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CHAPTER 10

UNBOUNDED OPERATORS

The generator of a semigroup

1. (Notation as in Exercise 14, Chapter 9)

The *generator* A of the C_0-semigroup $T(\cdot)$ is its strong right derivative at 0 with *maximal domain* $D(A)$: denoting the (right) differential ratio at 0 by A_h, i.e., $A_h := h^{-1}[T(h) - I] \quad (h > 0)$, we have

$$Ax = \lim_{h \to 0+} A_h x \qquad x \in D(A) = \{x \in X;\ \lim_h A_h x \text{ exists}\}. \tag{1}$$

Prove:

(a) $\bigcup_{t>0} V(t)X \subset D(A)$, and for each $t > 0$ and $x \in X$,

$$AV(t)x = T(t)x - x. \tag{2}$$

(This is a rewording of the conclusion of Exercise 14(e), Chapter 9.)

(b) $D(A)$ is dense in X. Hint: by Part (a), $V(t)x \in D(A)$ for any $t > 0$ and $x \in X$ and $AV(t)x = T(t)x - x$. Apply Exercises 14(d) and 13(c) in Chapter 9.

Solution.

By Exercise 14(d), Chapter 9, $T(\cdot)$ is strongly continuous on $[0, \infty)$, and therefore, by Exercise 13(c), Chapter 9, $t^{-1}V(t)x \to T(0)x = x$ as $t \to 0+$, for each $x \in X$. Since $t^{-1}V(t)x \in D(A)$ for all $t > 0$ (by Part (a)), this shows that $D(A)$ is dense in X.

(c) For $x \in D(A)$ and $t > 0$, $T(t)x \in D(A)$ and

$$AT(t)x = T(t)Ax = (d/dt)T(t)x, \tag{3}$$

where the right side denotes the strong derivative at t of $u := T(\cdot)x$. Therefore $u : [0, \infty) \to D(A)$ is a solution of class C^1 of the *abstract Cauchy problem*

$$(\text{ACP}) \qquad u' = Au \qquad u(0) = x. \tag{4}$$

Also

$$\int_0^t T(s)Ax\,ds = T(t)x - x \qquad (x \in D(A)). \tag{5}$$

Hint: for left derivation, use Exercise 14(c), Chapter 9.

Solution.

Fix $x \in D(A)$ and $t > 0$. Then for $h > 0$

$$A_h T(t)x = T(t)A_h x \to T(t)Ax \tag{6}$$

as $h \to 0+$, since $x \in D(A)$ and $T(t) \in B(X)$. Thus $T(t)x \in D(A)$ and $AT(t)x = T(t)Ax$. Furthermore, by the semigroup property, the left hand side of (6) is equal to $h^{-1}[T(t+h)x - T(t)x]$; therefore the strong right derivative of $T(\cdot)x$ at t exists and is equal to $T(t)Ax$. If $0 < h < t$,

$$||(-h)^{-1}[T(t-h)x - T(t)x] - T(t)Ax|| = ||T(t-h)A_h x - T(t)Ax||$$

$$\leq ||T(t-h)[A_h x - Ax]|| + ||[T(t-h) - T(t)]Ax||.$$

The first summand on the right hand side is $\leq M\, e^{a(t-h)}||A_h x - Ax||$ by Exercise 14(c), Chapter 9, and therefore converges to 0 as $h \to 0$, since $x \in D(A)$. The second summand converges to 0 by the strong continuity of $T(\cdot)$ (cf. Exercise 14(d), Chapter 9). We then conclude that the strong left derivative of $T(\cdot)x$ at t exists and is equal to $T(t)Ax$. This proves (3) and (4) (since $u' = T(\cdot)Ax$, u' is continuous by Exercise 14(d), Chapter 9). Integrating (3) over $[0,t]$, we obtain (5) by Exercise 13(d), Chapter 9.

(d) A is a closed operator. Hint: use the identity

$$V(t)Ax = AV(t)x = T(t)x - x \qquad (x \in D(A);\, t > 0) \tag{7}$$

(cf. Part (a)) and Exercise 13(c), Chapter 9.

Solution.

By (5) and Part (a), Relation (7) holds. Let $x_n \in D(A)$ be such that $x_n \to x$ and $Ax_n \to y$ in X. Then by (7), for all $h > 0$,

$$A_h x = \lim_n A_h x_n = \lim_n h^{-1}V(h)Ax_n = h^{-1}V(h)y.$$

Therefore, by Exercises 14(d) and 13(c), Chapter 9, $A_h x \to y$ as $h \to 0$. This proves that $x \in D(A)$ and $Ax = y$. Thus A is a closed operator.

(e) If $v : [0,\infty) \to D(A)$ is a solution of class C^1 of ACP, then $v = T(\cdot)x$. (This is the *uniqueness* of the solution of ACP when A is the generator of a C_0-semigroup.) In particular, the generator A determines the semigroup $T(\cdot)$ uniquely. Hint: apply Exercise 13(d), Chapter 9, to $V := T(\cdot)v(s-\cdot)$ on the interval $[0,s]$.

Solution.

If $V(\cdot) : [0,\infty) \to B(X)$ is differentiable in the s.o.t. at some point t and $u(\cdot) : [0,\infty) \to X$ is strongly differentiable at t, then $V(\cdot)u(\cdot)$ is strongly differentiable at t and $(Vu)'(t) = V'(t)u(t) + V(t)u'(t)$. Indeed, since the operators $h^{-1}[V(t+h)-V(t)]$ (for $t, t+h \geq 0$ and $0 < h < 1$) converge as $h \to 0$ in the s.o.t., their operator norms are bounded by some constant K by the Uniform Boundedness theorem. We write

$$h^{-1}[(Vu)(t+h)-(Vu)(t)] = h^{-1}[V(t+h)-V(t)]u(t)$$

$$+h^{-1}[V(t+h)-V(t)][u(t+h)-u(t)] + V(t)h^{-1}[u(t+h)-u(t)].$$

As $h \to 0$, the first summand converges to $V'(t)u(t)$; the second summand has norm $\leq K\,||u(t+h)-u(t)|| \to 0$; and the third summand converges to $V(t)u'(t)$.
We apply this observation to the operator function $T(\cdot)$ and the vector function $v(s-\cdot)$ ($s > 0$ fixed); then for V as in the hint, we have by Part (c) and ACP for v

$$V' = T(\cdot)Av(s-\cdot) - T(\cdot)v'(s-\cdot) = 0,$$

and therefore, by Exercise 13(d), Chapter 9,

$$0 = \int_0^s V'dt = V(s) - V(0) = T(s)x - v(s),$$

that is $v = T(\cdot)x$.

Semigroups continuous in the u.o.t.

2. (Notation as in Exercise 1.) Suppose $T(h) \to I$ in the u.o.t. (i.e., $||T(h)-I|| \to 0$ as $h \to 0+$). Prove:

(a) $V(h)$ is non-singular for h small enough (which we fix from now on). Define $A := [T(h)-I]V(h)^{-1}$ $(\in B(X))$.

Solution.

Observe first that $T(\cdot)$ is continuous in the u.o.t. on $[0,\infty)$. This is true at $t=0$ by hypothesis. If $t > 0$, then for $0 < h < t$, we have

$$||T(t+h)-T(t)|| = ||T(t)[T(h)-I]|| \leq ||T(t)||\,||T(h)-I|| \to 0$$

as $h \to 0$, and by Exercise 14(c), Chapter 9,

$$||T(t-h)-T(t)|| = ||T(t-h)[I-T(h)]|| \le ||T(t-h)||\,||T(h)-I|| \le Me^{a(t-h)}||T(h)-I|| \to 0$$

as $h \to 0$. Therefore the integral $V(h) := \int_0^h T(t)dt$ is well-defined as a Riemann integral in $B(X)$ in the u.o.t., and $\lim_{h\to 0+}(1/h)V(h) = I$ in the u.o.t.: given $\epsilon > 0$, there exists $\delta = \delta(\epsilon) > 0$ such that $||T(t) - I|| < \epsilon$ for $0 \le t < \delta$; then for $h < \delta$,

$$||(1/h)V(h) - I|| = ||(1/h)\int_0^h [T(t) - I]dt|| \le (1/h)\int_0^h ||T(t) - I||dt < \epsilon.$$

Taking in particular $\epsilon = 1$, we have $||(1/h)V(h) - I|| < 1$ for $0 < h < \delta(1)$, hence $(1/h)V(h)$ is non-singular (and therefore $V(h)$ in non-singular) for these h (cf. Remark 7.4).

(b) $T(t) - I = V(t)A$ for all $t \ge 0$ (with A as above). Conclude that A is the generator of $T(\cdot)$ (in particular, the generator is a bounded operator).

Solution.

Since $T(s)$ commutes with $V(h)$ (by the group property of $T(\cdot)$ and Exercise 13(b), Chapter 9), it follows that $V(t)$ commutes with $V(h)$ by the same exercise (here, $s, t, h \ge 0$ are arbitrary).
By Exercise 14(e), Chapter 9,

$$[T(h) - I]V(t) = V(t+h) - V(h) - V(t).$$

By symmetry of the right hand side in t and h, we conclude that

$$[T(h) - I]V(t) = [T(t) - I]V(h). \tag{8}$$

Fix now h as in Part (a), and define $A := [T(h) - I]V(h)^{-1}$ ($\in B(X)$). Then by (8)

$$AV(t)V(h) = AV(h)V(t) = [T(h) - I]V(t) = [T(t) - I]V(h),$$

hence $AV(t) = T(t) - I$, and therefore $A_t = A(1/t)V(t) \to A$ as $t \to 0+$ in the u.o.t. (cf. solution of Part (a)). This shows that A is the generator of $T(\cdot)$.

(c) Conversely, if the generator A of $T(\cdot)$ is a bounded operator, then $T(t) = e^{tA}$ (defined by the usual absolutely convergent series in $B(X)$) and $T(h) \to I$ in the u.o.t. Hint: the exponential is a continuous semigroup (in the u.o.t.) with generator A; use the uniqueness statement in Exercise 1(e).

