error, we arrive at a = 3, b = 2. So $x^2 + 5x + 6 = (x+3)(x+2)$. Therefore the roots of $x^2 + 5x + 6 = 0$ are $x = -3$ and $x = -2$.

Checking by using the formula, given $ax^2 + bx + c = 0$, the formula is:

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

Here we have a = 1, b = 5, c = 6. In the formula we thus get:

$$x = \frac{-5 \pm \sqrt{25-24}}{2} = -\frac{5}{2} \pm \frac{\sqrt{1}}{2} = -\frac{5}{2} \pm \frac{1}{2} = -3 \text{ or } -2.$$

(Note that the factors are x + 3 and x + 2, while the roots are x = –3 and x = –2. Do you understand why?)

(b) Using the same method as in (a) above, we have $(a+b) = -1$ and $ab = -6$. By trial and error, we arrive at a = -3, b = 2. So $x^2 - x - 6 = (x-3)(x+2)$. The roots are x = 3, x = –2. This is confirmed when we apply the formula, as in (a) above.

(c) Here we have $(a+b) = \frac{3}{8}$ and $ab = \frac{1}{32}$. By trial and error, we arrive at $a = \frac{1}{4}$, $b = \frac{1}{8}$. So $x^2 + \frac{3}{8}x + \frac{1}{32} = (x+\frac{1}{4})(x+\frac{1}{8})$. The roots are $x = -\frac{1}{4}$ and $x = -\frac{1}{8}$. This is confirmed when we apply the formula, as in (a) above.

(d) First, note that $6x^2 + 25x - 9$ can be written as $6(x^2 + \frac{25}{6}x - \frac{9}{6})$. These two expressions are *identically* equal; that is, they are equal for all possible values of x. Therefore, whatever values of x make $6(x^2 + \frac{25}{6}x - \frac{9}{6}) = 0$ will automatically make $6x^2 + 25x - 9 = 0$ too. Second, $6(x^2 + \frac{25}{6}x - \frac{9}{6}) = 0$ when $x^2 + \frac{25}{6}x - \frac{9}{6} = 0$.

Combining these two pieces of information, we can conclude that when $x^2 + \frac{25}{6}x - \frac{9}{6} = 0$, $6x^2 + 25x - 9 = 0$ too. In other words, the roots of the quadratic equation $x^2 + \frac{25}{6}x - \frac{9}{6} = 0$ are the same as the roots of the quadratic equation $6x^2 + 25x - 9 = 0$.

So we will try to find the roots of $x^2 + \frac{25}{6}x - \frac{9}{6} = 0$. Here we have $(a+b) = \frac{25}{6}$ and $ab = -\frac{9}{6} = -\frac{3}{2}$. After much trial and error, we arrive at $a = \frac{9}{2}$, $b = -\frac{1}{3}$. So $x^2 + \frac{25}{6}x - \frac{9}{6} = (x+\frac{9}{2})(x-\frac{1}{3})$. Therefore the given equation, $6x^2 + 25x - 9 = 6(x^2 + \frac{25}{6}x - \frac{9}{6}) = 6(x+\frac{9}{2})(x-\frac{1}{3})$. The roots are $x = -\frac{9}{2}$ and $x = \frac{1}{3}$. This is confirmed when we apply the formula, as in (a) above.

