Complex Numbers

2

2.1 Motivation and concept

Complex numbers arise naturally in mathematics, often when solving quadratic equations such as $x^2 + x + 1 = 0$, which has the solutions $x = \dfrac{1 \pm \sqrt{-4}}{2} = \dfrac{1}{2} \pm \sqrt{-1}$. Because the negative square root cannot be evaluated, as no ordinary number can be negative when squared, a new number conventionally called i (although engineers call this j) was invented with the property $i^2 = -1$ or $i = \sqrt{-1}$. The solution to the equation becomes $x = 1/2 \pm i$. This new number is one of a new class called *complex numbers*. These are not numbers in the elementary sense used in counting or measuring, but constitute new mathematical objects and have an existence of their own. These numbers are called 'complex' only because they contain two parts and can always be written in the form

$$z = a + ib \tag{2.1}$$

where a is called the *real* (*Re*) part and b the *imaginary* (*Im*) part of the number. The complex number $z = i$, if written in the form of equation (2.1), has a real part $a = 0$ and an imaginary part $b = 1$. The latter is rather a misnomer as b is just as 'real' as is a; it is just a number and perhaps, therefore, the best way to view a complex number is to consider it a number in two dimensions with amounts a and b in each of these dimensions. In that case, a complex number can be represented as a point on a graph rather than being a point on a line, as a normal number may be considered to be. The graph is called an Argand diagram, if drawn with the real part a along the conventional x-axis and b along the y; the area defined by a and b is also called the Argand or Gauss plane. The imaginary number i has a real part that is 0 and an imaginary part that is 1, and is represented by the point {0, 1} on the y-axis of an Argand diagram.

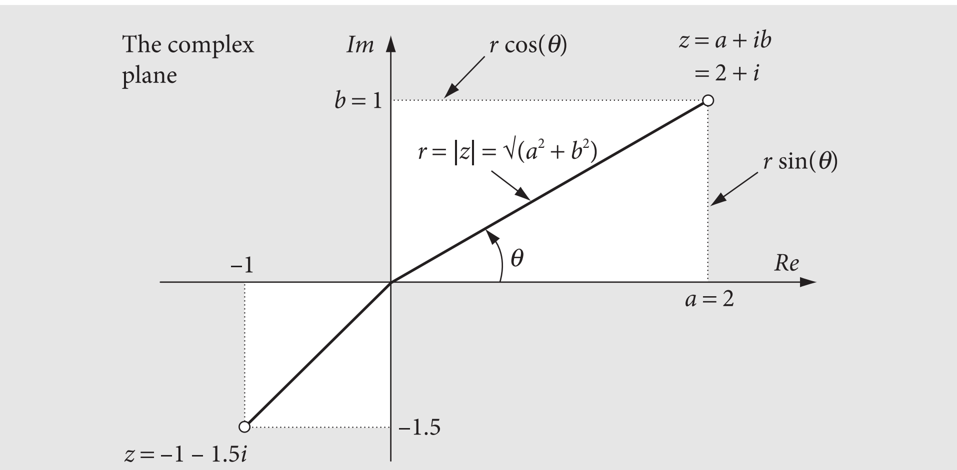

Fig. 2.1 The Argand diagram showing two complex numbers in the form $z = a + ib$, r is the modulus of the complex number z and θ the argument measured anticlockwise from the real axis.

The Argand diagram is not like a normal graph in which a function such as $y = x^3$ is plotted, because the value of y on the graph normally shows how large the function is at a given value of x. The Argand diagram shows one point in the real and imaginary plane for each complex number so is more like a map that locates a place with latitude or longitude.

Performing algebra with complex numbers is no more difficult than with 'normal' numbers, because the prime rule of algebra still applies:

'Whatever I do to one side of an equation I do to the other side.'

The normal rules for addition and multiplication apply but with the additional rule that additions and subtractions are kept separate for the real and imaginary parts, as is done for components of vectors. A complex number can be divided in the usual way by a real number. Dividing by a complex number has the additional step that the top and bottom of the expression are first multiplied by the complex conjugate of the denominator. This is explained below. Although i is a complex number, $i^2 = -1$ and is a real number:

$$i = \sqrt{-1}, \quad -i = -\sqrt{-1}, \quad i^2 = -1, \quad i = \frac{-1}{i}.$$

2.2 Complex conjugate

Complex numbers possess a new property compared to real numbers and this is the complex conjugate. If $z = a + ib$ then the *complex conjugate* is defined as

$$z^* = a - ib, \tag{2.2}$$

where, by convention, an asterisk is added and every i is replaced with $-i$; the result is that z^*z is always a real number;[1]

$$z^*z = (a + ib)^*(a + ib) = (a - ib)(a + ib) = a^2 + b^2. \tag{2.3}$$

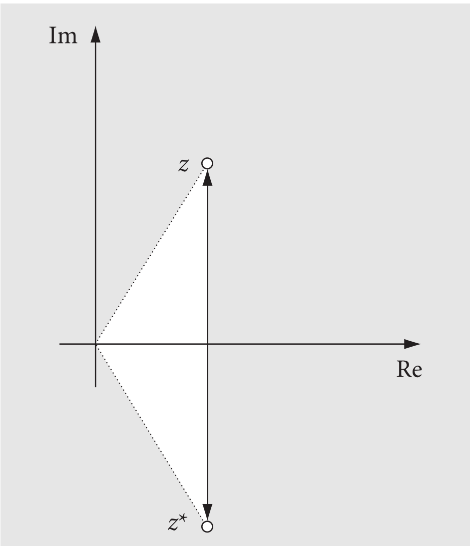

Fig. 2.2 The complex number z and its complex conjugate z^*.