Solution.

In any unital Banach algebra $\mathcal{A}$, if $a \in \mathcal{A}$, then e^{ta} defines a (semi)group in $\mathcal{A}$, continuous in the norm topology of $\mathcal{A}$ (indeed, it suffices to show that e^{ta} norm-converges to the identity when $t \to 0$; this follows from the elementary estimate $||e^{ta} - e|| \le |t|\,||a||e^{|t|\,||a||} \to 0$ as $t \to 0$). Furthermore,

$$||(1/t)[e^{ta} - e] - a|| \le |t|\,||a||^2 e^{|t|\,||a||} \to 0$$

as $t \to 0$, which proves that the generator of e^{ta} (in the norm topology!) is a. Specializing to $\mathcal{A} = B(X)$, if the generator A of the C_0-semigroup $T(\cdot)$ is in $B(X)$, then e^{tA} is also a C_0-semigroup with generator A. By the uniqueness statement in Exercise 1(e), $T(t) = e^{tA}$, and in particular $T(h) \to I$ in the u.o.t., as observed above.

The resolvent of a semigroup generator

3. Let $T(\cdot)$ be a C_0-semigroup on the Banach space X. Let A be its generator, and ω its type (cf. Exercise 14(f), Chapter 9). Fix $a > \omega$. Prove:

(a) The *Laplace transform*

$$L(\lambda) := \int_0^\infty e^{-\lambda t} T(t)\, dt$$

converges absolutely (in $B(X)$) and $||L(\lambda)|| = O(1/(\Re\lambda - a))$ for $\Re\lambda > a$. (Cf. Exercises 13(e) and 14(c), Chapter 9.)

(*Details*. Since $a > \omega = \lim_{t\to\infty} t^{-1} \log ||T(t)||$ (cf. Exercise 14(f), Chapter 9), there exists $t_0 > 0$ such that $||T(t)|| < e^{at}$ for all $t > t_0$. Let

$$M := \sup_{t\in[0,t_0]} e^{-at}||T(\cdot)||.$$

Then $1 \le M < \infty$ (cf. Exercise 13(a), Chapter 9), and

$$||T(t)|| \le Me^{at} \qquad (t \ge 0). \tag{9}$$

Hence

$$||e^{-\lambda t}T(t)|| \le M\, e^{-(\Re\lambda - a)t}, \tag{10}$$

and therefore

$$\int_0^\infty ||e^{-\lambda t}T(t)||\, dt \le \frac{M}{\Re\lambda - a}$$

for $\Re\lambda > a$. By Exercise 13(e), Chapter 9, it follows that $L(\lambda)$ converges (absolutely) in $B(X)$ and $||L(\lambda)|| \le M/(\Re\lambda - a)$ for $\Re\lambda > a$.)

(b) $L(\lambda)(\lambda I - A)x = x$ for all $x \in D(A)$ and $\Re\lambda > a$.

Solution.

Let $x \in D(A)$, $\Re\lambda > a$, and $b > 0$. By Exercise 1(c) and Exercise 13(d) in Chapter 9,

$$\int_0^b e^{-\lambda t} T(t)(\lambda I - A)x\,dt = -\int_0^b \left([e^{-\lambda t}]' T(t)x + e^{-\lambda t}[T(t)x]'\right) dt$$

$$= -\int_0^b [e^{-\lambda t} T(t)x]'\,dt = x - e^{-\lambda b} T(b)x \to x$$

as $b \to \infty$, by (10). On the other hand, by Part (a), the limit is equal to $L(\lambda)(\lambda I - A)x$, and the desired identity follows.

(c) $L(\lambda)X \subset D(A)$, and $(\lambda I - A)L(\lambda) = I$ for $\Re\lambda > a$.

Solution.

Fix λ such that $\Re\lambda > a$ and $x \in X$. Since $A_h \in B(X)$ for each $h > 0$, we have (cf. Exercise 13(b), Chapter 9, extended to convergent improper integrals)

$$A_h L(\lambda)x = \int_0^\infty e^{-\lambda t} A_h T(t)x\,dt = h^{-1}\Big(\int_0^\infty e^{-\lambda t} T(t+h)x\,dt - L(\lambda)x\Big)$$

$$= h^{-1}(e^{\lambda h} - 1)L(\lambda)x - e^{\lambda h}\Big[h^{-1}\int_0^h e^{-\lambda t} T(t)x\,dt\Big] \to \lambda L(\lambda)x - x$$

as $h \to 0$ (cf. Exercise 13(c), Chapter 9). This proves that $L(\lambda)x \in D(A)$ and $A\,L(\lambda)x = \lambda L(\lambda)x - x$, i.e., $(\lambda I - A)L(\lambda)x = x$.

(d) Conclude that $\sigma(A) \subset \{\lambda \in \mathbb{C};\ \Re\lambda \le \omega\}$ and $R(\lambda; A) = L(\lambda)$ for $\Re\lambda > \omega$.

(*Details.* Suppose $\lambda \in \mathbb{C}$ is such that $\Re\lambda > \omega$. Pick a in the interval $(\omega, \Re\lambda)$. By Parts (b) and (c), $\lambda \in \rho(A)$ and $R(\lambda; A) = L(\lambda)$.)

(e) For any $\lambda_k > a$ $(k = 1, \ldots, m)$,

$$\|\prod_k (\lambda_k - a) R(\lambda_k; A)\| \le M, \tag{11}$$

where M is a positive constant depending only on a and $T(\cdot)$. In particular

$$\|R(\lambda)^m\| \le \frac{M}{(\lambda - a)^m} \qquad (\lambda > a;\ m \in \mathbb{N}). \tag{12}$$

Hint: apply Part (d), and the multiple integral version of Exercise 13(e), Chapter 9.

Solution.

For all $x \in X$, we have by Part (d)

$$|| \prod_{k=1}^{m} (\lambda_k - a) R(\lambda_k; A) x ||$$

$$= || \int_0^\infty \cdots \int_0^\infty \prod_{k=1}^{m} (\lambda_k - a) e^{-\lambda_k t_k} T(t_k) x \, dt_1 \cdots dt_m ||$$

$$\leq M \int_0^\infty \cdots \int_0^\infty \prod_k (\lambda_k - a) e^{-(\lambda_k - a) t_k} dt_1 \cdots dt_m \; ||x|| = M \, ||x||,$$

where we used the estimate (9). This proves (11), and (12) follows by taking $\lambda_1 = \cdots = \lambda_m = \lambda > a$.

(f) Let A be *any* closed densely defined operator on X whose resolvent set contains a ray (a, ∞) and whose resolvent $R(\cdot)$ satisfies $||R(\lambda)|| \leq M/(\lambda - a)$ for $\lambda > \lambda_0$ (for some $\lambda_0 \geq a$). (Such an A is sometimes called an *abstract potential.*) Consider the function $A(\cdot) : (a, \infty) \to B(X)$:

$$A(\lambda) := \lambda A R(\lambda) = \lambda^2 R(\lambda) - \lambda I.$$

Then, as $\lambda \to \infty$,

$$\lim A(\lambda) x = Ax \qquad (x \in D(A));$$

$$\lim \lambda R(\lambda) = I \quad \text{and} \quad \lim AR(\lambda) = 0 \quad \text{in the s.o.t.}.$$

Note that these conclusions are valid if A is the generator of a C_0-semigroup, with $a > \omega$ fixed. Cf. Exercise 3, Parts (d) and (e).

Solution.

For $x \in D(A)$ and $\lambda > \lambda_0$,

$$||AR(\lambda)x|| = ||R(\lambda)Ax|| \leq \frac{M}{\lambda - a} ||Ax|| \to 0 \tag{13}$$

as $\lambda \to \infty$. Furthermore, for all $\lambda > \lambda_0$

$$||AR(\lambda)|| = ||\lambda R(\lambda) - I|| \leq \frac{M\lambda}{\lambda - a} + 1 \leq K \tag{14}$$

for some constant $K > 0$.
Let $x \in X$. Given $\epsilon > 0$, pick $x_0 \in D(A)$ such that $||x - x_0|| < \epsilon/(2K)$ (by density of $D(A)$ in X). By (13) for x_0, there exists $\lambda_1 > \lambda_0$ such that $||AR(\lambda)x_0|| < \epsilon/2$ for all $\lambda > \lambda_1$. Therefore, for all $\lambda > \lambda_1$, we have by (14)

$$||AR(\lambda)x|| \leq ||AR(\lambda)|| \, ||x - x_0|| + ||AR(\lambda)x_0|| < \epsilon.$$

Thus $AR(\lambda) \to 0$ in the s.o.t. Since $\lambda R(\lambda) = AR(\lambda) + I$, we have equivalently $\lambda R(\lambda) \to I$ in the s.o.t. Finally, for $x \in D(A)$, $A(\lambda)x = [\lambda R(\lambda)](Ax) \to Ax$.
(Note that the hypothesis that A be closed is not explicitly needed, but it actually follows from the assumption that $\rho(A) \neq \emptyset$. Indeed, if $\lambda \in \rho(A)$, then $R(\lambda)$ is in $B(X)$, hence is trivially closed. Therefore $\lambda I - A = R(\lambda)^{-1}$ is closed (cf. Section 10.1), and so $A = \lambda I - (\lambda I - A)$ is closed.)

4. Let A be a closed densely defined operator on the Banach space X such that $(a, \infty) \subset \rho(A)$ (for some $a \geq 0$) and (12) in Exercise 3(e) is satisfied. Define $A(\cdot)$ as in Exercise 3(f) and denote $T_\lambda(t) := e^{tA(\lambda)}$ (the usual power series). Prove:

(a) $||T_\lambda(t)|| \leq M \exp(t\frac{a\lambda}{\lambda - a})$ for all $\lambda > a$. Conclude that

$$||T_\lambda(t)|| \leq Me^{2at} \qquad (\lambda > 2a) \tag{15}$$

and

$$\limsup_{\lambda \to \infty} ||T_\lambda(t)|| \leq Me^{at}. \tag{16}$$

Solution.