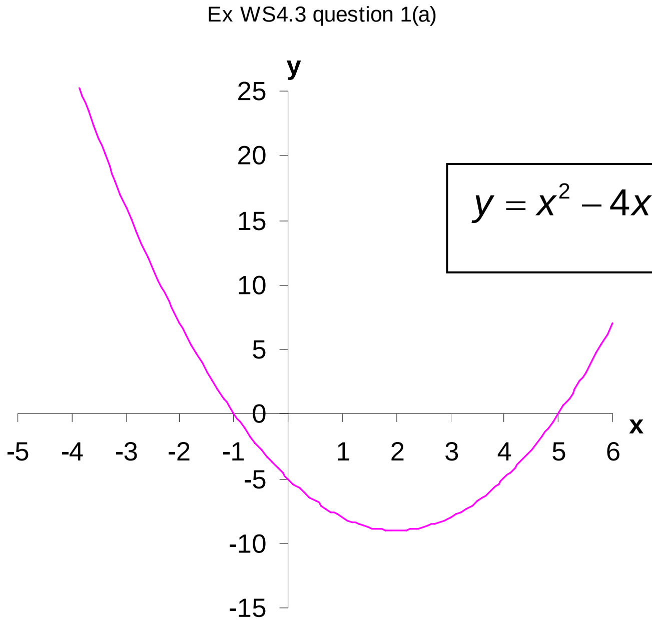
(e) Here we have $(a+b) = 14$ and $ab = 49$. After a little thought, we arrive at a = b = 7. So $x^2 + 14x + 49 = (x+7)(x+7)$. The single (repeated) root is x = –7.

This is confirmed when we apply the formula, as in (a) above. Here we have a = 1, b = 14, c = 49. In the formula we thus get:

$$x = \frac{-14 \pm \sqrt{14^2 - 4(49)}}{2} = -7 \pm \frac{\sqrt{196-196}}{2} = -7.$$

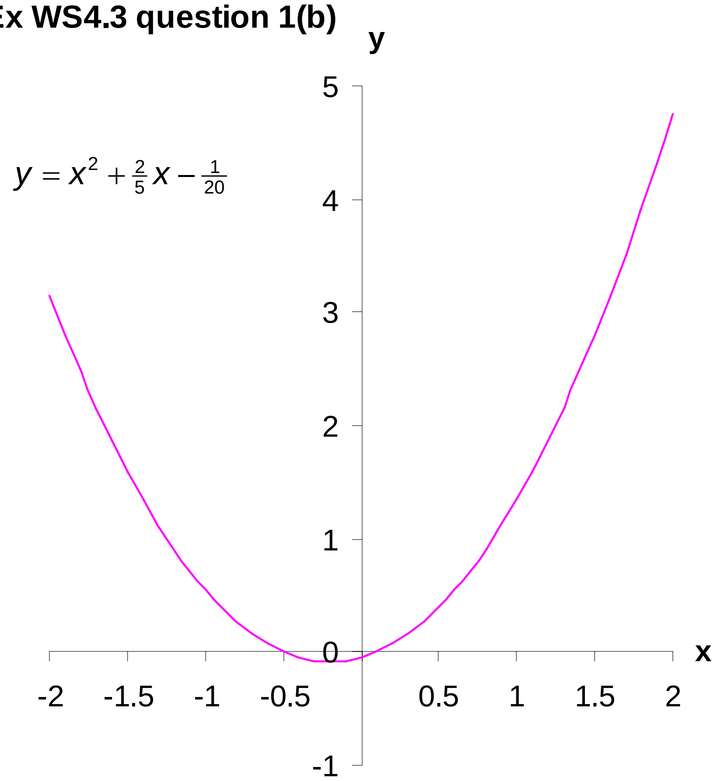
Exercise WS4.3

1.(a) By factorisation, or by using the formula, we find that the roots of $x^2 - 4x - 5 = 0$ are 5 and –1. Therefore we know that the graph of $y = x^2 - 4x - 5$ cuts the x-axis at $x = 5$ and –1. Also, the fact that the coefficient of x^2 is positive tells us that the curve is a parabola with its "arms" pointing upward (or, in other words, that its turning point is a minimum). This information helps us to sketch the graph; see below

