Exercise WS14.1

1. Consider the surface $z = 10x^{0.5} + 5y$, with *x* and *y* assumed not negative.

(a) Assign any fixed positive value to *y*, and thence sketch the graph of the resulting relationship between *x* and *z*. Repeat this for further fixed values of *y* until you have built up a picture of what the iso-*y* sections of this surface look like. (Hint: In chapter 4 we considered the graph of $y = x^{0.5}$.)

(b) Repeat (a) for various fixed values of *x* and thus build up a picture of what the iso-*x* sections look like. (Hint: this is easier than (a).)

(c) Find the intercepts on the three axes and, using this information together with what you have discovered in (a) and (b), make a three dimensional sketch graph of the surface.

2. Repeat question 1 for the surface $z = xy + 10x + 10y$, with *x* and *y* assumed not negative. (Hints: In this case it might help to start by considering the iso-*x* section for *x* = 0, and similarly for *y* = 0. Then consider how *z* behaves when *x* = *y*.)

Exercise WS14.2

For each of the following functions, find $\frac{\partial z}{\partial x}$ and $\frac{\partial z}{\partial y}$.

(Hint: All the rules of differentiation from chapters 6 and 13 for a function of one independent variable apply straightforwardly, with the other independent variable(s) treated as constants.)

1. (a) $z = 3x^2 + 2y^3 + 5$

(b) $z = x^{\frac{1}{2}} + \dfrac{x^2}{y^3}$

(c) $z = \dfrac{2x^3 + 3y^2}{2x + 4y^2}$

(d) $z = e^{x-y}$

2. Find $\frac{\partial z}{\partial u}$, $\frac{\partial z}{\partial v}$ and $\frac{\partial z}{\partial w}$ for each of the following:

(a) $z = u^5 + v^2 + w^3 + uvw$

(b) $z = u^{\frac{1}{2}} v^{\frac{1}{4}} w^{\frac{1}{3}}$

Exercise WS14.3

1. (a) Given $z = 5x^4 + 3x^2 y + y^2$, find the two first-order partial derivatives, $\frac{\partial z}{\partial x}$ and $\frac{\partial z}{\partial y}$, and the four second-order partial derivatives, $\frac{\partial^2 z}{\partial x^2}$, $\frac{\partial^2 z}{\partial y \partial x}$, $\frac{\partial^2 z}{\partial y^2}$, and $\frac{\partial^2 z}{\partial x \partial y}$.

   (b) Explain briefly what, in general, each of these partial derivatives measures.

2. For each of the functions in exercise WS14.2, question 1, find

   $\frac{\partial^2 z}{\partial x^2}$, $\frac{\partial^2 z}{\partial y^2}$, $\frac{\partial^2 z}{\partial y \partial x}$, and $\frac{\partial^2 z}{\partial x \partial y}$.

3. For each of the following functions, find $\frac{\partial z}{\partial x}$, $\frac{\partial z}{\partial y}$, and $\frac{\partial^2 z}{\partial y^2}$

   (a) $z = (x^2 + 2y)^{0.5}$

   (b) $z = y^2 e^{x^2}$

4. Suppose you are given the following information about a function $z = f(x, y)$.

   At a point P, $\frac{\partial z}{\partial x} > 0$; $\frac{\partial^2 z}{\partial x^2} > 0$; $\frac{\partial^2 z}{\partial y \partial x} < 0$; $\frac{\partial z}{\partial y} < 0$; $\frac{\partial^2 z}{\partial y^2} > 0$

   Use this information to sketch the shapes of the iso-*x* and iso-*y* sections at P and thus indicate the shape of the surface in the vicinity of P. (Hint: Figures 14.12 and 14.13 in the book are relevant.)

5. Repeat question 4 for a function $z = \text{h}(x, y)$, where at a point R,

$$\frac{\partial z}{\partial x} < 0 \ ; \ \frac{\partial^2 z}{\partial x^2} < 0 \ ; \ \frac{\partial^2 z}{\partial y \partial x} < 0 \ ; \ \frac{\partial z}{\partial y} > 0 \ ; \ \frac{\partial^2 z}{\partial y^2} > 0$$

Exercise WS14.4

1. A firm finds that its production function is

$$Q = 15KL - 4K^2 - 5L^2 + 6K + 4L$$

where Q is weekly output (in thousands of units) and K and L are weekly inputs of machine-hours and worker-hours respectively (measured in thousands).

(a) By differentiation, find the marginal products of labour and capital, and sketch their graphs.

(b) Confirm that your graphs in (a) are correctly drawn by examining the signs of the direct second derivatives $\frac{\partial^2 Q}{\partial L^2}$ and $\frac{\partial^2 Q}{\partial K^2}$. Is this production function characterised by diminishing marginal productivity?