For $\lambda > a$,

$$||T_\lambda(t)|| = ||\exp t[\lambda^2 R(\lambda) - \lambda I]|| \leq e^{-\lambda t} \sum_n \frac{t^n \lambda^{2n}}{n!} ||R(\lambda)^n||$$

$$\leq Me^{-\lambda t} \sum_n (1/n!)(\frac{t\lambda^2}{\lambda - a})^n = M \exp\left(t[\frac{\lambda^2}{\lambda - a} - \lambda]\right)$$

$$= M \exp(t\frac{a\lambda}{\lambda - a}).$$

If $\lambda > 2a$, it follows that $\frac{a\lambda}{\lambda - a} < 2a$, and (15) follows from the last estimate; (16) follows by letting $\lambda \to \infty$ is that same estimate.

(b) If $x \in D(A)$, then uniformly for t in bounded intervals,

$$\lim_{2a < \lambda, \mu \to \infty} ||T_\lambda(t)x - T_\mu(t)x|| = 0. \tag{17}$$

Hint: apply Exercise 13(d), Chapter 9, to the function $V(s) := T_\lambda(t-s)T_\mu(s)$ on the interval $[0,t]$; Exercise 1(c) to the semigroups $T_\lambda(\cdot)$ and $T_\mu(\cdot)$; Part (a), and Exercise 3(f).

Solution.

Fix $t > 0$, and consider the operator function $V(\cdot)$ of the hint in the interval $[0,t]$. Note that $V(t) = T_\mu(t)$ and $V(0) = T_\lambda(t)$. Therefore, by Exercise 13(d), Chapter 9,

$$T_\mu(t) - T_\lambda(t) = \int_0^t V'(s)\,ds. \tag{18}$$

The semigroups $T_\lambda(\cdot)$ and $T_\mu(\cdot)$ commute and have the generators $A(\lambda)$ and $A(\mu)$ respectively (both in $B(X)$ and commuting with both semigroups). By Exercise 1(c),

$$V'(s) = V(s)[A(\mu) - A(\lambda)]. \tag{19}$$

By Part (a), if λ, $\mu > 2a$, then for all $s \in [0,t]$, $||V(s)|| \le M^2 e^{4at}$. Hence by (18) and (19), and Exercise 13(b), Chapter 9,

$$||T_\mu(t)x - T_\lambda(t)x|| \le M^2 t e^{4at} ||A(\mu)x - A(\lambda)x||.$$

By Exercise 3(f), if $x \in D(A)$,

$$\sup_{0 \le t \le \tau} ||T_\mu(t)x - T_\lambda(t)x|| \le M^2 \tau e^{4a\tau} ||A(\mu)x - A(\lambda)x|| \to 0 \tag{20}$$

as λ, $\mu \to \infty$.

(c) For each $x \in X$, $\{T_\lambda(t)x; \lambda \to \infty\}$ is Cauchy (uniformly for t in bounded intervals). (Use Part (b), the density of $D(A)$, and (15) in Part (a).) *Define* then

$$T(t) = \lim_{\lambda \to \infty} T_\lambda(t) \tag{21}$$

in the s.o.t. Then $T(\cdot)$ is a strongly continuous semigroup such that $||T(t)|| \le Me^{at}$ and

$$T(t)x - x = \int_0^t T(s)Ax\,ds \qquad (x \in D(A)). \tag{22}$$

Hint: use (5) in Exercise 1(c) for the semigroup $e^{tA(\lambda)}$, and apply Exercise 3(f).

Solution.

Let $\epsilon > 0$. Given $\tau > 0$ arbitrary, denote $K = M e^{2a\tau}$. Given $x \in X$, pick $x_0 \in D(A)$ such that $||x - x_0|| < \epsilon/(4K)$. By Exercise 3(f), there exists $b > 2a$ such that

$$||A(\mu)x_0 - A(\lambda)x_0|| < \epsilon/(2K^2\tau) \tag{23}$$

for λ, $\mu > b$. Then for λ, $\mu > b$ and all $t \in [0, \tau]$, we have by (15),(20), and (23)

$$||T_\mu(t)x - T_\lambda(t)x|| \le ||T_\mu(t) - T_\lambda(t)|| \, ||x - x_0|| + K^2\tau ||A(\mu)x_0 - A(\lambda)x_0|| < \epsilon.$$

This proves that $T_\lambda(t)$ converges in the s.o.t. to an operator $T(t) \in B(X)$ (cf. Exercise 19, Chapter 6), uniformly for $t \in [0, \tau]$. Since $T_\lambda(\cdot)x$ is continuous, it follows from the uniform convergence that $T(\cdot)x$ is continuous in $[0, \tau]$ for each $\tau > 0$ and $x \in X$, that is $T(\cdot)$ is strongly continuous. By properties of limits, the semigroup property of $T_\lambda(\cdot)$ is inherited by $T(\cdot)$. For all $x \in X$ and $t \ge 0$, we have by (16)

$$||T(t)x|| = \lim_{\lambda \to \infty} ||T_\lambda(t)x|| \le \limsup_{\lambda \to \infty} ||T_\lambda(t)|| \, ||x|| \le M e^{at} ||x||,$$

hence $||T(t)|| \le M e^{at}$.
By (5) (Exercise 1(c)) applied to the semigroups $T_\lambda(\cdot)$,

$$T_\lambda(t)x - x = \int_0^t T_\lambda(s)A(\lambda)x \, ds. \tag{24}$$

For $x \in D(A)$, we have

$$|| \int_0^t T_\lambda(s)A(\lambda)x \, ds - \int_0^t T(s)Ax \, ds ||$$

$$\le \int_0^t ||T_\lambda(s)(Ax) - T(s)(Ax)|| \, ds + \int_0^t ||T_\lambda(s)[Ax - A(\lambda)x]|| \, ds.$$

The integrand of the first integral converges pointwise to 0 when $\lambda \to \infty$ (since $T_\lambda(s) \to T(s)$ in the s.o.t.) and is dominated by the integrable function $2Me^{2as}||Ax||$ (on $[0, t]$) for all $\lambda > 2a$ (cf. Part (a)). By the Dominated Convergence theorem, the first integral converges to zero as $\lambda \to \infty$. For $\lambda > 2a$, the second integral is $\le Me^{2at}t \, ||Ax - A(\lambda)x|| \to 0$ as $\lambda \to \infty$, by Exercise 3(f). We then conclude that for $x \in D(A)$

$$\lim_{\lambda \to \infty} \int_0^t T_\lambda(s)A(\lambda)x \, ds = \int_0^t T(s)Ax \, ds,$$

and (22) follows from (24) by letting $\lambda \to \infty$.

(d) If A' is the generator of the semigroup $T(\cdot)$ defined in Part (c), then $A \subset A'$. Since $\lambda I - A$ and $\lambda I - A'$ are both one-to-one and onto for $\lambda > a$ and coincide on $D(A)$, conclude that $A' = A$.

Solution.

Note that since $||T(t)|| \leq Me^{at}$ (cf. Part (c)), $\omega = \lim_{t\to\infty} t^{-1}\log||T(t)|| \leq a$ (cf. Exercise 14(f), Chapter 9), and therefore $(a,\infty) \subset \rho(A')$, by Exercise 3(d). Also $(a,\infty) \subset \rho(A)$ by hypothesis. Fix $\lambda > a$; then $R(\lambda; A)$ and $R(\lambda; A')$ both exist.
Let $x \in D(A)$ and $t > 0$. By (22) and Exercise 13(c), Chapter 9

$$(1/t)[T(t)x - x] = (1/t)\int_0^t T(s)\,(Ax)\,ds \to Ax$$

as $t \to 0$. Thus $x \in D(A')$, and $A'x = Ax$. This shows that $A \subset A'$.
On the other hand, let $x' \in D(A')$; denote $y' = (\lambda I - A')x'$ and $x = R(\lambda; A)y' (\in D(A))$. Then $Ax = A'x$ (by the preceding observation), and therefore

$$x' = R(\lambda; A')y' = R(\lambda; A')(\lambda I - A)x = R(\lambda; A')(\lambda I - A')x = x \in D(A).$$

Hence $A' = A$.

(e) An operator A with domain $D(A) \subset X$ is the generator of a C_0-semigroup satisfying $||T(t)|| \leq Me^{at}$ for some real a iff it is closed, densely defined, $(a,\infty) \subset \rho(A)$ and (12) is satisfied. (Collect information from above!) This is the *Hille-Yosida theorem.* In particular (case $M = 1$ and $a = 0$), A is the generator of a contraction semigroup iff it is closed, densely defined, and $\lambda R(\lambda)$ exist and are contractions for all $\lambda > 0$. (Terminology: the bounded operators $A(\lambda)$ are called the *Hille-Yosida approximations* of the generator A.)

(*References* (in proper order).
Necessity: Exercises 1(d), (b), 3(d)-(e).
Sufficiency: Exercise 4 (c)-(d).
Note that in the special case of contraction semigroups, the *virtually* weaker condition $||\lambda R(\lambda)|| \leq 1$ (for all $\lambda > 0$) is equivalent to the condition $||[\lambda R(\lambda)]^n|| \leq 1$ (for all $\lambda > 0$ and $n \in \mathbb{N}$) of the general Hille-Yosida theorem.)

Core for the generator

5. Let $T(\cdot)$ be a C_0-semigroup on the Banach space X, and let A be its generator. Prove:

(a) $T(\cdot)$ is a C_0-semigroup on the Banach space $[D(A)]$. (Recall that the norm on $[D(A)]$ is the graph norm $||x||_A := ||x|| + ||Ax||$.)

Solution.