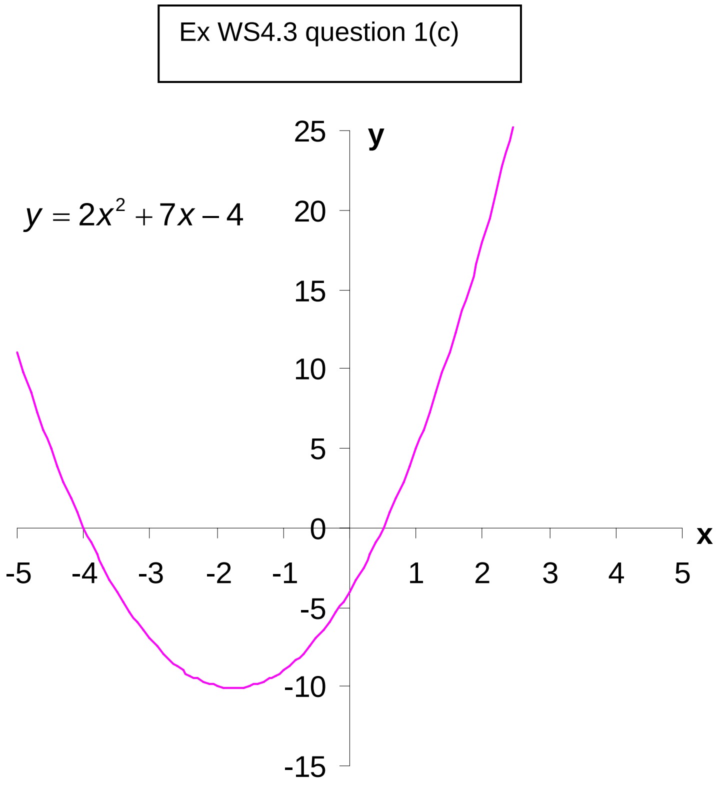


1.(b) By factorisation, or by using the formula, we find that the roots of $x^2 + \frac{2}{5}x - \frac{1}{20} = 0$ are –0.5 and 0.2. Therefore the graph of cuts the x-axis at x = –0.5 and 0.2. As in (a) above, the fact that the coefficient of x^2 is positive tells us that the curve is a parabola with its "arms" pointing upward (or, in other words, that its turning point is a minimum). This helps us to sketch the graph; see below.

Ex WS4.3 question 1(b)



1.(c) This is the same as the two previous cases, with roots in this case of 0.5 and –4. See sketch below.



2.(a) From 1(a) above, we know that the roots (solutions) of $x^2 - 4x - 5 = 0$ are 5 and –1. This is confirmed by the fact that the graph of $y = x^2 - 4x - 5$ in 1(a) cuts the *x*-axis at *x* = 5 and –1. At these cuts, *y* = 0 and therefore $x^2 - 4x - 5 = 0$.

(b) As in 2(a) above, with roots –0.5 and 0.2 (from 1(b) above).

(c) As in 2(a) above, with roots 0.5 and –4 (from 1(c) above).

3. Preliminary: suppose we are given the quadratic expression $x^2 + \frac{b}{a}x + \frac{c}{a}$, where *a, b* and *c* are parameters (constants). From the book, we know that

$$x^2 + \frac{b}{a}x + \frac{c}{a} \equiv (x + \alpha)(x + \beta) \qquad (1)$$

where $\alpha + \beta = \frac{b}{a}$ and $\alpha\beta = \frac{c}{a}$. Here –α and –β are the roots of the quadratic equation $x^2 + \frac{b}{a}x + \frac{c}{a} = 0$.

Multiplying both sides of (1) by *a* we get

$$a(x^2 + \frac{b}{a}x + \frac{c}{a}) \equiv a(x+\alpha)(x+\beta) \qquad (2)$$

where as before $\alpha + \beta = \frac{b}{a}$ and $\alpha\beta = \frac{c}{a}$, and –α and –β are the roots of the quadratic equation $x^2 + \frac{b}{a}x + \frac{c}{a} = 0$. However, since *both* sides of equation (2) are zero when either *x* = –α or *x* = –β, it follows that –α and –β are also the roots of the quadratic equation $a(x^2 + \frac{b}{a}x + \frac{c}{a}) = 0$. (This is true by definition, since a root is any value of *x* that makes the quadratic expression zero). Finally, since $a(x^2 + \frac{b}{a}x + \frac{c}{a}) \equiv ax^2 + bx + c$, it follows that –α and –β are also the roots of the quadratic equation $ax^2 + bx + c = 0$. (This conclusion is simply a generalisation of the answer to Ex WS4.2 (d)).