In geometrical terms, forming the complex conjugate is equivalent to a reflection in the real axis because only the imaginary part is inverted.

In quantum mechanics, the wavefunction is often found to be a complex quantity and, therefore, the complex conjugate is always used to calculate expectation values such as $\langle x \rangle = \int \psi^* x \psi dx$ and probabilities $p = \int \psi^* \psi dx$ because only a mathematically real quantity is measured in an experiment, not an imaginary one.

The quantity $z + z^*$ is always a real number equal to $2\text{Re}(z)$ or $2\text{Re}(z^*)$ which is the same. It is worth remembering the rules

$$(z_1 + z_2)^* = z_1^* + z_2^*, \qquad (z_1 z_2)^* = z_1^* z_2^*.$$

In some textbooks and some scientific papers, formulae involving complex numbers are written in a form that does not include the complex conjugate but instead has the notation +*c.c.* at the end of the equation to indicate that the complex conjugate is to be added. This is primarily a method of increasing the readability of formulae. An electric field describing linearly polarized light could be written as

$$E(t, x) = E_0(e^{i(\omega t - kx)} + c.c.)$$

instead of $E(t, x) = E_0(e^{i(\omega t - kx)} + e^{-i(\omega t - kx)})$. Similarly,

$$\chi(t) = E_0\left(\frac{e^{i\omega t}}{\omega_a^2 - \omega^2 + 2i\omega/T} + c.c\right)$$

represents

$$\chi(t) = E_0\left(\frac{e^{i\omega t}}{\omega_a^2 - \omega^2 + 2i\omega/T} + \frac{e^{-i\omega t}}{\omega_a^2 - \omega^2 - 2i\omega/T}\right)$$

[1] Some texts use a bar over the number to represent the complex conjugate although this rare.

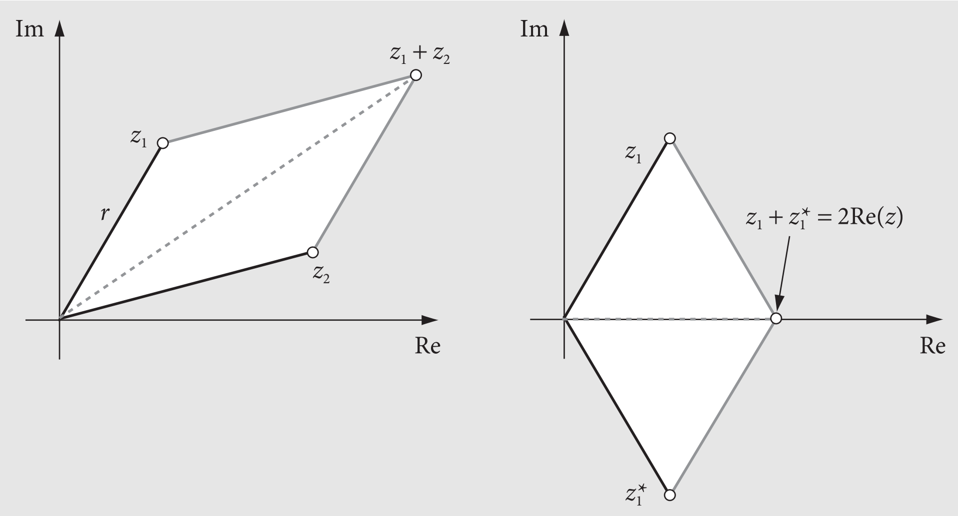

Fig. 2.3 Left: Adding two complex numbers together to form $z_1 + z_2$, dashed line. Right: Adding $z_1 + z_1^*$.

2.2.1 Adding complex numbers

The real and imaginary parts are added separately as shown in Fig. 2.3. This is somewhat like adding two vectors.

2.2.2 Multiplying and dividing complex numbers

Multiplying complex numbers is straightforward using the normal rules of algebra but remembering to use $i^2 = -1$ where necessary.

$$(3 + 5i)(1 - 2i) = 3 - 6i + 5i - 10i^2 = 13 - i.$$

Dividing numbers is a little more difficult. Always multiply top and bottom of the whole expression by the complex conjugate of the denominator, because this makes the denominator a real number, and is equivalent to multiplying by 1. An example makes this clearer.

$$\frac{3 + 5i}{1 - 2i} = \left(\frac{3 + 5i}{1 - 2i}\right)\frac{(1 - 2i)^*}{(1 - 2i)^*} = \left(\frac{3 + 5i}{1 - 2i}\right)\left(\frac{1 + 2i}{1 + 2i}\right) = \frac{13 - i}{5}.$$

2.2.3 Modulus and Argument

The second new property held by complex numbers is variously called the *modulus*, *magnitude, absolute value*, or *norm* of the complex number. This is calculated in a similar way to that of a vector and is the length of the complex number measured from the origin, Figs 2.1–2.4.