For each $t \geq 0$, $T(t)$ maps $D(A)$ into $D(A)$ by Exercise 1(c). If $x \in D(A)$, then by Exercise 1(c)

$$||T(t)x||_A = ||T(t)x|| + ||AT(t)x|| = ||T(t)x|| + ||T(t)Ax|| \leq ||T(t)||\,||x||_A,$$

that is, $T(t) \in B([D(A)])$. The semigroup property of $T(\cdot)$ on $D(A)$ is trivial. We finally verify the C_0-condition with respect to the graph norm: for $x \in D(A)$ and $t > 0$, we have by Exercise 1(c) and the C_0 property of $T(\cdot)$ on X

$$||T(t)x - x||_A = ||T(t)x - x|| + ||T(t)(Ax) - Ax|| \to 0$$

as $t \to 0$.

(b) Let D be a $T(\cdot)$-invariant subspace of $D(A)$, dense in X. For each $x \in D$, consider $V(t)x := \int_0^t T(s)x\,ds$ (defined in the Banach space $\overline{D}$, the closure of D in $[D(A)]$). Given $x \in D(A)$, let $x_n \in D$ be such that $x_n \to x$ (in X, by density of D in X). Then $V(t)x_n \to V(t)x$ in the graph-norm. Conclude that $V(t)x \in \overline{D}$ for each $t > 0$, and therefore $x \in \overline{D}$, i.e., *D is dense in* $[D(A)]$. (A dense subspace of $[D(A)]$ is called a *core* for A.) Thus, *a $T(\cdot)$-invariant subspace of $D(A)$ which is dense in X is a core for A*. (On the other hand, a core D for A is trivially dense in X, since $D(A)$ is dense in X and D is $||\cdot||_A$-dense in $D(A)$.)

(*Details.* By Part (a), $T(\cdot)$ is a C_0-semigroup in the Banach space $[D(A)]$, and therefore (cf. Exercises 14(d) and 13(b), Chapter 9) $V(t) \in B([D(A)])$. For $x \in D(A)$, $V(t)x$ is defined as an integral in both the X-norm and the graph norm.
Let $x \in D \subset D(A)$ and $t > 0$. Since D is $T(\cdot)$-invariant, the Riemann sums defining $V(t)x$ belong to D, and therefore their limit $V(t)x$ in the graph norm belongs to $\overline{D}$.
Let $x \in D(A)$. Since D is dense in X, there exist $x_n \in D$ such that $x_n \to x$ in X. We have by Exercise 1(a)

$$\begin{aligned}||V(t)x_n - V(t)x||_A &= ||V(t)(x_n - x)|| + ||AV(t)(x_n - x)|| \\ &= ||V(t)(x_n - x)|| + ||[T(t) - I](x_n - x)|| \\ &\leq [||V(t)|| + ||T(t) - I||]||x_n - x|| \to 0\end{aligned}$$

as $n \to \infty$. Since $V(t)x_n \in \overline{D}$, we conclude that $V(t)x \in \overline{D}$. By Exercise 13(c), Chapter 9 (applied in the Banach space $\overline{D}$), $(1/t)V(t)x \in \overline{D}$ converge to x in the graph norm as $t \to 0$, and therefore $x \in \overline{D}$. This shows that D is dense in $[D(A)]$.)

(c) A C^∞-vector for A is a vector $x \in X$ such that $T(\cdot)x$ is of class C^∞ ("strongly") on $[0, \infty)$. Let D^∞ denote the space of all C^∞-vectors for A. Then

$$D^\infty = \bigcap_{n=1}^{\infty} D(A^n). \tag{25}$$

Solution.

Let $x \in D^\infty$. We show by induction on n that

$$x \in D(A^n) \quad \text{and} \quad [T(\cdot)x]^{(n)} = T(\cdot)A^n x \tag{26}$$

for all $n \in \mathbb{N}$. Since $T(\cdot)x$ is differentiable at 0, $x \in D(A)$ and by Exercise 1(c), $[T(\cdot)x]' = T(\cdot)Ax$. This verifies (26) for $n = 1$. Assume (26) for some n. It follows from (26) for n that $T(\cdot)(A^n x)$ is differentiable, with derivative equal to $[T(\cdot)x]^{(n+1)}$. In particular (differentiability at zero!) $A^n x \in D(A)$, i.e., $x \in D(A^{n+1})$, and by Exercise 1(c),

$$[T(\cdot)x]^{(n+1)} = [T(\cdot)(A^n x)]' = T(\cdot)A(A^n x) = T(\cdot)A^{n+1}x,$$

which verifies (26) for $n+1$. By (26), we have the inclusion

$$D^\infty \subset \bigcap_{n=1}^{\infty} D(A^n). \tag{27}$$

On the other hand, suppose $x \in D(A^n)$ for all n. We show by induction on n that for all n, $T(\cdot)x$ is n-times differentiable, with n-th derivative given by (26). Since $x \in D(A)$, $T(\cdot)x$ is differentiable and has the derivative $T(\cdot)Ax$, by Exercise 1(c). Assume the above proposition for some n. Since $x \in D(A^{n+1})$ by hypothesis, $A^n x \in D(A)$, and therefore, by Exercise 1(c), $T(\cdot)(A^n x)$ is differentiable, and its derivative is equal to $T(\cdot)A(A^n x) = T(\cdot)A^{n+1}x$. By our induction hypothesis, this means that $[T(\cdot)x]^{(n+1)}$ exists and equals $T(\cdot)A^{n+1}x$, and the induction is completed. This shows that $x \in D^\infty$, that is, the reversed inclusion in (27) has been proved, and (25) follows.

(d) Let $\phi_n \in C_c^\infty(\mathbb{R})$ be non-negative, with support in $(0, 1/n)$ and integral equal to 1. Given $x \in X$, let $x_n = \int \phi_n(t)T(t)x\,dt$. Then

(i) $x_n \to x$ in X;

(ii) $x_n \in D(A)$ and $Ax_n = -\int \phi_n'(t)T(t)x\,dt$;

(iii) $x_n \in D(A^k)$ and $A^k x_n = (-1)^k \int \phi_n^{(k)}(t)T(t)x\,dt$ for all $k \in \mathbb{N}$. In particular, $x_n \in D^\infty$.

Conclude that D^∞ is dense in X and is a core for A. (Cf. Part (b).)

Solution.

(i) We have

$$||x_n - x|| = ||\int \phi_n(t)[T(t)x - x]\,dt|| \le \int_0^{1/n} \phi_n(t)\,||T(t)x - x||\,dt.$$

Let $\epsilon > 0$ be given. Dy the C_0-condition, there exists $n_0 \in \mathbb{N}$ such that $||T(t)x - x|| < \epsilon$ for $0 \le t \le 1/n_0$. Then for all $n > n_0$, $||x_n - x|| < \epsilon \int_0^{1/n} \phi_n(t)\,dt = \epsilon$.

(ii) Fix $x \in X$ and $n \in \mathbb{N}$, and let $0 < h \le 1$. The change of variable $t + h \to t$ gives

$$T(h)x_n = \int \phi_n(t)T(t+h)x\,dt = \int \phi_n(t-h)T(t)x\,dt.$$

Hence

$$||A_h x_n + \int \phi_n'(t)T(t)x\,dt|| = ||-\int \Big[(-h)^{-1}[\phi_n(t-h) - \phi_n(t)] - \phi_n'(t)\Big]\,T(t)x\,dt||$$

$$\le \int_{-1}^{1} \Big|(-h)^{-1}[\phi_n(t-h) - \phi_n(t)] - \phi_n'(t)\Big|\,||T(t)x||\,dt.$$

By the Mean Value theorem and the continuity of $T(\cdot)x$, the last integrand is $\le$ $2||\phi_n'||_u \max_{[-1,1]} ||T(\cdot)x||$, and converges pointwise to 0 as $h \to 0$. By the Dominated Convergence theorem, the integral converges to 0. This proves (ii).

(iii) We prove this by induction on k. Part (ii) gives the case $k = 1$. Assume (iii) for some k. Replacing x_n by $A^k x_n$ and ϕ by $\phi^{(k)}$ in the argument we used to prove (ii), we conclude that $A^k x_n \in D(A)$ and $A(A^k x_n) = -\int (\phi^{(k)})'(t)T(t)x\,dt$, that is, by the induction hypothesis, $x_n \in D(A^{k+1})$ and $A^{k+1}x_n = (-1)^{k+1}\int \phi^{(k+1)}(t)T(t)x\,dt$, as desired.

By (iii) and Part (c), $x_n \in D^\infty$ for all $n \in \mathbb{N}$. Hence, by Part (i), D^∞ is *dense in* X. We show next that D^∞ is $T(\cdot)$-*invariant*. (Then by Part (b), we conclude that D^∞ is a core for A.) Indeed, fix $x \in D^\infty$ and $t > 0$. By Part (c), $x \in D(A^n)$ for all n, and Exercise 1(c) implies inductively that $T(t)x \in D(A^n)$ and $A^n T(t)x = T(t)A^n x$ for all n. In particular (by Part (c)) $T(t)x \in D^\infty$.

The Hille-Yosida space of an arbitrary operator

6. Let A be an unbounded operator on the Banach space X with $(a, \infty) \subset \rho(A)$, for some real a. Denote its resolvent by $R(\cdot)$. Let $\mathcal{A}$ be the multiplicative semigroup generated by the set $\{(\lambda - a)R(\lambda);\ \lambda > a\}$. Let $Z := Z(\mathcal{A})$ (cf. Theorem 9.11), and consider A_Z, the part of A in Z. The *Hille-Yosida space* for A, denoted W, is the closure of $D(A_Z)$ in the Banach subspace Z. Prove:

(a) W is $R(\lambda)$-invariant for each $\lambda > a$ and $R(\lambda; A_W) = R(\lambda)|_W$. In particular, A_W is closed as an operator in the Banach space W.

Solution.