We are now ready to answer the question. In the question, we are given a function $y = ax^2 + bx + c$. This cuts the *x* axis when $ax^2 + bx + c = 0$. We are told that this occurs when *x* = –1 or 3.

From our preliminaries above we know that

$$ax^2 + bx + c \equiv a(x^2 + \frac{b}{a}x + \frac{c}{a}) \equiv a(x+\alpha)(x+\beta)$$

where $\alpha + \beta = \frac{b}{a}$ and $\alpha\beta = \frac{c}{a}$.

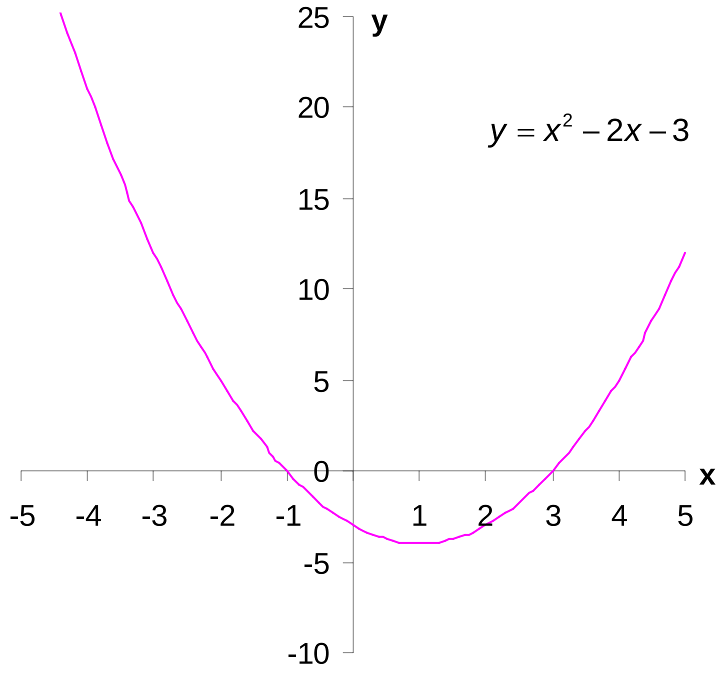
This expression equals zero when *x* = –α or –β. The expression we are given equals zero when *x* = –1 or 3. Comparing the two expressions, we see that α = 1 and β = –3. So $\alpha + \beta = 1 - 3 = \frac{b}{a}$ and $\alpha\beta = 1(-3) = \frac{c}{a}$. Since we are also told that *a* = 1, we have 1–3 = –2 = *b*, and 1(-3) = –3 = *c*. So the function we are given must be $y = x^2 - 2x - 3$.

Check: by factorisation: $x^2 - 2x - 3 = (x+1)(x-3)$. So $y = x^2 - 2x - 3$ = 0, and the curve cuts the *y*-axis when *x* = –1 or +3, as given.

See graph below.

Can you re-solve this question with *a* = –1 instead of *a* = 1?

Ex WS4.3 question 3



4. Given $ax^2 + bx + c = 0$, the formula for its solution is:

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

If a and c have the same sign (both positive or both negative), then ac is positive and therefore $-4ac$ is negative. If also b = 0, then b –4ac is negative. So in the formula above, the expression under the square root sign is negative and the given quadratic equation has no real roots.

5. We know from the book that if the equation $ax^2 + bx + c = 0$ has a single (repeated) root, then the graph of $y = ax^2 + bx + c$ is tangent to the x-axis. In this example the single root is x = 4.

