Exercise WS19.1

1. Given: $\mathbf{A} = \begin{bmatrix} 4 & 3 \\ 1 & 2 \end{bmatrix}$ and $\mathbf{B} = \begin{bmatrix} 1 & 4 \\ 3 & 2 \end{bmatrix}$

   Find:

   (a) $\mathbf{A} + \mathbf{B}$

   (b) $2\mathbf{A} - 3\mathbf{B}$

   (c) $\mathbf{AB}$

   (d) $\mathbf{AB} + \mathbf{BA}$

   (e) $\mathbf{A}^{\mathrm{T}}\mathbf{B}$ (where $\mathbf{A}^{\mathrm{T}}$ denotes the transpose of $\mathbf{A}$)

   (f) $\mathbf{B}^{\mathrm{T}}\mathbf{A}$

   (g) $\mathbf{A}^2\mathbf{B}$

2. Given: $\mathbf{A} = \begin{bmatrix} a_1 & a_2 & a_3 \end{bmatrix}$ and $\mathbf{B} = \begin{bmatrix} b_1 \\ b_2 \\ b_3 \end{bmatrix}$

   Find $\mathbf{AB}$ and $\mathbf{BA}$

3. Given: $\mathbf{A} = \begin{bmatrix} a_1 \\ a_2 \\ a_3 \end{bmatrix}$ and $\mathbf{B} = \begin{bmatrix} b_1 \\ b_2 \\ b_3 \end{bmatrix}$

   Find (if they exist) $\mathbf{AB}$, $\mathbf{A}^{\mathrm{T}}\mathbf{B}$, $\mathbf{BA}$, $\mathbf{B}^{\mathrm{T}}\mathbf{A}$, $\mathbf{A}^{\mathrm{T}}\mathbf{B}^{\mathrm{T}}$, and $\mathbf{B}^{\mathrm{T}}\mathbf{A}^{\mathrm{T}}$.

   In each case, explain why the matrix product does (or does not) exist.

4. Given: $\mathbf{A} = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \end{bmatrix}$ and $\mathbf{B} = \begin{bmatrix} b_{11} & b_{12} \\ b_{21} & b_{22} \\ b_{31} & b_{32} \end{bmatrix}$

   Find (if they exist) $\mathbf{AB}$ and $\mathbf{BA}$

5. An investor buys 1000 shares in company *x*, 500 shares in company *y* and 100 shares in company *z*. The respective prices (in euros) are 150, 60, and 50 per share.

   (a) Define two vectors, one for quantities of shares purchased and the other for their prices, such that the product of the two vectors will give the total cost of the portfolio of shares.

   (b) Calculate the total cost of the portfolio, using matrix multiplication.

   (c) If the price of shares in company *z* doubles, and the prices of shares in companies *x* and *y* fall by 20%, calculate by matrix multiplication the change in the value of the portfolio.

Exercise WS19.2

1. Find the inverses (if they exist) of:

$$\mathbf{A}=\begin{bmatrix}4 & 3\\ 1 & 2\end{bmatrix}\ ;\ \mathbf{B}=\begin{bmatrix}1 & 4\\ 3 & 2\end{bmatrix}\ ;\ \mathbf{C}=\begin{bmatrix}1 & 4\\ -2 & -8\end{bmatrix}\ ;\ \mathbf{D}=\begin{bmatrix}4 & 3\\ 1 & 0.5\end{bmatrix}$$

2. Using your answers from (1) above, solve where possible the following sets of simultaneous equations. If you find that no solution is possible, explain why.

   (a) $4x + 3y = 5$

   $x + 2y = 2$

   (b) $x + 4y = -7$

   $3x + 2y = -1$

   (c) $x + 4y = 1$

   $-2x - 8y = 1$

   (d) $4x + 3y = 1$

   $x + 0.5y = -2$

3. Find the determinant, all minors and cofactors, and the inverse of each of the following matrices:

(a) $\begin{bmatrix} 3 & 0 & -1 \\ 2 & 1 & 2 \\ 4 & 0 & 0 \end{bmatrix}$

(b) $\begin{bmatrix} 1 & 1 & 1 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{bmatrix}$

(c) $\begin{bmatrix} -1 & -1 & -1 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{bmatrix}$