The *part of A in Z*, A_Z, is defined by

$$D(A_Z) = \{x \in D(A) \cap Z;\ Ax \in Z\};$$

$$A_Z = A|_{D(A_Z)}. \tag{28}$$

(Cf. discussion following Definition 10.7.) By the Renorming theorem (Theorem 9.11), Z (with the norm $||\cdot||_Z := ||\cdot||_{\mathcal{A}}$) is a Banach subspace of X invariant under any bounded operator S in $\mathcal{A}'$, and $||S|_Z||_{B(Z)} \leq ||S||$. In the present case, $R(\lambda) \in \mathcal{A}'$, and therefore Z is $R(\lambda)$-invariant, for all $\lambda > a$, and $||R(\lambda)|_Z||_{B(Z)} \leq ||R(\lambda)||$.
Let $x \in Z$ and $\lambda > a$. Since $x \in Z$, $R(\lambda)x \in Z$; also $R(\lambda)x \in D(A)$ trivially, and $AR(\lambda)x = \lambda R(\lambda)x - x \in Z$. Thus

$$R(\lambda)Z \subset D(A_Z).$$

On the other hand, if $x \in D(A_Z)$, then $(\lambda I - A)x \in Z$, and therefore $x = R(\lambda)(\lambda I - A)x \in R(\lambda)Z$. Hence

$$R(\lambda)Z = D(A_Z) \tag{29}$$

for any $\lambda > a$. In particular, the subspace $D(A_Z)$ of Z is $R(\lambda)$-invariant. By the continuity of $R(\lambda)|_Z$ on the Banach space Z, it follows that the closure W of $D(A_Z)$ in Z is $R(\lambda)$-invariant.
If $x \in D(A_Z)$, $(\lambda I - A_Z)x = (\lambda I - A)x \in Z$, and therefore $R(\lambda)|_Z(\lambda I - A_Z)x = R(\lambda)(\lambda I - A)x = x$. On the other hand, for all $x \in Z$, $R(\lambda)x \in D(A_Z)$ by (29), and $(\lambda I - A_Z)R(\lambda)x = (\lambda I - A)R(\lambda)x = x$. We then conclude that

$$R(\lambda; A_Z) = R(\lambda)|_Z. \tag{30}$$

The same argument (using the $R(\lambda)$-invariance of W, and replacing Z by W) shows that

$$R(\lambda; A_W) = R(\lambda)|_W. \tag{31}$$

(b) $||R(\lambda; A_W)^m||_{B(W)} \leq \frac{1}{(\lambda-a)^m}$ for all $\lambda > a$ and $m \in \mathbb{N}$.

Solution.

It follows from (31) and the $R(\lambda)$-invariance of W that

$$R(\lambda; A_W)^m = R(\lambda)^m|_W \qquad (\lambda > a;\ m \in \mathbb{N}).$$

Therefore, by the Renorming theorem (Theorem 9.11(iii)), since $[(\lambda - a)R(\lambda)]^m \in \mathcal{A}$, we have

$$(\lambda - a)^m ||R(\lambda; A_W)^m||_{B(W)} = ||[(\lambda - a)R(\lambda)]^m|_W||_{B(W)}$$
$$\leq ||[(\lambda - a)R(\lambda)]^m|_Z||_{B(Z)} \leq 1 \qquad (32)$$

for all $\lambda > a$ and $m \in \mathbb{N}$.

(c) $\lim_{\lambda\to\infty} \lambda R(\lambda; A_W)w = w$ in the Z-norm. Conclude that $D(A_W)$ is dense in W.

Solution.

By (30) and (32) for $m = 1$,

$$||R(\lambda; A_Z)||_{B(Z)} \leq 1/(\lambda - a) \qquad (\lambda > a).$$

Therefore, for $x \in D(A_Z)$,

$$||\lambda R(\lambda)x - x||_Z = ||\lambda R(\lambda; A_Z)x - x||_Z = ||R(\lambda; A_Z)Ax||_Z$$

$$\leq ||R(\lambda; A_Z)||_{B(Z)}||Ax||_Z \leq ||Ax||_Z/(\lambda - a) \to 0$$

as $\lambda \to \infty$.
Given $w \in W$ and $\epsilon > 0$, it follows from the definition of W (as the closure in Z of $D(A_Z)$) that there exists $x \in D(A_Z)$ such that $||w - x||_Z < \epsilon$. Then by (31) and (32)

$$||\lambda R(\lambda; A_W)w - w||_Z \leq ||[\lambda R(\lambda; A_W) - I](w - x)||_Z$$

$$+||\lambda R(\lambda)x - x||_Z \leq (\frac{\lambda}{\lambda - a} + 1)\epsilon + ||\lambda R(\lambda)x - x||_Z \to \epsilon$$

as $\lambda \to \infty$. This proves that $\lambda R(\lambda; A_W)w \to w$ in the $Z - norm$ as $\lambda \to \infty$, for each $w \in W$. Since $R(\lambda; A_W)w \in D(A_W)$ (because $R(\lambda; A_W)w = R(\lambda)|_W w = R(\lambda)w \in D(A)$; also $R(\lambda)w \in W$ by $R(\lambda)$-invariance of W; and $AR(\lambda; A_W)w = AR(\lambda)w = \lambda R(\lambda)w - w \in W$), it follows that $D(A_W)$ is dense in the Banach space W.

(d) A_W *generates a* C_0*-semigroup* $T(\cdot)$ *on the Banach space* W, such that $||T(t)||_{B(W)} \leq e^{at}$.

(*Observation.* This is a consequence of the Hille-Yosida theorem (cf. Exercise 4(e)) applied to the operator A_W in the Banach space W. The sufficient conditions are satisfied, by Parts (a)-(c).)

(e) If Y is a resolvent-invariant Banach subspace of X such that A_Y generates a C_0-semigroup on Y with the growth condition $||T(t)||_{B(Y)} \leq e^{at}$, then Y is a Banach subspace of W. (This is the *maximality* of the Hille-Yosida space.)

Details. Relation (31) is clearly valid for Y. Therefore, by (11) with $M = 1$, we have for any $y \in Y$ and $\lambda_k > a$,

$$||\prod_k(\lambda_k - a)R(\lambda_k)y|| \leq ||\prod_k(\lambda_k - a)R(\lambda_k; A_Y)y||_Y$$

$$\leq ||\prod_k(\cdots)||_{B(Y)}||y||_Y \leq ||y||_Y.$$

Hence $||y||_Z \leq ||y||_Y$, and therefore Y is a Banach subspace of Z. In particular $D(A_Y) \subset D(A_Z)$. Since A_Y generates a C_0- semigroup in Y, Y is the Y-closure of $D(A_Y)$, which is contained in the Z-closure of $D(A_Z)$, and the latter is precisely W. This proves that Y is a Banach subspace of W.

Convergence of semigroups

7. Let $\{T_s(\cdot);\, 0 \leq s < c\}$ be a family of C_0-semigroups on the Banach space X, such that

$$||T_s(t)|| \leq Me^{at} \qquad (t \geq 0;\; 0 \leq s < c) \tag{33}$$

for some $M \geq 1$ and $a \geq 0$. Let A_s be the generator of $T_s(\cdot)$, and denote $T(\cdot) = T_0(\cdot)$ and $A = A_0$. Note that (33) implies that

$$||R(\lambda; A_s)|| \leq M/(\lambda - a) \qquad (\lambda > a;\; s \in [0, c)). \tag{34}$$

Fix a core D for A. We say that A_s *graph-converge on* D to A (as $s \to 0$) if for each $x \in D$, there exists a vector function $s \in (0, c) \to x_s \in X$ such that $x_s \in D(A_s)$ for each s and $[x_s, A_s x_s] \to [x, Ax]$ in $X \times X$. Prove:

(a) A_s graph-converge to A on D iff, for each $\lambda > a$ and $y \in (\lambda I - A)D$, there exists a vector function $s \to y_s$ such that $[y_s, R(\lambda; A_s)y] \to [y, R(\lambda; A)y]$ in $X \times X$ (as $s \to 0$). Hint: $y_s = (\lambda I - A_s)x_s$ and (34).

Solution.

Suppose A_s graph-converge to A on D, and let $\lambda > a$. Let $y \in (\lambda I - A)D$, and $x := R(\lambda; A)y$. Since $x \in D$, there exist $x_s \in D(A_s)$ $(s \in (0, c))$ such that $[x_s, A_s x_s] \to [x, Ax]$ in $X \times X$. Define $y_s := (\lambda I - A_s)x_s$. Then

$$[y_s, R(\lambda; A_s)y_s] = [(\lambda I - A_s)x_s, x_s] \to [(\lambda I - A)x, x] = [y, R(\lambda; A)y]$$

in $X \times X$, as $s \to 0+$.
Conversely, suppose that for each $\lambda > a$ and $y \in (\lambda I - A)D$, there exist y_s ($s \in (0, c)$) such that

$$[y_s, R(\lambda; A_s)y_s] \to [y, R(\lambda; A)y] \tag{35}$$

in $X \times X$ as $s \to 0+$. Let $x \in D$, and define $y := (\lambda I - A)x$. Then $y \in (\lambda I - A)D$, and therefore there exist y_s as above. Define $x_s = R(\lambda; A_s)y_s$. Then $x_s \in D(A_s)$ for all $s \in (0, c)$, and by (35)

$$[x_s, A_s x_s] = [R(\lambda; A_s)y_s, \lambda R(\lambda; A_s)y_s - y_s] \to [R(\lambda; A)y, \lambda R(\lambda; A)y - y] = [x, Ax],$$

that is, A_s graph-converge to A on D.
The criterion in (35) may be simplified (for y and y_s as before) to

$$[y_s, R(\lambda; A_s)y)] \to [y, R(\lambda; A)y] \tag{36}$$

as $s \to 0+$. Indeed, let $\lambda > a$ and $y \in (\lambda I - A)D$. Let y_s satisfy (35). Then

$$||R(\lambda; A_s)y - R(\lambda; A)y|| \le ||R(\lambda; A_s)(y - y_s)|| + ||R(\lambda; A_s)y_s - R(\lambda; A)y)||.$$

The first summand above is $\le M(\lambda - a)^{-1}||y - y_s|| \to 0$ as $s \to 0+$, where we used (34) and (35). The second summand converges to 0 as $s \to 0+$, by (35). Therefore y_s satisfy (36). The same kind of estimate shows that the criterion (36) implies (35). This proves (a).

