Exercise WS17.1

1. (a) Sketch and label some isoquants for a production function that has decreasing returns to scale at low and high levels of output but increasing returns to scale at intermediate levels of output.

   (b) Sketch and label some isoquants for a production function that has decreasing returns to scale when either very capital intensive or very labour intensive methods of production are used, but increasing returns to scale when intermediate technologies (neither very capital intensive nor very labour intensive methods) are used.

   (c) Could the production functions in (a) or (b) be homogeneous?

2. Which of the following functions are homogeneous, and if so, of what degree?

   (a) $z = 3x^3 - 4xy^2 + 9x^{-3}y^6$

   (b) $z = \sqrt[3]{x^2 y}$

   (c) $z = (x^2 + xy - y^2)^{0.5}$

   (d) $z = \dfrac{x^2}{3x^2 + 4xy - y^2}$

3. For each of the following production functions, assess the degree of homogeneity and whether it has constant, increasing or decreasing returns to scale.

   (a) $Q = 100K^{0.3}L^{0.7}$

   (b) $Q = 50K^{0.6}L^{0.5}$

   (c) $Q = AK^{\alpha}L^{\beta}$

   (d) $Q = aKL - bK^2 - cL^2$ (where *a*, *b*, and *c* are parameters)

   (e) $Q = aKL - bK^2 - cL^2 + dK + eL$ (where *a*, *b*, *c*, *d,* and *e* are parameters)

Exercise WS17.2

1. Refer to your answers to Ex WS17.1, question 3. For each production function that you found to be homogeneous;

   (a) Verify that Euler's theorem holds;

   (b) Assuming constant input prices, roughly sketch the shape of the average and marginal cost functions. (As usual, don't aim at a high degree of accuracy in your sketches; just try to get the general shape right.)

2. A firm's production function is $Q = KL$

   (a) Assess whether this function is homogeneous; and, if so, of what degree. Are returns to scale constant, increasing, or decreasing? What happens to output if both inputs are, say, halved?

   (b) If the prices of capital and labour are r and w respectively, find the cost-minimising capital/labour ratio. Show that, for this particular production function, this ratio depends only on $\frac{w}{r}$, the ratio of input prices, and is independent of the level of output. (Assume the firm is perfectly competitive in the markets for capital and labour, so that w and r are exogenous; that is, for the firm they are given constants.)

   (c) From (b), show diagrammatically how K and L vary as output varies, assuming $\frac{w}{r}$ constant.

   (d) Using your answer to (b), derive an expression for total cost, TC, as a function of Q alone, with w and r as parameters.

   (e) By differentiating your answer to (d), examine how TC varies with output. What does this tell us about the shape of the average (AC) and marginal cost (MC) curves associated with this production function? How is your answer related to your answer to (a)? (Hint: You will find it interesting and instructive to find the elasticity of the TC function, though this is not essential.)

   (f) Sketch the graph of the TC, AC and MC functions derived in (d) and (e). (As usual, don't aim at a high degree of accuracy; just try to get the general shape right.)

3. Repeat the previous question for the production function $Q = 16KL - 4K^2 - 5L^2$, with the following minor modifications:

   In (b), show that the cost-minimising K to L ratio now depends on the parameters of the production function as well as on $\frac{w}{r}$, but remains independent of the level of output. Do you think it likely that this feature (that is, the cost-minimising K to L ratio being independent of the level of output) is likely to be found in the real world?

Also in (b), find the cost-minimising K to L ratio if w = 3 and r = 4.

If you find (d) difficult, simplify by continuing to assume that w = 3 and r = 4. You should then find that TC as a function of Q is given by

$$TC = (w + r)\frac{Q^{0.5}}{7^{0.5}}$$

and therefore, for example, that when $K = L = 10$, $Q = 700$ and $TC = 70$. What are the values of Q and TC when $K = L = 20$?

4. What is meant by the adding-up problem? Illustrate it for a firm operating under conditions of perfect competition and with the production function of question (2) above; that is, $Q = KL$.

Exercise WS17.3

1. The demand functions for two goods, X and Y, are

$$X = 100P_X^{-1}P_Y^{2} \qquad \text{and} \qquad Y = 500P_X^{0.1}P_Y^{-2}$$

(a) Find the own-price and cross-price elasticities.

(b) Comment on their numerical values.

(c) Sketch the graphs of these demand functions when log scales are taken on both axes. Show that, with log scales on the axes, the slopes measure the elasticities.

2. In year 0, government debt was D_0 and GDP was Y_0. In the same year the debt ratio, R, (the ratio of government debt to GDP) was $R_0 \equiv \frac{D_0}{Y_0} = 0.35$. Each year, government debt grows due to new borrowing, which equals 2% of the previous year's GDP. GDP itself is also growing at 3.5% per year.

(a) Show that, in general, the debt ratio will increase through time if the growth rate of debt exceeds the growth rate of GDP. (Hint: Example 17.10 in the book is relevant.)

(b) Treating growth of all variables as occurring discretely in annual jumps, find an expression for the growth rate of the debt ratio and calculate after how many years the debt ratio will reach 40%.

(c) What is the nature of the approximation (if any) in your answers to (a) and (b)?

(d) Repeat (b) treating growth of all variables as continuous.

(e) Does the debt ratio stabilise after a certain number of years?

(f) Does the debt ratio seem to you to be an important economic variable?

(Hints: You may find (c) easier than (b), in which case you should feel free to answer it first! If you get completely stuck or want to check your answer, you could simply construct a table of values of D and Y in successive years, starting with, say, $D = 35$ and $Y = 100$.)

3. Assume that the aggregate production function for the UK economy is $Q = AK^{0.4}L^{0.7}$ where A is a parameter.

(a) If capital and labour inputs both grow at the same rate (say, x%) per year, what growth rate of output should be expected? Explain the methodology underlying your method of calculation. How may we explain this ability of output to grow at a rate different from the growth rate of inputs?

(b) In recent years the labour input has been growing at about 0.5% per year (due mainly to immigration) and, due to net investment, the capital stock has been growing at a rate of about 1.5% per year. What consequent growth rate of aggregate output would you expect to observe?

(c) Compare your answer to (b) with the actual growth rates of GDP in recent years. (Visit the Office of National Statistics, http://www.statistics.gov.uk/ if you don't have the data to hand.)

(d) If the growth of the labour input ceased (perhaps due to restrictions on immigration) by how much would investment need to increase to maintain the growth rate of aggregate output?

(e) The parameter A may be regarded as a measure of overall productive efficiency, in the sense that any increase in A increases output with unchanged inputs of K and L. Suppose due to increased technological and managerial skills, A began to increase at y% per year. Show the effect of this on the growth rate of output, when K and L grow at x% per year.