The modulus r of the complex number $z = a + ib$ is

$$r = +\sqrt{a^2 + b^2}. \tag{2.4}$$

It is variously written as

$$r = |z| = |a + ib| = +\sqrt{z^* z} = |z^*|. \tag{2.5}$$

The square of a complex number is the square of the modulus;

$$|a + ib|^2 = (a + ib)^*(a + ib) = (a - ib)(a + ib) = a^2 + b^2 = z^* z = |z|^2$$

and is always a positive number.

In Fig. 2.1 and Fig. 2.4 the line from the origin to the complex number is at an angle θ given by

$$\tan(\theta) = b/a, \qquad \theta = \tan^{-1}(b/a) \tag{2.6}$$

measured anticlockwise from the real axis. This angle θ is called the *argument*, *amplitude*, *polar angle*, or *phase* of the complex number and is measured in radians, a full circle being

2π radians. The use of the word 'amplitude' to mean an angle is very confusing, and should probably be avoided.

The location of any complex number is $\{a, b\}$ in Cartesian type coordinates, or alternatively, in polar type coordinates is $\{r, \theta\}$. The complex number is then described as

$$z = r[\cos(\theta) + i\sin(\theta)].$$

This interpretation is also illustrated in Fig. 2.1 for a point $z = a + ib$ where r is the distance of the point from the origin. The distance along the real axis is $a = r\cos(\theta)$ and along the imaginary axis, $b = r\sin(\theta)$. Equating the real and imaginary parts gives

$$z = a + ib = r[\cos(\theta) + i\sin(\theta)]. \tag{2.7}$$

For example, if the complex number is $z = i$, it has a real part that is 0 and an imaginary part of 1, and is represented by a point $\{0, 1\}$ which is on the imaginary axis. Its modulus is 1 and its argument $\pi/2$. If the number is $z = -1 - i$ then the point is found at $\{-1, -1\}$ on the Argand diagram. Its argument is $-5\pi/4$ (225°) and its modulus $\sqrt{(-1-i)(-1+i)} = \sqrt{2}$.

2.3 Summary

If the complex number is $z = a + ib = r[\cos(\theta) + i\sin(\theta)]$ where a and b are real numbers, then

$a = \mathrm{Re}(z)$ is the *real part* of z

$b = \mathrm{Im}(z)$ is the *imaginary part* of z

$r = |z| = \sqrt{z^*z}$ is the *modulus* of z, or *absolute* value, *magnitude* or *norm*.

$\theta = \tan^{-1}(b/a)$ is the *argument* of z, also called the *polar angle* or *phase*.

$z^* = a - ib = r[\cos(\theta) - i\sin(\theta)]$ is the *complex conjugate* of z.

$zz^* = |z|^2 = |z^*|^2$ is the *absolute* value is always a positive real number.

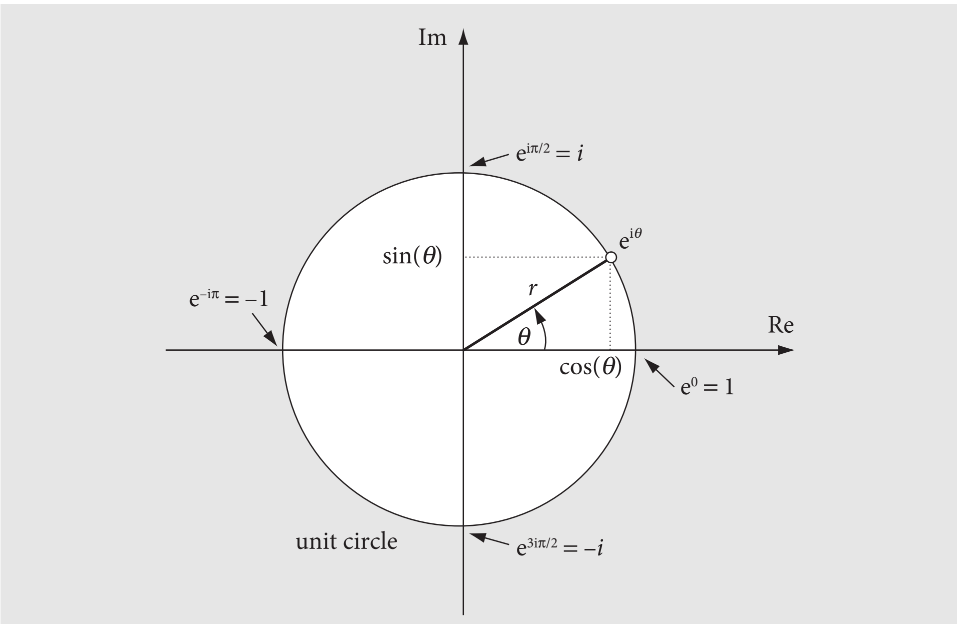

Fig. 2.4 As the angle (argument) θ varies anticlockwise from 0 to 2π, the complex number changes from 1 to i to -1 to $-i$ according to Euler's theorem, equation (2.19). A unit circle has radius of 1.