(b) If A_s graph-converge to A on D, then as $s \to 0$, $R(\lambda; A_s) \to R(\lambda; A)$ in the s.o.t. for all $\lambda > a$ (the later property is called *resolvents strong convergence*). Hint: show that $(\lambda I - A)D$ is dense in X, and use Part (a) and (34).

Solution.

Let $\lambda > a$, and $y \in (\lambda I - A)D$. By Part (a), there exist y_s ($s \in (0, c)$) such that (36) is satisfied. In particular, $R(\lambda; A_s)y \to R(\lambda; A)y$.
We show next that $(\lambda I - A)D$ is dense in X. Let $x \in X$. Define $z = R(\lambda; A)x$. Since $z \in D(A)$ and D is dense in $[D(A)]$ (by definition of a core!), there exist $x_n \in D$ such that $x_n \to z$ in the graph-norm, i.e., $x_n \to z$ and $Ax_n \to Az$. Hence $y_n := (\lambda I - A)x_n \to (\lambda I - A)z = x$. Since $y_n \in (\lambda I - A)D$, this proves the density of $(\lambda I - A)D$ in X.
Finally, given $x \in X$ arbitrary and $\epsilon > 0$, pick $y \in (\lambda I - A)D$ such that $||x - y|| < \frac{(\lambda - a)\epsilon}{3M}$. For this y, by the first part of the proof, there exists $\delta > 0$ such that $||R(\lambda; A_s)y - R(\lambda; A)y|| < \epsilon/3$ for $0 < s < \delta$. Then for $0 < s < \delta$,

$$||R(\lambda; A_s)x - R(\lambda; A)x|| \le ||R(\lambda; A_s)(x - y)|| + ||R(\lambda; A_s)y - R(\lambda; A)y||$$

$$+||R(\lambda; A)(y - x)||.$$

The first and third summand above are $< \epsilon/3$ by (34). The middle summand is $< \epsilon/3$ by the choice of δ. Hence $R(\lambda; A_s)x \to R(\lambda; A)x$ for all $x \in X$, as $s \to 0+$.

(c) Conversely, resolvents strong convergence implies graph-convergence on D. (Given $y \in (\lambda I - A)D$, choose $y_s = y$ constant!)

(*Details.* Given $y \in (\lambda I - A)D$ and choosing $y_s = y$ for all $s \in (0, c)$, we have by strong resolvent convergence

$$[y_s, R(\lambda; A_s)y] = [y, R(\lambda; A_s)y] \to [y, R(\lambda; A)y],$$

and therefore, by Part (a), A_s graph-converge to A on D.)

(d) If $T'(\cdot)$ is also a C_0-semigroup satisfying (33), and A' is its generator, then

$$R(\lambda; A')[T'(t) - T(t)]R(\lambda; A) = \int_0^t T'(t-u)[R(\lambda; A') - R(\lambda; A)]T(u)\,du \tag{37}$$

for $\lambda > a$ and $t \geq 0$. Hint: verify that the integrand in (37) is the derivative with respect to u of the function $-T'(t-u)R(\lambda; A')T(u)R(\lambda; A)$.

Solution.

Observe that the formula for the derivative of a product is valid for functions with values in a Banach algebra (provided the order of the factors is kept); the proof is identical with the scalar case proof.
Fix $t \geq 0$ and $\lambda > a$. For $u \in [0, t]$, let

$$V(u) := -T'(t-u)R(\lambda; A')T(u)R(\lambda; A). \tag{38}$$

Then by Exercise 1(c) and the commutativity of $T(u)$ and $R(\lambda; A)$,

$$\frac{dV(u)}{du} = T'(t-u)A'R(\lambda; A')T(u)R(\lambda; A) - T'(t-u)R(\lambda; A')T(u)AR(\lambda; A)$$

$$= T'(t-u)[\lambda R(\lambda; A') - I]T(u)R(\lambda; A) - T'(t-u)R(\lambda; A')T(u)[\lambda R(\lambda; A) - I]$$

$$= T'(t-u)[R(\lambda; A') - R(\lambda; A)]T(u).$$

By Exercise 13(d), Chapter 9, and the commutativity of $T'(t)$ and $R(\lambda; A')$

$$\int_0^t T'(t-u)[R(\lambda; A') - R(\lambda; A)]T(u)\,du = V(t) - V(0)$$

$$= R(\lambda; A')[T'(t) - T(t)]R(\lambda; A).$$

(e) Resolvents strong convergence implies *semigroups strong convergence*, i.e., for each $0 < \tau < \infty$,

$$\sup_{t \leq \tau} ||T_s(t)x - T(t)x|| \to 0 \tag{39}$$

as $s \to 0$. Hint: by (33), it suffices to consider $x \in D(A) = R(\lambda; A)X$. Write $[T_s(t) - T(t)]R(\lambda; A)y = R(\lambda; A_s)[T_s(t) - T(t)]y + T_s(t)[R(\lambda; A) - R(\lambda; A_s)]y + [R(\lambda; A_s) - R(\lambda; A)]T(t)y$. Estimate the norm of the first summand for $y \in D(A)$ (hence $y = R(\lambda; A)x$) using (37), and use the density of $D(A)$ and (33)-(34). The second summand $\to 0$ strongly, uniformly for $t \le \tau$, by (33)-(34). For the third summand, consider again $y \in D(A)$, for which one can use the relation $T(t)y = y + \int_0^t T(u)Ay\, du$. Cf. Exercise 13(b), Chapter 9 and the Dominated Convergence theorem.

Solution.

Fix $\tau > 0$ and $\lambda > a$, and denote briefly by H_s, K_s, L_s the operators in the three summands of the decomposition suggested in the hint.

Estimate for H_s. Let $y \in D(A)$, and set $x := (\lambda I - A)y$. Then by Part (d) with $T'(\cdot) = T_s(\cdot)$ and (33), we have for all $t \in [0, \tau]$

$$||H_s y|| = ||R(\lambda; A_s)[T_s(t) - T(t)]R(\lambda; A)x||$$

$$\le \int_0^t ||T_s(t-u)||\, ||[R(\lambda; A_s) - R(\lambda; A)]T(u)x||\, du$$

$$\le M \int_0^\tau e^{a(\tau - u)} ||[R(\lambda; A_s) - R(\lambda; A)]T(u)x||\, du := h_s(\tau; y).$$

By strong resolvent convergence, the integrand in $h_s(\tau; y)$ converges pointwise to 0 as $s \to 0+$, and by (33) and (34), it is bounded (for all $s \in [0, c)$ and $u \in [0, \tau]$) by $\frac{2M^2}{\lambda - a} e^{a\tau} ||x||$. By the Dominated Convergence theorem, $h_s(\tau; y) \to 0$ as $s \to 0+$.
We also have the operator-norm estimate (by (33) and (34))

$$||H_s|| \le \frac{2M^2}{\lambda - a} e^{a\tau} := k(\tau)$$

for all s, t as above.
For $x \in X$ arbitrary and $\epsilon > 0$ given, we pick $y \in D(A)$ such that $||x - y|| < \epsilon/(2k(\tau))$. For this y, since $h_s(\tau; y) \to 0$ as $s \to 0+$, there exists $\delta > 0$ such that $h_s(\tau; y) < \epsilon/2$ for all $s < \delta$. Hence

$$\sup_{t \in [0,\tau]} ||H_s x|| \le ||H_s(x - y)|| + ||H_s y|| \le k(\tau)||x - y|| + h_s(\tau; y) < \epsilon$$

for all $s < \delta$.

Estimate for K_s. For all $y \in X$, we have by (33)

$$\sup_{t \in [0,\tau]} ||K_s y|| \le M e^{a\tau} ||[R(\lambda; A) - R(\lambda; A_s)]y|| \to 0$$

as $s \to 0+$, by strong resolvent convergence.

Estimate for L_s. Let $y \in D(A)$. By Exercise 1(c) and Exercise 13(d), Chapter 9,

$$\int_0^t T(u)Ay\,du = \int_0^t [T(u)y]'\,du = T(t)y - y.$$

Hence, using Exercise 13(b), Chapter 9,

$$\sup_{t\in[0,\tau]} ||L_s y|| = ||[R(\lambda; A_s) - R(\lambda; A)][\int_0^t T(u)Ay\,du + y]||$$

$$\leq \int_0^\tau ||[R(\lambda; A_s) - R(\lambda; A)][T(u)Ay]||\,du + ||[R(\lambda; A_s) - R(\lambda; A)]y||.$$

The second summand above converges to 0 as $s \to 0+$, by strong resolvent convergence. The integrand in the first summand converges pointwise to 0 (for the same reason), and is bounded on $[0, \tau]$ by $\frac{2M^2}{\lambda - a} e^{a\tau} ||Ay||$ (by (33) and (34)). Therefore, by the Dominated Convergence theorem, the first summand above converges to 0 as $s \to 0+$.
By (33) and (34), we also have the operator-norm estimate $||L_s|| \leq \frac{2M^2}{\lambda - a} e^{a\tau} = k(\tau)$ for all $s \in [0, c]$ and $t \in [0, \tau]$. Given $x \in X$ arbitrary and $\epsilon > 0$, we pick $y \in D(A)$ such that $||x - y|| < \epsilon/[2k(\tau)]$ (by density of $D(A)$ in X). For this y, the preceding conclusion shows that there exists $\delta > 0$ such that $\sup_{t\leq\tau} ||L_s y|| < \epsilon/2$ for $0 < s < \delta$. Then for these s,

$$\sup_{t\in[0,\tau]} ||L_s x|| \leq k(\tau)\,||x - y|| + \sup_{t\leq\tau} ||L_s y|| < \epsilon.$$

This completes the proof of (39).