Using the method of question 3 above, we have

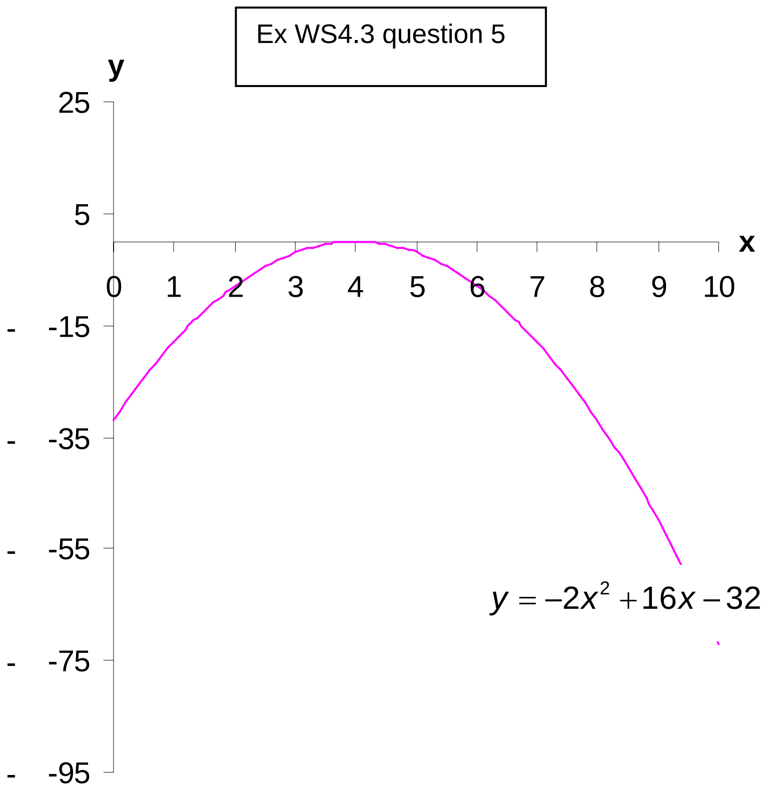
$$ax^2 + bx + c \equiv a(x^2 + \tfrac{b}{a}x + \tfrac{c}{a}) \equiv a(x + \alpha)(x + \beta)$$

where $x = -\alpha$ and $x = -\beta$ are the roots. In this example we are told that $-\alpha = -\beta = 4$, and $a = -2$. Substituting this information into the identities above, we get:

$ax^2 + bx + c \equiv a(x+\alpha)(x+\beta) = -2(x-4)(x-4)$. Multiplying out, we get

$ax^2 + bx + c = -2(x-4)(x-4) = -2(x^2 - 8x + 16) = -2x^2 + 16x - 32$

The given function is therefore $y = -2x^2 + 16x - 32$ (see graph below).