(d) $\begin{bmatrix} 2 & 1 & 1 \\ 1 & 1 & 2 \\ 4 & 3 & 5 \end{bmatrix}$

Exercise WS19.3

1. Use Cramer's rule to find *x* in the equation systems below.

(a)
$$3x \quad\quad - z = -2$$
$$2x + y + 2z = 0$$
$$4x \quad\quad\quad = 1$$

(b)
$$x + y + z = 1$$
$$y + z = 0$$
$$z = -1$$

2. Solve the following system of linear equations using (a) matrix inversion, and (b) Cramer's rule.

$$\begin{bmatrix} 2 & 1 & 1 \\ 1 & 1 & 2 \\ 4 & 3 & 4 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}$$

3. The demand functions for apples (subscripted *a*) and bananas (subscripted *b*) are inter-connected by the following demand functions:

$$Q_a^D = 100 - 15P_a + 2P_b \qquad \text{and} \qquad Q_b^D = 67 + 3P_a - 3P_b$$

The supply functions for apples and bananas (similarly subscripted) are:

$$Q_a^S = -4 + 25P_a \qquad \text{and} \qquad Q_b^S = -4 + 7P_b$$

(a) By imposing the equilibrium condition that quantity supplied equals quantity demanded in both markets, find a pair of simultaneous equations in the variables P_a and P_b.

(b) Show that this pair of simultaneous equations can be written in matrix form as

$$\mathbf{Ap} = \mathbf{b} \qquad \text{where } \mathbf{A} = \begin{bmatrix} 40 & -2 \\ -3 & 10 \end{bmatrix} \;;\; \mathbf{p} = \begin{bmatrix} P_a \\ P_b \end{bmatrix} \;;\; \mathbf{b} = \begin{bmatrix} 104 \\ 71 \end{bmatrix}$$

(c) By inverting the matrix **A**, find the equilibrium values of P_a and P_b. Hence find the equilibrium quantities of apples and bananas.

4. A businessperson owns 3 shops. Shop *S* is in the southern part of town; shop *E* is in the east; and shop *W* is in the west. Each sells 3 products: cigarettes, newspapers and ice-cream. Last month shop *S* sold 1000 packets of cigarettes, 2000 newspapers and 3000 ice creams. Shop *E* sold 2000 packets of cigarettes, 500 newspapers and 1000 ice creams. Shop *W* sold 4000 packets of cigarettes, 1000 newspapers and 500 ice creams. The cigarettes were sold at a price of 10 euros per packet; the newspapers at 3 euros each; and the ice creams at 2 euros each.

(a) Set up a matrix **A** showing the sales of each product by each shop.

(b) Show that the total revenue of each shop can be obtained, in vector form, by appropriate multiplication of **A** by a vector of prices, **p**.

(c) Does it matter in (b) whether **p** is defined as a row vector or a column vector?

(d) What operation on the vector of total revenues would result in a scalar that gave the combined total revenue of all three shops? (Hint: Consider a vector [1, 1, 1].)

5. Consider the following reduced-form macroeconomic model:

$Y = C + I^* + G^*$ (equilibrium condition) (1)

$C = a(Y - T) + b$ (consumption behavioural relationship) (2)

$T = tY$ (income tax revenue, T, as a function of the income tax rate, t) (3)

The unknown or endogenous variables are Y = aggregate output and income, C = consumption by households (assumed equal to their planned or desired consumption $\hat{C}$) and T = income tax revenue. The pre-determined or exogenous variables are I^* and G^* (private investment and government expenditure respectively) and b. The parameters are a and t.

(a) Show that, after suitable rearrangement, this set of simultaneous equations can be written as **Ax** = **b**, where:

$$\mathbf{A} = \begin{bmatrix} 1 & -1 & 0 \\ a & -1 & -a \\ t & 0 & -1 \end{bmatrix} \quad ; \quad \mathbf{x} = \begin{bmatrix} Y \\ C \\ T \end{bmatrix} \quad ; \quad \mathbf{b} = \begin{bmatrix} I^* + G^* \\ -b \\ 0 \end{bmatrix}$$

Note that **A** is a matrix containing the parameters of the model, **x** is a vector containing the endogenous variables and **b** is a vector containing the exogenous variables.

(b) By calculating $\mathbf{A}^{-1}$, find the equilibrium values of the endogenous variables as a function of the values of the exogenous variables and the parameter values.

(c) Give an economic interpretation to the elements of $\mathbf{A}^{-1}$.

(d) Using your answer to (b), find the equilibrium values of the endogenous variables when $a = 0.75$, $t = 0.2$, $b = 500$, and $I^* + G^* = 1000$. Check your answer by solving the model by simultaneous equation methods.

(e) Use your answer to (b) to calculate the effect on Y, C, and T of a small increase in each of the exogenous variables, I^*, G^*, and b.