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CHAPTER 9

INTEGRAL REPRESENTATION

[The first three exercises provide the proofs of theorems used in this chapter.]

Runge's theorem

1. Let $S^2 = \overline{\mathbb{C}}$ denote the Riemann sphere, and let $K \subset \mathbb{C}$ be compact. Fix a point a_j in each component V_j of $S^2 - K$, and let $\mathcal{R}(\{a_j\})$ denote the set of all rational functions with poles in the set $\{a_j\}$.

If μ is a complex Borel measure on K, we define its *Cauchy transform* $\tilde{\mu}$ by

$$\tilde{\mu}(z) = \int_K \frac{d\mu(w)}{w - z} \qquad (z \in S^2 - K). \tag{1}$$

Prove

(a) $\tilde{\mu}$ is analytic in $S^2 - K$.

Details. Fix $z \in \mathbb{C} - K$, and let $r = d(z, K)$. Consider $\zeta \in B(z, r/2)$, $\zeta \neq z$. We have

$$\frac{\tilde{\mu}(\zeta) - \tilde{\mu}(z)}{\zeta - z} - \int_K \frac{d\mu(w)}{(w - z)^2} = (\zeta - z) \int_K \frac{d\mu(w)}{(w - \zeta)(w - z)^2}. \tag{2}$$

For $w \in K$, $|w - z| \geq r$ and $|w - \zeta| \geq |w - z| - |\zeta - z| > r - r/2 = r/2$. Therefore the integrand in (2) is $\leq 2/r^3$ for $w \in K$, and consequently the left hand side of (2) has modulus $\leq (2/r^3)||\mu|| |\zeta - z| \to 0$ (as $\zeta \to z$). This proves the analyticity of $\tilde{\mu}$ in $\mathbb{C} - K$ (and also the formula $\tilde{\mu}'(z) = \int_K \frac{d\mu}{(w-z)^2}$ for all $z \in \mathbb{C} - K$). Since $\tilde{\mu}(\infty) = 0$, the differential ratios at ∞ vanish, and $\tilde{\mu}$ is then trivially analytic at ∞.

(b) For $a_j \neq \infty$, let $d_j = d(a_j, K)$ and fix $z \in B(a_j, r) \subset V_j$ (necessarily $r < d_j$). Observe that

$$\frac{1}{w - z} = \sum_{n=0}^{\infty} \frac{(z - a_j)^n}{(w - a_j)^{n+1}}, \tag{3}$$

and the series converges uniformly for $w \in K$.

For $a_j = \infty$, we have

$$\frac{1}{w-z} = -\sum_{n=0}^{\infty} \frac{w^n}{z^{n+1}} \qquad (|z| > r), \tag{4}$$

and the series converges uniformly for $w \in K$.

Details. For $w \in K$ and $z \in B(a_j, r)$, we have

$$(w-z)^{-1} = (w-a_j)^{-1}(1 - \frac{z-a_j}{w-a_j})^{-1}. \tag{5}$$

Since $|w - a_j| \geq d(a_j, K) = d_j > r$, $\left|\frac{z-a_j}{w-a_j}\right| < r/d_j < 1$. Therefore the geometric series $\sum_n \left(\frac{z-a_j}{w-a_j}\right)^n$ is majorized by the convergent constant geometric series $\sum_n (r/d_j)^n$. By the Weierstrass test for uniform convergence, the series converges absolutely and uniformly for $w \in K$ to the sum $(1 - \frac{z-a_j}{w-a_j})^{-1}$, and (3) follows from (5).
In (4), we may take $r := \sup_K |w|$. If $|z| > r$, the series in (4) is majorized by the convergent series $(1/|z|)\sum(r/|z|)^n$ for $w \in K$; it then converges absolutely and uniformly for $w \in K$ to $-z^{-1}[1-(w/z)]^{-1} = 1/(w-z)$.

(c) If $\int_K h d\mu = 0$ for all $h \in \mathcal{R}(\{a_j\})$, then $\tilde{\mu}(z) = 0$ for all $z \in B(a_j, r)$, hence for all $z \in V_j$, for all j, and therefore $\tilde{\mu} = 0$ on $S^2 - K$.