(f) Conversely, semigroups strong convergence implies resolvents strong convergence. Hint: Use the Laplace integral representation of the resolvents.

Collecting, we conclude that *generators graph-convergence on D, resolvents strong convergence, and semigroups strong convergence are equivalent* (when Condition (33) is satisfied).

Solution.

Since (33) implies that the semigroups $T_s(\cdot)$ have type $\leq a$ for all $s \in [0, c)$, it follows from Exercise 3(d) that their resolvents are equal to their Laplace transforms for all $\lambda > a$. Hence for all $x \in X$ and $\lambda > a$ (cf. Exercise 13(e), Chapter 9)

$$||R(\lambda; A_s)x - R(\lambda; A)x|| = ||\int_0^\infty e^{-\lambda t}[T_s(t)x - T(t)x]\,dt||$$

$$\leq \int_0^\infty e^{-\lambda t}||T_s(t)x - T(t)x||\, dt.$$

By semigroup strong convergence, the last integrand converges pointwise to 0 as $s \to 0+$; by (33), it is dominated by $2M\, e^{-(\lambda-a)t} \in L^1([0,\infty))$. By the Dominated Convergence theorem, the last integral converges to 0 as $s \to 0+$. This proves resolvents strong convergence.

Exponential formulas

8. Let A be the generator of a C_0-semigroup $T(\cdot)$ on the Banach space X.
Let $F : [0,\infty) \to B(X)$ be contraction-valued, such that $F(0) = I$ and the (strong) right derivative of $F(\cdot)x$ at 0 coincides with Ax, for all x in a core D for A. Prove:

(a) Fix $t > 0$ and define A_n as in Exercise 21(f), Chapter 6. Then $e^{tA_n} - F(t/n)^n \to 0$ in the s.o.t. as $n \to \infty$.

(*Details.* By definition of a core, D is graph-dense in $D(A)$, hence dense in $D(A)$, because $||\cdot||_A \geq ||\cdot||$. Since $D(A)$ is dense in X (cf. Exercise 1(b)), it follows that D is dense in X. For $t > 0$ fixed, set

$$A_n := A_n(t) := (n/t)[F(t/n) - I]. \tag{40}$$

Then for each $x \in D$,

$$A_n x = \frac{F(t/n)x - F(0)x}{t/n} \to Ax \tag{41}$$

as $n \to \infty$, strongly in X. In particular $\sup_n ||A_n x|| < \infty$, and (a) follows from Exercise 21(f), Chapter 6.)

(b) $s \to e^{sA_n}$ is a (uniformly continuous) contraction semigroup, for each $n \in \mathbb{N}$. (Cf. Exercise 21(a), Chapter 6.)

(*Details.* We need only to verify that e^{sA_n} is a contraction. By Exercise 21(a), Chapter 6, applied to the contraction $C = F(t/n)$ $(t > 0$ fixed), $e^{sA_n} = e^{(sn/t)(C-I)}$ is a contraction.)

(c) Suppose $T(\cdot)$ is a *contraction* C_0-semigroup. As $n \to \infty$, the semigroups e^{sA_n} converge strongly to the semigroup $T(s)$, uniformly on compact intervals. (Cf. conclusion of Exercise 7 above; note that $A_n x \to Ax$ for all $x \in D$.) Conclude that $F(t/n)^n \to T(t)$ in the s.o.t., for each $t \geq 0$.

Solution.

(Exercise 7 was formulated for a family of C_0-semigroups $T_s(\cdot)$ with $s \in [0,c)$, using the notation $T_0(\cdot) = T(\cdot)$ and considering convergence as $s \to 0+$. It can be formulated likewise when the parameter s varies in an interval (c,∞), $T(\cdot)$ (with generator A) is a given C_0-semigroup satisfying (33), and one considers convergence as $s \to \infty$. In particular, the equivalence of generators graph-convergence on a core, resolvents strong convergence, and semigroups strong convergence (under Condition (33)) is valid for *sequences* $\{T_n(\cdot)\} \cup \{T(\cdot)\}$ of C_0-semigroups, as $n \to \infty$.)
Since the (final) claim is trivial for $t = 0$, we *fix* $t > 0$, and consider $A_n := A_n(t)$ as before. By (41), $A_n x \to Ax$ for all $x \in D$; thus, taking $x_n = x \in D$, we have $[x_n, A_n x_n] = [x, A_n x] \to [x, Ax]$ as $n \to \infty$, that is, A_n graph-converge to A on the core D for A. The uniformly continuous semigroups $T_n(s) := e^{sA_n}$ are contraction semigroups (cf. Part (b)), with generator A_n; $T(\cdot)$ is a contraction C_0-semigroup with generator A. Thus (33) is satisfied (with $M = 1$ and $a = 0$) by the semigroups family $\{T_n(\cdot);\ n \in \mathbb{N}\} \cup \{T(\cdot)\}$. By Exercise 7, the graph-convergence of A_n to A on D is equivalent to the strong semigroup convergence of $T_n(\cdot)$ to $T(\cdot)$. Thus

$$||T(s)x - e^{sA_n}x|| \to 0 \tag{42}$$

as $n \to \infty$ for all $x \in X$, uniformly for s in compact intervals.
By Exercise 21(e), Chapter 6 (with $C := F(t/n)$),

$$||e^{tA_n}x - F(t/n)^n x|| \le n^{1/2}||[F(t/n) - I]x|| = tn^{-1/2}||A_n x|| \to 0$$

as $n \to \infty$, for all $x \in D$. Since $||e^{tA_n} - F(t/n)^n|| \le 2$, (cf. Part (b) and hypothesis on F) and D is dense in X, the preceding conclusion is valid for all $x \in X$. Hence by (42), for all $x \in X$ and $t > 0$,

$$||T(t)x - F(t/n)^n x|| \le ||T(t)x - e^{tA_n}x|| + ||e^{tA_n}x - F(t/n)^n x|| \to 0$$

as $n \to \infty$.

(d) Let $T(\cdot)$ be a C_0-semigroup such that $||T(t)|| \le e^{at}$, and consider the *contraction* semigroup $S(t) := e^{-at}T(t)$ (with generator $A - aI$; $a \ge 0$). Choose F as follows: $F(0) = I$ and for $0 < s < 1/a$,

$$F(s) := (s^{-1} - a)R(s^{-1}; A) = (s^{-1} - a)R(s^{-1} - a; A - aI).$$

Verify that F satisfies the hypothesis stated at the beginning of the exercise, and conclude that

$$T(t) = \lim_{n\to\infty} [\frac{n}{t} R(\frac{n}{t}; A)]^n \tag{43}$$

in the s.o.t., for each $t > 0$.

Solution.

Since $S(\cdot)$ is a contraction semigroup with generator $A-aI$, $\lambda R(\lambda; A-aI)$ are contractions for all $\lambda > 0$ (cf. Exercise 4(e)), hence F is contraction-valued. For $x \in D(A)$ and $s > 0$

$$s^{-1}[F(s)x - F(0)x] = s^{-1}R(s^{-1}-a; A-aI)(A-aI)x$$

$$= s^{-1}R(s^{-1}; A)(A-aI)x \to (A-aI)x$$

as $s \to 0+$, by Exercise 3(f). This shows that F satisfies the required hypothesis relative to the semigroup $S(\cdot)$, with $D = D(A) = D(A-aI)$. By Part (c), it follows that for all $t > 0$ and $x \in X$,

$$S(t)x = \lim_n F(t/n)^n x = \lim_n [(\frac{n}{t} - a)R(\frac{n}{t}; A)]^n x$$

$$= \lim_n [1 - \frac{at}{n}]^n [\frac{n}{t} R(\frac{n}{t}; A)]^n x,$$

hence

$$\lim_n [\frac{n}{t} R(\frac{n}{t}; A)]^n x = e^{at} S(t)x = T(t)x.$$

(e) Let $T(\cdot)$ be *any* C_0-semigroup. By Exercise 14(c), Chapter 9, $||T(t)|| \le Me^{at}$ for some $M \ge 1$ and $a \ge 0$. Consider the equivalent norm

$$|x| := \sup_{t \ge 0} e^{-at} ||T(t)x|| \qquad (x \in X).$$

Then $|T(t)x| \le e^{at}|x|$, and therefore (43) is valid over $(X, |\cdot|)$, hence over X (since the two norms are equivalent). Relation (43) (true for any C_0-semigroup!) is called *the exponential formula* for semigroups.

(*Details.* Trivially $|x| \ge e^{-a0}||T(0)x|| = ||x||$, and since $||T(t)|| \le Me^{at}$, $|x| \le M\,||x||$. Thus the norms $|\cdot|$ and $||\cdot||$ are equivalent. For all $t \ge 0$, we have by the semigroup property (denoting $u = s + t$)

$$|T(t)x| = \sup_{s \ge 0} e^{-as}||T(s)[T(t)x]|| = e^{at} \sup_{u \ge t} e^{-au}||T(u)x|| \le e^{at}|x|.)$$

(f) Let A, B, C generate contraction C_0-semigroups $S(\cdot)$, $T(\cdot)$, $U(\cdot)$ respectively, and suppose $C = A + B$ on a core D for C. Then

$$U(t) = \lim_{n \to \infty} [S(t/n)T(t/n)]^n \qquad (t \ge 0) \tag{44}$$

in the s.o.t. Hint: Choose $F(t) = S(t)T(t)$ in Part (c).

Solution.

We verify that $F(\cdot) := S(\cdot)T(\cdot)$ satisfies the required hypothesis. Clearly, F is contraction-valued and $F(0) = I$. Let $x \in D$. By assumption, $x \in D(A+B) = D(A) \cap D(B)$, and $Ax + Bx = Cx$. For $h > 0$, we write

$$h^{-1}[F(h) - I]x = [S(h) - I]\,Bx + [S(h) - I]\,[h^{-1}(T(h) - I)x - Bx]$$
$$+h^{-1}[S(h) - I]x + h^{-1}[T(h) - I]x.$$

When $h \to 0$, the first summand converge to 0 (by the C_0-condition); the second summand converges to 0 (by the definition of B, since $x \in D(B)$ and $||S(h) - I|| \leq 2$); the third and fourth summands converge to Ax and Bx respectively, since $x \in D(A) \cap D(B)$. Hence the strong right derivative of $F(\cdot)x$ at 0 is equal to $Ax + Bx = Cx$. We now get (44) by applying Part (c).)