2.4 Using Maple

When using Maple to perform calculations with complex numbers, a non-conventional notation is used, and `I` (capital *i*) represents *i* in mathematical notation. Also, to evaluate expressions with complex numbers, it is necessary to use `evalc(..)` to force a calculation to happen. For example,

```
> evalc( exp(I*Pi));                -1
```

The real `Re(..)` imaginary, `Im(..)` and absolute (modulus) `abs(..)` values are next calculated with a function defined as *f*. Notice how `evalc(..)` has to be used to force the result.

```
> f:= exp(-I*x/2)*I*Pi/2;
```

$$f := \frac{1}{2}\,\mathrm{I}\,e^{-\frac{1}{2}\mathrm{I}\,x}\,\pi$$

```
> real_part:= Re(f);              value:= evalc(Re(f));
  Imaginary_part:= Im(f);         value:= evalc(Im(f));
  absolute_value:= abs(f);        value:= evalc(abs(f));
```

$$real\_part := -\frac{1}{2}\pi\,\Im\left(e^{-\frac{1}{2}\mathrm{I}\,x}\right) \qquad value := \frac{1}{2}\pi\sin\left(\frac{1}{2}x\right)$$

$$Imaginary\_part := \frac{1}{2}\pi\,\Re\left(e^{-\frac{1}{2}\mathrm{I}\,x}\right) \qquad value := \frac{1}{2}\pi\cos\left(\frac{1}{2}x\right)$$

$$absolute\_value := \frac{1}{2}e^{\frac{1}{2}\Im(x)}\,\pi \qquad value := \frac{1}{2}\pi$$

In Section 2.8, it is shown how easy it is to evaluate these apparently complicated expressions.

2.5 Questions

Full solutions are available at **www.oxfordtextbooks.co.uk/orc/beddard**.

Q2.1 If $z_1 = 2 + i$ and $z_2 = -1 - 3i/2$,

(a) calculate $z_1 + z_2$ and **(b)** $z_1 - z_2$. **(c)** What is $-iz_1z_2$?

Q2.2 **(a)** If $i^2 = -1$, what are i^3, i^4, i^5, and i^6?

(b) What relationship links positive powers of i?

Q2.3 If $z = a - ib$ what is i^3z?

Q2.4 If $z = 3 + 4i$ find z^2 and the modulus and argument of z^2.

Q2.5 Calculate $z = (2 - 5i)(3 + i) + 3i$, and find the modulus and argument of the result.

Q2.6 Express the number $\dfrac{5 - i}{2 - 3i}$ in the form $z = a + ib$ and find its modulus and argument.

Q2.7 Simplify $z = (2 - 5i)(3 + i)/(3 - i)$ and find the modulus and argument of the result.

Q2.8 Find the modulus and argument of

(a) $\cos(\theta) - i\sin(\theta)$, and

(b) $1 - i\tan(\theta)$, where $0 < \theta < \pi/2$ in both cases.

Q2.9 If $w = z^2$ and $z = x + iy$ and $w = u + iv$, find u and v.

2.6 DeMoivre's theorem and powers of complex numbers

A complex number z can be written as

$$z = r[\cos(\theta) + i\sin(\theta)],$$

and if n is any number, what is $z^n = r^n[\cos(\theta) + i\sin(\theta)]^n$? The trigonometric part can be shown to have the simple form,

$$[\cos(\theta) + i\sin(\theta)]^n = \cos(n\theta) + i\sin(n\theta), \tag{2.8}$$

therefore,

$$z^n = r^n[\cos(n\theta) + i\sin(n\theta)] \tag{2.9}$$

which is called DeMoivre's theorem and is essential to calculating powers of complex numbers. One of the unexpected things that can be done is to find the n^{th} root of 1, i, −3 or any other number for that matter.

To demonstrate that DeMoivre's theorem is correct, calculate the product of two complex numbers expressed in angular form, and then let $\theta_1 = \theta_2$. Suppose, for simplicity, that $r_1 = r_2 = 1$, then the product of two numbers is

$$\begin{aligned}&[\cos(\theta_1) + i\sin(\theta_1)][\cos(\theta_2) + i\sin(\theta_2)]\\ &= \cos(\theta_1)\cos(\theta_2) + i\cos(\theta_1)\sin(\theta_2) + i\sin(\theta_1)\cos(\theta_2) - \sin(\theta_1)\sin(\theta_2)\\ &= \cos(\theta_1 + \theta_2) + i\sin(\theta_1 + \theta_2).\end{aligned}$$

The double angle formula (Chapter 1.5.1) was used in the last step, and letting $\theta_1 = \theta_2$ produces

$$[\cos(\theta) + i\sin(\theta)]^2 = \cos(2\theta) + i\sin(2\theta).$$

as predicted by DeMoivre's theorem. This result can be generalized to any power of a real or complex value n.