Details. Fix $z \in B(a_j, r)$, $a_j \neq \infty$. By uniform convergence of the series in (3) (for $w \in K$), we have

$$\tilde{\mu}(z) = \sum_n (z-a_j)^n \int_K (w-a_j)^{-n-1} d\mu(w). \tag{6}$$

The integrands in (6) belong to $\mathcal{R}(\{a_j\})$, and therefore all the integrals vanish by hypothesis. Consequently $\tilde{\mu}(z) = 0$ in the disc $B(a_j, r)$, hence in V_j, by Part (a). The same is true for $a_j = \infty$, using (4) instead of (3). We then conclude that $\tilde{\mu} = 0$ on $S^2 - K$.

(d) Let $\Omega \subset \mathbb{C}$ be open such that $K \subset \Omega$. If f is analytic in Ω and μ is as in Part (c), then $\int_K f d\mu = 0$. (Hint: represent $f(z) = (1/2\pi i)\int_\Gamma \frac{f(w)}{w-z} dw$ for all $z \in K$, where $\Gamma \in \Gamma(K, \Omega)$, cf. 9.18 for notation, and use Fubini's theorem.)

Details. Fix $\Gamma \in \Gamma(K, \Omega)$, and let $\delta = d(\Gamma, K)(>0)$ and $M = \sup_\Gamma |f|$ $(< \infty)$. Then $|f(z)|/|z-w| \leq M/\delta$ for $z \in \Gamma$ and $w \in K$, hence

$$\int_K \int_\Gamma \frac{|f(z)|}{|z-w|} d|z|\, d|\mu|(w) < \infty.$$

By Tonelli's theorem (Theorem 2.18), $f(z)/(z-w) \in L^1(d|z| \times |\mu|)$ (where the measure $d|z|$ is on the Lebesgue measurable space for Γ, and the Borel measure $|\mu|$ on K has been completed). By Fubini's theorem (Theorem 2.17), Cauchy's formula, the hypothesis on μ, and Part (c),

$$2\pi i \int_K f\,d\mu = \int_K \int_\Gamma \frac{f(z)}{z-w} dz\, d\mu(w) = \int_\Gamma \int_K \frac{d\mu(w)}{z-w} f(z) dz$$
$$= \int_\Gamma \tilde{\mu}(z) f(z) dz = 0.$$

(e) Prove that $\mathcal{R}(\{a_j\})$ is $C(K)$-dense in $H(\Omega)$ (the subspace of $C(K)$ consisting of the analytic functions in Ω restricted to K). Hint: Theorem 4.9, Corollary 5.3, and Part (d). The result in Part (e) is *Runge's theorem*. In particular, the rational functions with poles off K are $C(K)$-dense in $H(\Omega)$.

Details. Suppose $\mathcal{R}(\{a_j\})$ is not $C(K)$-dense in $H(\Omega)$. Pick $f \in H(\Omega)$ not in the $C(K)$-closure of $\mathcal{R}(\{a_j\})$. By Corollary 5.3, there exists $x^* \in C(K)^*$ such that $x^*h = 0$ for all $h \in \mathcal{R}(\{a_j\})$ and $x^*f = 1$. By the Riesz Representation theorem (Theorem 4.9), there exists a regular complex Borel measure μ on K such that $\int_K h\,d\mu = 0$ for all $h \in \mathcal{R}(\{a_j\})$ and $\int_K f\,d\mu = 1$. This contradicts Part (d).

(f) If $S^2 - K$ is *connected*, the polynomials are $C(K)$- dense in $H(\Omega)$. Hint: apply Part (e) with $a = \infty$ in the single component of $S^2 - K$.

[*Paraphrazing.* When $S^2 - K$ is connected (i.e., when K is *simply connected*), it has a single component, namely the component of ∞, and $\mathcal{R}(\{\infty\})$ is the algebra $\mathcal{P}$ of all complex polynomials on S^2. Hence (f) follows from Part (e).]