(c) By examining the signs of the cross partial derivatives $\frac{\partial^2 Q}{\partial L \partial K}$ and $\frac{\partial^2 Q}{\partial K \partial L}$ determine the effect on the MPL of an increase in the capital input and the effect on the MPK of an increase in the labour input. Give an economic interpretation to this finding. Would you expect it to be generally true of all or most production functions?

(d) Find the equation of the average product of labour (APL). By differentiation, show that APL reaches a maximum when $L = \left(\frac{2}{5}K(2K-3)\right)^{0.5}$. Show that, when K = 10, the value if L that maximises the APL is L = 8.246. Find the maximised value of the APL. Show that, in the usual way, MPL = APL at this point. Hence sketch the graphs of the APL and MPL on the same diagram.

(e) Repeat (d) for the average product of capital, APK.

(f) Comment on the general plausibility of this functional form as a production function.

2. A firm's production function has the Cobb-Douglas form $Q = 100K^{0.5}L^{0.75}$.

(a) Find the marginal products of capital and labour and show that they can be written respectively as

$$\frac{\partial Q}{\partial K} = 0.5\frac{Q}{K} \quad \text{and} \quad \frac{\partial Q}{\partial L} = 0.75\frac{Q}{L}$$

(Hint: There is a little trick involved here; see the appendix to chapter 14.)

(b) Are the marginal products of capital and labour ever negative? Explain how you reached your answer. Give an economic interpretation to your answer.

(c) How are the marginal products of capital and labour related to the respective average products? Explain how you reached your answer.

(d) Sketch the graphs of the marginal and average product functions.

(e) Show that the four second-order partial derivatives can be written as

$$\frac{\partial^2 Q}{\partial K^2} = -0.25\frac{Q}{K^2} \;\; ; \;\; \frac{\partial^2 Q}{\partial L^2} = -\tfrac{3}{16}\frac{Q}{L^2} \;\; ; \;\; \frac{\partial^2 Q}{\partial L \partial K} = \tfrac{3}{8}\frac{Q}{KL} \;\; ; \text{ and } \;\; \frac{\partial^2 Q}{\partial K \partial L} = \tfrac{3}{8}\frac{Q}{KL}$$

(Hint: use the same trick as in (a) above.)

(f) Is diminishing marginal productivity a characteristic of this production function? If so, does it operate at some, or all, levels of input and output? Explain how you reached your answer.

(g) What is the effect on the marginal product of one input if the other input is increased to a new fixed value? Explain how you reached your answer.

3. Continuing with the production function in the previous question, find the equation of a typical isoquant (say, $Q = 100$) with K as the dependent variable. Show that the isoquants are downward sloping and convex in the KL plane. Explain the relationship between the marginal products of capital and labour and the slope of an isoquant.

4. An individual's utility function has the Cobb-Douglas form

$$U = X^{0.5}Y^2$$

where U is an index of her utility and X and Y are the weekly quantities of the two goods consumed.

(a) Find the marginal utilities of the two goods. Does the consumer ever experience satiation with respect to either good?

(b) How does the marginal utility of each good vary as consumption of it increases with consumption of the other good held constant? What is the economic significance of the signs of these partial derivatives?

(c) Use the information obtained above to sketch the graph of a typical iso-*Y* section; that is, the curve showing how *U* varies as consumption of *X* increases with consumption of *Y* held constant. Similarly, sketch the graph of an iso-*X* section.

(d) By examining the signs of the cross-partial derivatives, determine whether an increase in consumption of *X* increases the marginal utility of *Y*, and vice versa. What is the economic interpretation of the signs of these cross partial derivatives?

(e) (i) Find the equation of a typical indifference curve, with either *K* or *L* as the dependent variable. (ii) Do any indifference curves ever cut either of the axes? (iii) How does the answer to this question relate to the issue of satiation in (a) above?

(f) (i) By differentiation, show that the indifference curve is negatively sloped and convex in the *KL* plane. (ii) Give an economic interpretation to the slope and curvature of the indifference curve. (iii) Sketch the graph of the indifference curve.

5. A person's utility function is $U = (X + 2)(Y + 1)$.

(a) Find the marginal utilities of goods *X* and *Y*.

(b) Find the direct second derivatives and the cross partial derivatives. What do these four derivatives tell us about:

(i) how the marginal utility of *X* changes when consumption of *X* increases with *Y* constant;

(ii) how the marginal utility of *X* changes when consumption of *Y* increases with *X* constant;

(iii) how the marginal utility of *Y* changes when consumption of *Y* increases with *X* constant; and

(iv) how the marginal utility of *Y* changes when consumption of *X* increases with *Y* constant?

(c) Find the equation of a typical indifference curve, with either *Y* or *X* as the dependent variable. Do the indifference curves cut either or both axes?

(d) By differentiation, determine the slope and convexity or concavity of the indifference curve. Sketch some indifference curves.

(e) In the light of your answers to (a) and (d), what can we infer about the relationship between diminishing marginal utility and the convexity of indifference curves?