The product z_1z_2 and quotient z_1/z_2 of two complex numbers are written in this form as

$$z_1z_2 = r_1r_2[\cos(\theta_1 + \theta_2) + i\sin(\theta_1 + \theta_2)], \tag{2.10}$$

where the angles add, and provided that $z_2 \neq 0$,

$$\frac{z_1}{z_2} = \frac{r_1}{r_2}[\cos(\theta_1 - \theta_2) + i\sin(\theta_1 - \theta_2)]$$

where the angles subtract. There is a geometrical interpretation to multiplying two complex numbers. If their moduluses are unity, $z_1 = \cos(\theta_1) + i\sin(\theta_1)$ and $z_2 = \cos(\theta_2) + i\sin(\theta_2)$, then multiplication results in rotation about the origin, equation (2.10). Geometrically this is shown in Fig. 2.5.

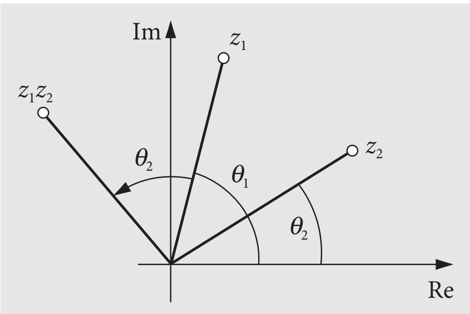

Fig. 2.5 Geometrical interpretation of the multiplication of two complex numbers.

2.6.1 Roots of a complex number

Suppose that w is a real or complex number whose roots we need to find, then mathematicians have shown that, in general, the answer will be a complex number. If the n roots of a number z are expressed as $w = z^{1/n}$, then the equation to examine is $w^n = z$.

We will let both sides of this equation be different complex numbers. Expressing the left-hand side in angular form using DeMoivre's theorem with a polar angle ϕ gives

$$w^n = R^n[\cos(n\phi) + i\sin(n\phi)]. \tag{2.11}$$

The right-hand side of the equation is

$$z = r[\cos(\theta) + i\sin(\theta)] \tag{2.12}$$

since any complex number can be written in this way. Therefore,

$$R^n = r$$

where both R and r are real numbers. The angles ϕ and θ are related in the most general way as

$$n\phi = \theta + 2\pi k \tag{2.13}$$

where $k = 0, 1, 2, \dots n-1$ because sine and cosine are cyclic functions; $\sin(\theta) = \sin(\theta + 2\pi) = \sin(\theta + 4\pi)$ and so forth, therefore there will be more than one root to the equation. Using $n\phi = \theta$ only allows one root to be found. Using equations (2.11) and (2.13), gives

$$w = R^{1/n}\left[\cos\left(\frac{\theta + 2\pi k}{n}\right) + i\sin\left(\frac{\theta + 2\pi k}{n}\right)\right]. \tag{2.14}$$

In the special case of calculating the n^{th} root of unity, $w^n = 1$ and $z = 1$, then from equation (2.12), $r = 1$, $\theta = 0$ and therefore,

$$w = \left[\cos\left(\frac{2\pi k}{n}\right) + i\sin\left(\frac{2\pi k}{n}\right)\right]. \tag{2.15}$$

There is always one real root and the other roots fall on the vertices of a polygon which is formed inside a circle of unit radius and touches the circle at its vertices.

To illustrate the method, $w^5 = 1$ is solved to find the five fifth roots of unity. The equation to solve is $w^n = z$ with $n = 5$ and $z = 1$. The roots are the solution of equation (2.15) with $n = 5$,

$$z = 1^{1/5} = \cos(2k\pi/5) \pm i\sin(2k\pi/5)$$

where $k = 0, 1, 2, 3, 4$. The *principal value* of the equation is the one solved with $k = 0$. The five roots are then

$$w = 1, \cos(2\pi/5) + i\sin(2\pi/5), \quad \cos(4\pi/5) + i\sin(4\pi/5),$$
$$\cos(6\pi/5) + i\sin(6\pi/5), \quad \cos(8\pi/5) + i\sin(8\pi/5),$$

and as $\sin(2\pi/5) = -\sin(8\pi/5)$ and so forth, only the positive terms need be used. Only one of the roots is not a complex number and as this first root lies on the real axis, the angle to the next root is

$$\theta = \tan^{-1}[\sin(2\pi/5)/\cos(2\pi/5)] \equiv 72°$$

and the other roots are separated from each other by the same angle as expected for a pentagon.

In the Maple calculation, the sequence command `seq` is used to generate pairs of numbers, which are plotted with the `pointplot` procedure. The circle function is used and `display` plots both curves together. To do this, each plot is given a name, `c1` and `p1`. The `constrained` instruction ensures that the x- and y-axes are the same size on the page and so produce a circle. The code here can be used to find and plot any root by defining `n` with `theta:=0`.