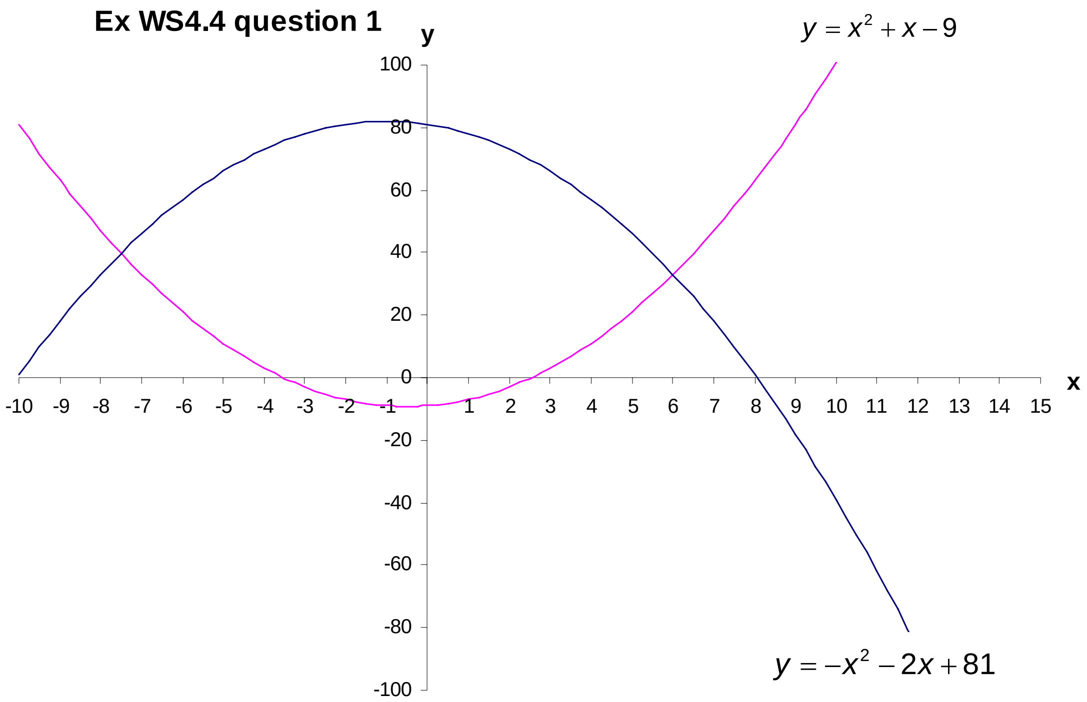
Exercise WS4.4

1. $x^2 + x - 9 = -x^2 - 2x + 81$

$\rightarrow \quad 2x^2 + 3x - 90 = 0$

Using formula, roots are –7.5 or 6.

Sketching graphs. We know that $y = x^2 + x - 9$ is a parabola. Since coefficient of x^2 is positive, its "arms" face upward (that is, its turning point is a minimum). Its intercept on y axis is at $y = –9$. By same reasoning, $y = -x^2 - 2x + 81$ is a parabola with "arms" facing downward (that is, its turning point is a maximum) and y intercept at $y = 81$. We also know that the two curves cut at $x = –7.5$ (with $y = 39.75$) and $x = 6$ (with $y = 33$). This information is enough to draw a rough sketch (see below).



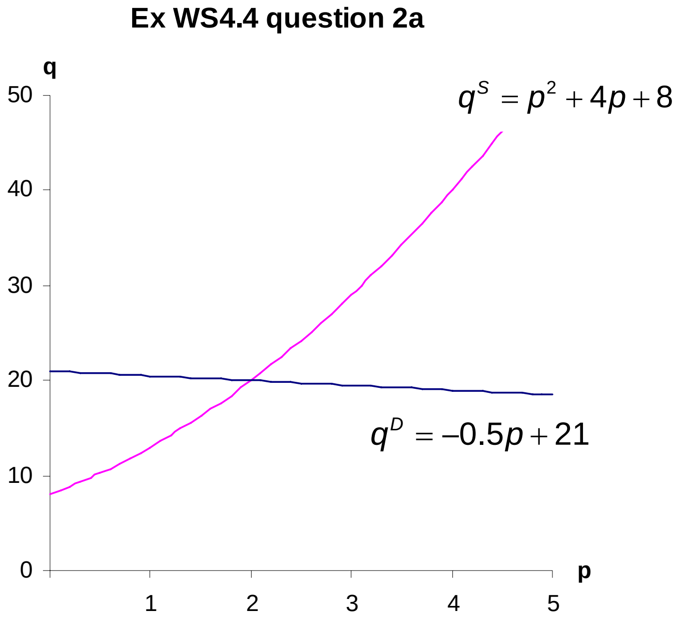
2.(a)(i) Setting $q^S = q^D$ enables us to equate the right hand sides of the two equations, giving $p^2 + 4p + 8 = -0.5p + 21$. This re-arranges as

$$p^2 + 4.5p - 13 = 0$$

Solving using the formula gives p = –6.5 or 2. We consider only positive values of p. So the equilibrium price is p = 2, with $q^S = q^D = 20$

