Exercise WS16.1

1. (a) For each of the following functions, find the optimum value of z (that is, maximum or minimum) subject to the given constraint, first by the method of direct substitution, and then by setting the slope of the iso-z section equal to the slope of the constraint.

(b) In each case, attempt to assess informally the shapes of the two surfaces and whether the optimum is a maximum or minimum of z.

(i) $z = xy$ subject to the constraint $x^2 + y^2 = 16$. (Note that there are four solutions.)

(ii) $z = 3x^2 - 10xy + 12y^2$ subject to the constraint $y = 20 - \frac{1}{2}x$

(iii) $z = x^2 + 3xy + y^2$ subject to the constraint $y = ax + b$, where a and b are parameters. How do the signs of a and b influence the solution(s)?

Exercise WS16.2

Solve each of the problems in exercise WS16.1 using the Lagrange multiplier method. In question (i), find the value of λ and verify by recalculating the solution that the value of λ gives (with a small error) the change in z when the constraint is relaxed by 1 unit.

Exercise WS16.3

1. A carpenter plans to make a rectangular wooden box with a lid (that is, a closed box) and with a capacity of 27 cubic metres. The necessary wood costs 25 euros per square metre. Find the dimensions of the box that will minimise the cost of wood. Find also the minimised cost.

2. A firm's production function is $Q = K^{0.6}L^{0.6}$ where K is capital input measured in machine-hours and L is labour input measured in worker-hours. The firm is perfectly competitive and hires its machines at a constant rental rate of $r = 5$ euros per hour and its workers at a constant wage rate of $w = 2$ euros per hour.

(a) Using the Lagrange multiplier method, find the maximum hourly output that the firm can produce, given a fixed budget of 1000 euros.

(b) How is the solution changed if the budget is doubled? Is the maximum output also doubled?

(c) How is the solution changed if the market wage rate rises to 4 euros?

(d) Illustrate your solutions graphically.

3. A firm's production function is $Q = K^{0.6}L^{0.5}$ where K is capital input measured in machine-hours and L is labour input measured in worker-hours. The firm is perfectly competitive and hires its machines at a constant rental rate of $r = 2$ euros per hour and its workers at a constant wage rate of $w = 6$ euros per hour.

(a) Using the Lagrange multiplier method, find the minimum cost of producing 100 units of output.

(b) In the same way, find the minimum cost of producing 200 units of output. Comparing your answers to (a) and (b), does increasing output by 100% increase costs by more or less than 100% in this example?

(c) Show algebraically and diagrammatically that to achieve minimum cost for a given output, $Q = \overline{Q}$, the firm must choose values for L and K such that the isoquant for $Q = \overline{Q}$, is tangent to an isocost line.

(d) Show that, whatever level of output the firm produces, it will always employ capital and labour in the proportion $\frac{K}{L} = \frac{18}{5}$. Show how this proportion is related to the input prices and the parameters of the production function.

Exercise WS16.4

1. A firm's production function is $Q = K^{0.4}L^{0.5}$. The firm is perfectly competitive and hires its machines at a constant rental rate of $r = 4$ euros per hour and its workers at a constant wage rate of $w = 5$ euros per hour. It can also sell as much output as it wishes at the ruling market price of $P = 20$ euros.

(a) Find the most profitable output, the profits at this output, and the capital and labour inputs.

(b) How would the most profitable output change if P rose from 20 to, say, 21 (with input prices unchanged)? What does this tell us about the firm's supply function (that is, Q as a function of P)? Does supply appear to be elastic or inelastic?

(c) Find the relationship between K and L when total costs are minimised. Illustrate this graphically, showing the isoquants and iso-cost lines. Show the point of maximum profit (from (a) above) on the same diagram.

(d) Find the relationship between total cost and output when profits are maximised. (Hint: By definition, $TC = wL + rK$, and we can eliminate K and L using our answer to (c). It may help to recall that, by definition, a TC function gives the *minimum* total cost of a given output.) What can you deduce about marginal cost, as a function of output? How does marginal cost compare with marginal revenue?

2. A firm's production function is $Q = 5KL - 2K^2 - 3L^2$. The firm is perfectly competitive and the relevant prices, as defined in question 1 above, are $w = r = P = 1$.

(a) Find the function (equation) giving the combinations of *K* and *L* such that $MPK = \frac{r}{P}$. Sketch the graph of this function with *K* as the dependent variable.

(b) Similarly, find the function giving the combinations of *K* and *L* such that $MPL = \frac{w}{P}$. Sketch the graph of this function with *K* as the dependent variable.
(This equation and its counterpart in (a) are known as the first-order conditions for maximum profit.)

(c) Show that the first-order conditions for a maximum (or minimum) of profits are satisfied when *L* = 9 and *K* = 11. Find the output and profits. (Hint: You can do this either by the Lagrange multiplier method or by solving the two equations from (a) and (b).

(d) Consider whether *L* = 9, *K* = 11 is a maximum or a minimum of profits. Consider also whether costs are minimised, given the output, at this point.

Exercise WS16.5

1. Ann's utility function is $U = 3\ln X + 2\ln Y$ where *X* and *Y* are weekly consumption levels of goods *X* and *Y*. The market prices are P_X = 2 euros and P_Y = 1 euro, and her weekly budget is *B* = 100 euros.

(a) Find the quantities that Ann should buy each week in order to maximise her utility.

(b) Show graphically her equilibrium as a tangency between an indifference curve and her budget line.

(c) What is her weekly expenditure on each good (i) in money terms, and (ii) as a proportion of her budget?

(d) Suppose P_X rises from 2 to 3 euros. Find the new utility-maximising quantities. Comment on the effect of a rise in P_X on the demand for good *Y*.

(e) Following this price increase, what are Ann's new weekly expenditures on each good (i) in money terms, and (ii) as a proportion of her budget?

(f) From your answer to (a), find Ann's demand function for good *X* (that is, the functional relationship between the quantity of *X* the she wishes to buy and the price of *X*, with other variable(s) treated as parameters). From the demand function, find the price elasticity of demand for *X*.

(g) Repeat (f) for good Y.

2. Ben's utility function is $U = X^3Y^2$. His weekly budget is 100 euros and he buys in the same markets as Ann in the previous question, and therefore faces the same prices; that is, P_X = 2 euros and P_Y = 1 euro.

   Show that Ben will buy the same quantities of each good as Ann, and that in general his demand functions for the two goods are identical to Ann's. Can you suggest why this is true, despite the difference in their utility functions?