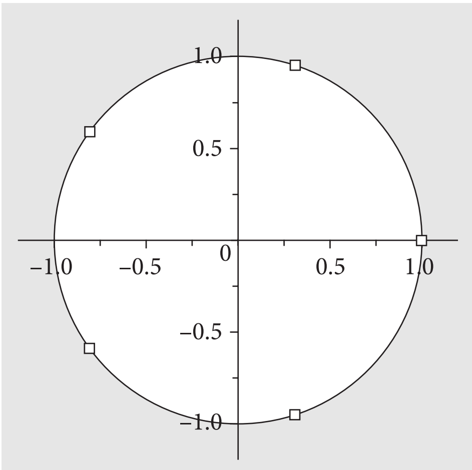

Fig. 2.6 The five roots of the equation $z^5 = 1$. The points form a pentagon.

Algorithm 2.1 *n* roots of unity

```
> with(plottools): with(plots):
  c1:= circle([0,0],1):               # name for plot circle
  n:= 5:   theta:= 0:                 # define values; n>0
  s1:= seq([cos((theta+2*Pi*k)/n ),sin((theta+2*Pi*k)/n )],
              k = 0..n-1):            # pairs of points
  p1:= pointplot([s1], symbol=BOX, scaling = constrained ):
  display([ p1, c1 ]);                # plot both p1 and c1
```

2.7 Questions

Full solutions are available at **www.oxfordtextbooks.co.uk/orc/beddard**.

Q2.10 Find the four roots of $(-3)^{1/4}$.

Strategy: This problem is the same as solving the equation $w^4 = -3$ and as there are four roots they must form a square on an Argand diagram whose corners lie on a circle of radius $3^{1/4}$. The roots of a negative number are sought so these must all be complex with a zero real part; i.e. with an imaginary part only.

Q2.11 Find the square roots of i, i.e. $w^2 = i$. Find their magnitude and plot them on an Argand diagram.

Q2.12 Solve $w^4 = 16$.

Strategy: Because the equation is fourth order, there are four solutions and not just the two real ones $w = \pm 2$. Use the method of previous questions.

Q2.13 Calculate the modulus and argument of $2 + 3i$ then calculate its square roots. What is the radius of the circle on which the roots lie and at what angles?

Strategy: The complex number $2 + 3i$ is best converted into its trigonometric form to calculate the modulus and argument.

2.8 Euler's theorem

The exponential series is $e^x = 1 + x + \dfrac{x^2}{2!} + \dfrac{x^3}{3!} + \cdots$, and similarly a series can be formed in the complex number w,

$$e^w = 1 + w + \frac{w^2}{2!} + \frac{w^3}{3!} + \cdots.$$

Now suppose that $w = i\theta$, where θ is real, then rearrange into real and imaginary terms;

$$e^{i\theta} = 1 + i\theta - \frac{\theta^2}{2!} - i\frac{\theta^3}{3!} + \frac{\theta^4}{4!} \cdots = \left[1 - \frac{\theta^2}{2!} + \frac{\theta^4}{4!} - \cdots\right] + i\left[\theta - \frac{\theta^3}{3!} + \frac{\theta^5}{5!} - \cdots\right].$$

The real and imaginary parts are expansions of the cosine and sine functions respectively, therefore, if z is a complex number

$$z = e^{i\theta} = \cos(\theta) + i\sin(\theta). \tag{2.16}$$

This equation was discovered in 1748 by the Swiss mathematician Euler, and is extremely important as it crops up everywhere from quantum mechanics to X-ray crystallography and other phenomena connected with waves.

Writing $\theta = -\theta$ produces

$$e^{-i\theta} = \cos(\theta) - i\sin(\theta)$$

and therefore, for a general complex number with (modulus) r as a real number,

$$re^z = re^{i\theta} = r[\cos(\theta) + i\sin(\theta)].$$

DeMoivre's theorem can be derived from these equations: the power of a complex number w is

$$w^n = r^n e^{in\theta} = r^n[\cos(n\theta) + i\sin(n\theta)]. \tag{2.17}$$

Adding and subtracting $e^{i\theta}$ and $e^{-i\theta}$ gives

$$\cos(\theta) = \frac{e^{i\theta} + e^{-i\theta}}{2} \qquad \text{and} \qquad \sin(\theta) = \frac{e^{i\theta} - e^{-i\theta}}{2i} \tag{2.18}$$

which are equations that prove most useful in manipulating trig functions.

Calculating $e^{i\theta}$ with $\theta = \pi$ and $r = 1$ produces

$$e^{i\pi} = -1 \qquad \text{or} \qquad e^{i\pi} + 1 = 0 \tag{2.19}$$

which some consider the most beautiful equation in mathematics, as it connects the most important numbers of mathematics (0, 1, i, e, and π) and uses the most important operations (multiplication, exponentiation, negation, and addition). Furthermore, an integer is produced by raising an irrational number π times the imaginary unit i to the power of another irrational number, e. It is not at all obvious why this connection exists from an arithmetical standpoint, but from a geometrical one it is clearer. Consider a circle of unit radius on an Argand diagram; as the angle θ increases from 0 to 2π, the modulus (radius) is 1 when $\theta = 0$, and is i when θ is π/2, and −1 when the angle is π and so on; see Fig. 2.4.