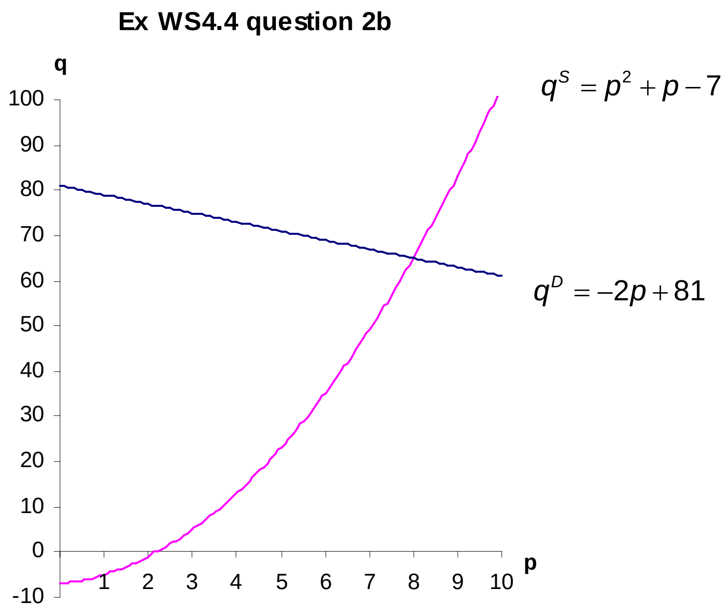
(Note that the supply function is quadratic but the demand function linear.)

(ii) Sketch graph; see below.



2.(b)(i) Using the same method as (a) above, the equilibrium is at p = 8, with $q^S = q^D = 65$.

(ii) Sketch graph:

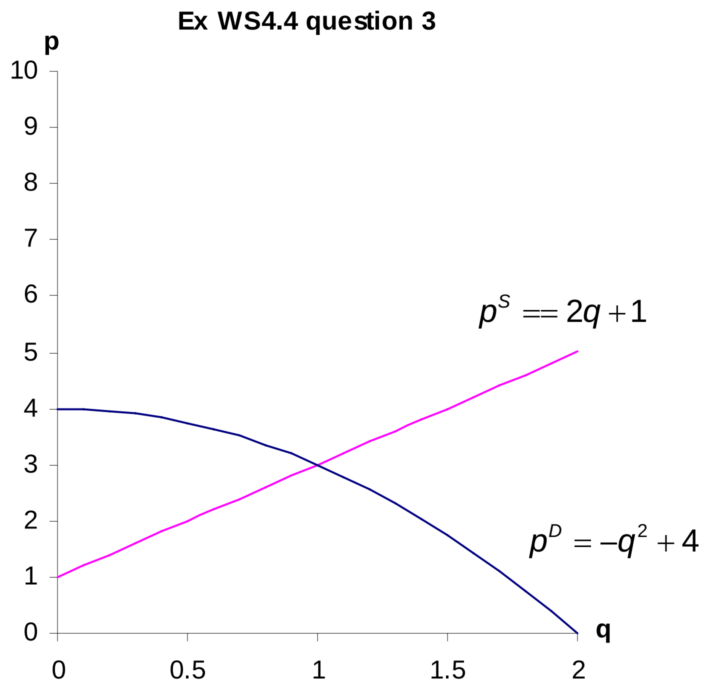


3. In this example we are working with the inverse demand and supply functions, but the method is the same. Setting $p^S = p^D$ enables us to write:

$2q + 1 = -q^2 + 4$, which re-arranges as $q^2 + 2q - 3 = 0$

The positive solution to this quadratic equation is $q = 1$, with $p^S = p^D = 3$.

Sketch graph:



Note that in this example the supply function is linear and the demand function quadratic.