Groups of operators

9. A group of operators on the Banach space X is a map $T(\cdot) : \mathbb{R} \to B(X)$ such that

$$T(s+t) = T(s)T(t) \qquad (s, t \in \mathbb{R}).$$

We assume that it is of class C_0, i.e., *the semigroup* $T(\cdot)|_{[0,\infty)}$ is of class C_0. Let A be the generator of this semigroup. Prove:

(a) The semigroup $S(t) := T(-t), \quad t \geq 0$, is of class C_0, and has the generator $-A$.

(*Details*. For $0 < t < h$ and $x \in X$,

$$S(t)x - x = T(-h)[T(h-t)x - T(h)x] \to 0$$

as $t \to 0$, by strong continuity of $T(\cdot)|_{[0,\infty)}$ at the point h (cf. Exercise 14(d), Chapter 9). Thus $S(\cdot)$ is a C_0- semigroup. Let A' be its generator. For $x \in D(A)$ and $0 < t < h$,

$$t^{-1}[S(t)x - x] = -T(-h)(-t)^{-1}[T(h-t)x - T(h)x] \to -T(-h)[T(\cdot)x]'(h)$$
$$= -T(-h)T(h)Ax = -Ax$$

as $t \to 0$, by Exercise 1(c). Hence $-A \subset A'$, and therefore $A' = -A$, by symmetry.)

(b) $\sigma(A)$ is contained in the strip

$$\Omega : -\omega' \leq \Re\lambda \leq \omega,$$

where ω, ω' are the types of the semigroups $T(\cdot)$ and $S(\cdot)$, respectively. Fix $a > \omega$ and $a' > \omega'$, and let $\Omega' = \{\lambda \in \mathbb{C};\ -a' \leq \Re\lambda \leq a\}$. For $\lambda \notin \Omega'$,

$$||R(\lambda; A)^n|| \leq \frac{M}{d(\lambda, \Omega')^n}. \tag{45}$$

If A generates a *bounded* C_0-group, then $\sigma(A) \subset i\mathbb{R}$ and

$$||R(\lambda; A)^n|| \leq \frac{M}{|\Re\lambda|^n}, \tag{46}$$

where M is a bound for $||T(\cdot)||$.

(*Details.* By Exercise 3(d) applied to the C_0-semigroups $T(\cdot)$ and $S(\cdot)$ (on $[0,\infty)$), with the respective generators A and A', we have by Part (a),

$$\sigma(A) \subset \{\lambda; \Re\lambda \leq \omega\}$$

and

$$\sigma(A) = -\sigma(A') \subset \{-\lambda; \Re\lambda \leq \omega'\} = \{\mu; \Re\mu \geq -\omega'\}.$$

Hence $\sigma(A) \subset \Omega$. Furthermore, by the estimate in the solution of Exercise 3(e) with complex λ such that $\Re\lambda > a$ (see also Exercise 3(a))

$$||R(\lambda; A)^n|| \leq \frac{M}{(\Re\lambda - a)^n}$$

for all n, with M depending on a. A similar estimate is valid for A', for $\Re\lambda > a' > \omega'$. Since $A' = -A$, this yields the estimates $||R(\lambda; A)^n|| \leq M'/(-\Re\lambda - a')^n$ for $\Re\lambda < -a'$. The estimate (45) follows.
If $||T(\cdot)|| \leq M$, we have $\omega = \omega' = 0$, hence $\Omega = i\mathbb{R}$, and therefore $\sigma(A) \subset i\mathbb{R}$. The estimate (46) follows as in Exercise 3(e), with complex λ.)

(c) An operator A generates a C_0-group of operators iff it is closed, densely defined, has spectrum in a strip Ω' as in Part (b), and (45) is satisfied for all *real* $\lambda \notin [-a', a]$. Hint: apply the Hille-Yosida theorem (cf. Exercise 4(e)) separately $\lambda > a$ and $\lambda > a'$.

Solution.

The necessity of the conditions follows from the necessity part of the Hille-Yosida theorem for the C_0-semigroup $T(\cdot) : [0,\infty) \to B(X)$ and from Part (b).
By the sufficiency part of the Hille-Yosida theorem, A generates a C_0-*semigroup* $T(\cdot)$. The hypothesis also implies that $(a', \infty) \subset \rho(-A)$ and

$$||R(\lambda; -A)^n|| = ||R(-\lambda; A)^n|| \leq \frac{M}{(\lambda - a')^n}$$

for all $n \in \mathbb{N}$ and $\lambda > a'$. By the Hille-Yosida theorem (cf. Exercise 4(e)), $-A$ generates a C_0-semigroup $S(\cdot) : [0,\infty) \to B(X)$.
Let $x \in D(A)$. By Exercise 1(c) (cf. also solution of Exercise 1(e))

$$\frac{d}{dt}T(t)S(t)x = T(t)AS(t)x - T(t)AS(t)x = 0 \qquad (t \geq 0),$$

and $T(0)S(0)x = x$. Hence $T(t)S(t)x = x$ for all $x \in D(A)$, and since $D(A)$ is dense in X (cf. Exercise 1(b)), it follows that $T(t)S(t) = I$ for all $t \geq 0$. Extending the definition of $T(\cdot)$ to $(-\infty, 0)$ by setting $T(-t) = S(t)$ for $t > 0$, $T(\cdot) : \mathbb{R} \to B(X)$ clearly satisfies the group relation.

(d) Let $T(\cdot)$ be a C_0-group of *unitary* operators on a Hilbert space X. Let $H = -iA$, where A is the generator of $T(\cdot)$. Then H is a (closed, densely defined) *symmetric* operator with *real* spectrum. In particular, $iI - H$ and $-iI - H$ are both *onto*, so that the deficiency indices of H are both zero. Therefore H is selfadjoint (cf. discussion following Definition 10.10).

(*Details.* Since $T(t)$ is a unitary (hence norm-preserving) operator, $||T(t)|| = 1$ for all $t \in \mathbb{R}$, and therefore $\sigma(A) \subset i\mathbb{R}$ by Part (b). Hence $\sigma(H) = -i\sigma(A) \subset \mathbb{R}$.
With notation as in Part (a), we have for $t > 0$

$$T(t)^* = T(t)^{-1} = T(-t) = S(t).$$

Hence

$$(t^{-1}[T(t) - I]x, y) = (x, t^{-1}[S(t) - I]y)$$

for all $x, y \in X$. Since the generator of $S(\cdot)$ is $-A$ (cf. Part (a)), if $x, y \in D(A)$, we obtain by letting $t \to 0+$ that $(Ax, y) = (x, -Ay)$, and consequently $(Hx, y) = -i(Ax, y) = i(x, Ay) = (x, Hy)$ for all $x, y \in D(H)$. Thus H is symmetric. As indicated in the statement of the exercise, it follows that H is selfadjoint.)

(e) Define e^{itH} by means of the operational calculus for the selfadjoint operator H. This is a C_0-group with generator $iH = A$, and therefore $T(t) = e^{itH}$ (cf. Exercise 1(e): the generator determines the semigroup uniquely). This representation of unitary groups is *Stone's theorem.*

Solution.

Let E be the resolution of the identity for the selfadjoint operator H. Define (cf. Section 10.4)

$$T'(t) := e^{itH} := \int_{\mathbb{R}} e^{its} dE(s) \qquad (t \in \mathbb{R}). \tag{47}$$

Since the operational calculus $f \in \mathbb{B}(\mathbb{R}) \to f(H) \in B(X)$ is a *-representation, (47) defines a unitary group. Also for all $x \in X$,

$$||[T'(t) - I]x||^2 = \int_{\mathbb{R}} |e^{its} - 1|^2 d||E(s)x||^2. \tag{48}$$

The integrand in (48) converges pointwise to 0 as $t \to 0$ and is bounded by 4. By the Dominated Convergence theorem (with respect to the *finite* positive Borel measure

$||E(\cdot)x||^2$ on $\mathbb{R}$), it follows from (48) that $||T'(t)x - x|| \to 0$ as $t \to 0$, and we conclude that $T'(\cdot)$ is a C_0-group of unitary operators. Let A' be its generator. By Exercise 3(d), for $\Re\lambda > 0$ and $x \in X$,

$$(R(\lambda; A')x, x) = \int_0^\infty e^{-\lambda t}(T'(t)x, x)\, dt = \int_0^\infty \int_{\mathbb{R}} e^{(is-\lambda)t} d(E(s)x, x)\, dt. \tag{49}$$

The absolute value of the integrand in the corresponding double integral is equal to $e^{-(\Re\lambda)t}$, and this function is integrable with respect to $dt \times d(E(s)x, x)$, by Tonelli's theorem. Therefore, by Fubini's theorem,

$$(R(\lambda; A')x, x) = \int_{\mathbb{R}} \int_0^\infty e^{-(\lambda - is)t} dt\, d(E(s)x, x)$$

$$= \int_{\mathbb{R}} (\lambda - is)^{-1} d(E(s)x, x) = (R(\lambda; iH)x, x).$$

By Exercise 12(c), Chapter 8, it follows that $R(\lambda; A') = R(\lambda; iH)$ for $\Re\lambda > 0$, hence $A' = iH = A$, and therefore $T'(\cdot) = T(\cdot)$, by the conclusion of Exercise 1(e). (Note that although $(E(\cdot)x, x)$ is a Borel measure, and the corresponding measure space is not complete, the Tonelli and Fubini theorems as stated in Theorems 2.17 and 2.18 are still applicable, because all functions involved are continuous, and we may work in the completion of the measure space.)