Euler's formula is important in science, because it permits the description of a sinusoidally varying real quantity by means of complex exponentials. This change simplifies equations, because it is far easier to manipulate exponentials than trig functions. For example, the general form of a sinusoidally varying quantity, such as a plane wave, is $f(t) = a_0\cos(\omega t - \theta)$, where a_0 is the amplitude, ω the frequency, and θ the phase. These are all constants, and t is time and is a real variable. The equivalent complex function is

$$g(t) = a_0 e^{i(\theta - \omega t)} = a_0[\cos(\omega t - \theta) - i\sin(\omega t - \theta)]$$

therefore $f(t) = \mathrm{Re}[g(t)]$. Very often in chemistry and physics, the complex form is used without explicitly stating that it is only the real part that represents the waveform. Fig. 2.7 compares various waveforms.

As an example of using Euler's equation, we will evaluate $w = \ln(-1)$ even though it doesn't exist—at least as a pure real number, then $w = \ln(i)$ and $w = \ln(z/3)$ are calculated where z is any complex number. The strategy in problems of this type is to convert the number −1, or i, or whatever it is into an exponential form using Euler's theorem.

(i) In the first example, $w = \ln(-1)$ or $e^w = -1$ and w has to be found to solve this equation. A general complex number can always be written as $z = re^{i\theta}$, therefore to find w, let $w = i\theta$. The absolute value (modulus) r of e^w is $\sqrt{e^{i\theta}e^{-i\theta}} = 1$. Because $e^{i\theta} = \cos(\theta) + i\sin(\theta)$, when $\theta = \pi$, $e^{i\theta} = -1$ making the *principal value* of $\ln(-1) = \ln(1e^{i\pi}) = i\pi$, which is a complex number. Note that there are other values of θ separated by $2k\pi i$, where k is an integer because $e^{i\theta}$ is a cyclic function.

(ii) In this example, $w = \ln(i)$ or $e^w = i$ and let $w = i\theta$. As $e^{i\theta} = \cos(\theta) + i\sin(\theta)$, when $\theta = \pi/2$ this equation produces $e^{i\pi/2} = i$ or $\ln(i) = \dfrac{i\pi}{2}$.

(iii) If $w = \ln(z/3)$, then $3e^w = z$, and if z is any complex number then we look for a value of θ such that $3e^{i\theta} = z$. Generally a complex number is represented by $z = re^{i\theta}$, then in this example $w = \ln(z) = \ln(3e^{i\theta}) = \ln(3) + i(\theta + 2\pi k)$ and $2\pi k$ is added because the function is cyclic and k is any integer; recall that the Euler equation can be put into a cosine and sine form, so it is a repetitive function. The principal value occurs when $k = 0$.

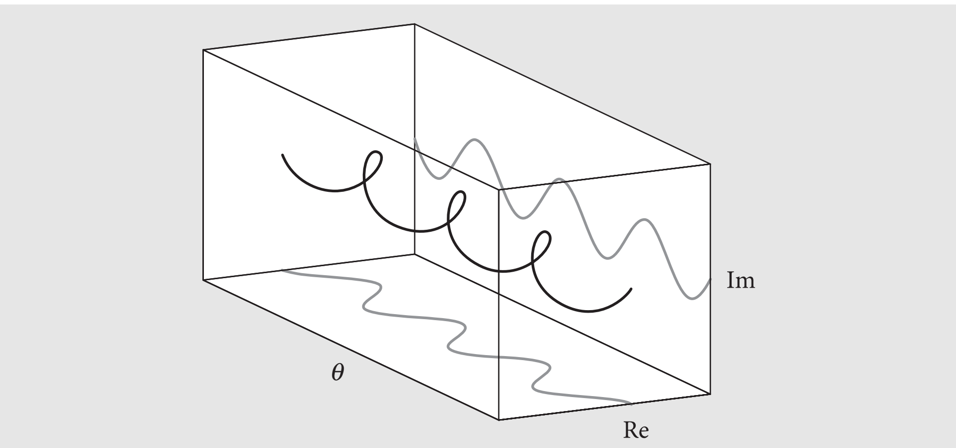

Fig. 2.7 Two visualizations of the complex number $e^{i\theta} = \cos(\theta) + i\sin(\theta)$ illustrate that it has a wavelike form.

Returning to example (i), $w = \ln(-1)$, if the -1 is treated as a complex number with an imaginary part that is zero, then the answer can be written down directly as

$$w = \ln(-1) = \ln(re^{i\theta}) = \ln(1) + i(\pi + 2\pi k)$$

and, since $r = 1$ and $\ln(1) = 0$, this gives the same result as in (i) $\ln(-1) = i\pi$ for the principal value.

2.9 Questions

Full solutions are available at **www.oxfordtextbooks.co.uk/orc/beddard**.

Q2.14 Calculate e^i.

Q2.15 Find the real and imaginary parts of **(a)** ie^{-ix}, **(b)** $e^{in\pi}$, and **(c)** $e^{in\pi/2}$, where n is an integer.

Q2.16 Calculate **(a)** i^i and **(b)** $i^{1/i}$.

Strategy: Using different bases such as $a^x = e^{x\ln(a)}$ any number can be raised to any power. With complex numbers always try to put the number in terms of Euler's equation.

Q2.17 The cosine function is defined as $\cos(x) = \dfrac{e^{ix} + e^{-ix}}{2}$. What is $\cos^{-1}(x)$?

Strategy: This is a case where x and y are swapped about. If $\cos^{-1}(x)$ then $\cos(y) = x$. It is true also that $\cos(y) = \dfrac{e^{iy} + e^{-iy}}{2}$. Next eliminate the cosine and solve for y and so find $\cos^{-1}(x)$.

Q2.18 Show that the identity $(\cos(x) + \sin(x))^2 = 1 + \sin(2x)$ can be (relatively easily) proved using complex numbers.

Q2.19 Show that $2\sin\left(\dfrac{a+b}{2}\right)\cos\left(\dfrac{a-b}{2}\right) = \sin(a) + \sin(b)$.

Q2.20 Starting with Euler's theorem and letting $\theta = a + b$, calculate $\sin(a + b)$ and $\cos(a + b)$ by equating real and imaginary parts.

Q2.21 Find $\sin(\theta)$ in exponential form then calculate $|\sin(i\theta)|$, and compare it with $|\sin(\theta)|$. Plot values of $|\sin(ix)|$ and $|\sin(x)|$ over the range $x = -4$ to 4.

Q2.22 **(a)** If $z = \cos(x) + i\sin(x)$ show that $\dfrac{dz}{dx} = iz$. **(b)** Integrate this result and prove Euler's theorem.

Q2.23 Calculate the real and imaginary parts of $\dfrac{1}{\sqrt{2\pi}}\left(\dfrac{1 - e^{-i\omega t}}{i\omega}\right)$. This function is the Fourier transform of a square wave of length t; see Chapter 9.5 and 9.6.

Strategy: use $i = -1/i$ and multiply out the terms.

Q2.24 In an NMR experiment, the FID signal has the form $s(t) = \sum_j a_j e^{i\omega_j t - t/\tau_j}$ where ω is the frequency of the transition, τ the average of the T1 and T2 lifetimes, and a the amplitude of each signal and there are j parts to the total signal. For simplicity, assume that τ_j has a constant value τ.

(a) Calculate the real, imaginary, and absolute value of s if $j = 2$.

(b) Plot the real part of the signal if $a_1 = a_2 = 2$ and $2\pi\omega = 1$ Hz and 0.2 Hz and $\tau_1 = \tau_2 = 50$ s and also when $\tau_1 = \tau_2 = 500$ s. Comment on the two results.

(c) Repeat (a) when $a_1 = i$, which means that the initial amplitude is complex, and $a_2 = 1$.

In spite of the fact that the signal from an experiment cannot be a complex number, this is what appears to be the case here. The reason for this is that in a real NMR experiment two signals are measured, one by a coil on the spectrometer's x-axis and the other by a similar coil on the y-axis. These are at right angles to the z-axis along which the permanent magnetic field is directed. These x and y signals are measured in *quadrature*, i.e. 90° out of phase to one another. One signal is taken to be the real component, and one the imaginary. They are then combined to produce $s(t)$ given above.

Strategy: The question asks you to find the components which when combined make $s(t)$. Use the Euler formula to do this and to simplify the complex exponential. As the signal represents the FID from an NMR experiment it oscillating in a sinusoidal way.

Q2.25 Derive the identities **(a)** $4\cos(\theta)\sin^2(\theta) = \cos(\theta) - \cos(3\theta)$ and

(b) $4\sin(\theta)\cos^2(\theta) = \sin(\theta) + \sin(3\theta)$

Strategy: Always use the exponential forms of sine and cosine wherever possible for complicated trig functions. These are

$$\cos(\theta) = \frac{e^{i\theta} + e^{-i\theta}}{2}, \qquad \sin(\theta) = \frac{e^{i\theta} - e^{-i\theta}}{2i}.$$

Q2.26 If C is the series whose n^{th} term is $\cos(nx)/n!$, and S the series $\sin(nx)/n!$, calculate the sum from $n = 1$ to infinity of $C + iS$, and hence find the sum C.

Strategy: Convert to the exponential form using $e^{ix} = \cos(x) + i\sin(x)$, sum the terms then convert back to trig form and separate out the real part of the result.

Q2.27 In the study of the dielectric properties of liquids and in electrochemical techniques that use potentiometry, the response of the solution to different electrical frequencies is studied. The general term for these experiments is impedance spectroscopy. In an experiment where a capacitor C and resistor R are in parallel, the impedance Z, which is a complex quantity, is given by $Z = (R^{-1} + i\omega C)^{-1}$ where ω is the frequency applied to the sample.

(a) Convert Z into the form $Z = Z' - iZ''$.

(b) Plot Z'' as ordinate, and z' as abscissa. Show that the resulting curve is a semicircle. Use $R = 5$ kΩ and C = 1 μF. Decide where high frequency is on the plot. This is not obvious from the graph because ω is not on one of the axes.

Strategy: Multiply top and bottom of the expression by the complex conjugate. Look up the parametric method of plotting graphs in the Maple appendix.