Hartogs-Rosenthal's theorem

2. (Notation as in Exercise 1.) Let m denote the $\mathbb{R}^2$-Lebesgue measure.

(a) The integral defining the Cauchy transform $\tilde{\mu}$ converges absolutely m-a.e. (Hint: show that, for each $N \in \mathbb{N}$,

$$\int_{|z|\le N} \int_K \frac{d|\mu|(w)}{|w-z|} dx dy < \infty \tag{7}$$

by using Tonelli's theorem and polar coordinates.)

Details. Let $z = x + iy \in \mathbb{C}$ and $R = \sup_{w\in K} |w|$ (finite, since K is compact). Write $w - z = re^{i\theta}$, with $r = |w-z| \le |z| + R$ for $w \in K$, and $0 \le \theta \le 2\pi$. Using polar coordinates in the plane, we get for any $N \in \mathbb{N}$

$$\int_K \int_{|z|\le N} \frac{dx\,dy}{|w-z|} d|\mu|(w) \le \int_K \int_{\theta=0}^{2\pi} \int_{r=0}^{N+R} (1/r) r dr\, d\theta\, d|\mu|(w)$$

$$= 2\pi(N+R)\,||\mu|| < \infty.$$

Hence (7) follows from Tonelli's theorem (Theorem 2.18), and therefore

$$\int_K \frac{d|\mu|(w)}{|w-z|} < \infty \tag{8}$$

m-a.e. in $|z| \le N$ for each N (cf. Section 1.4, page 15). Consequently (8) is valid m-almost everywhere in the plane. (Compare Exercise 1(a)!)

(b) Let $\mathcal{R}(K)$ denote the space of rational functions with poles off K. Then $\int_K h d\mu = 0$ for all $h \in \mathcal{R}(K)$ iff $\tilde{\mu} = 0$ off K. (Hint: use Cauchy's formula and Fubini's theorem for the non-trivial implication.)

Details. For each $z \notin K$, $h_z(w) := 1/(w-z) \in \mathcal{R}(K)$; if $\int_K h\, d\mu = 0$ for all $h \in \mathcal{R}(K)$, then $\tilde{\mu}(z) = \int_K h_z(w)\, d\mu(w) = 0$ for all $z \notin K$.
Conversely, suppose $\tilde{\mu} = 0$ off K, and let $h \in \mathcal{R}(K)$ and $w \in K$. Since the poles of h are off K, there exists an open set Ω containing K such that $h \in H(\Omega)$. Let $\Gamma \in \Gamma(K,\Omega)$ (cf. Notation 9.18). By Cauchy's integral formula, $2\pi i\, h(w) = \int_\Gamma h(z)/(z-w)\, dz$. Since the integrand is continuous on the compact set Γ, we have trivially

$$\int_K \int_\Gamma \frac{|h(z)|}{|z-w|} d|z|\, d|\mu|(w) < \infty.$$

Therefore, by the Tonelli and Fubini theorems (Theorems 2.18 and 2.17), we have

$$2\pi i \int_K h\, d\mu = \int_K \int_\Gamma \frac{h(z)}{z-w} dz\, d\mu(w) = \int_\Gamma \int_K \frac{d\mu(w)}{z-w}\; h(z)\, dz$$

$$= \int_\Gamma \tilde{\mu}(z)\, h(z)\, dz = 0,$$

since $\tilde{\mu} = 0$ on $\Gamma \subset \Omega - K$.

(c) It can be shown that if $\tilde{\mu} = 0$ m-a.e., then $\mu = 0$. Conclude that if $m(K) = 0$ and μ is a complex Borel measure on K such that $\int_K h d\mu = 0$ for all $h \in \mathcal{R}(K)$, then $\mu = 0$. Consequently, if $m(K) = 0$, then $\mathcal{R}(K)$ is dense in $C(K)$ (cf. Theorem 4.9 and Corollary 5.6). This is the Hartogs-Rosenthal theorem.

(*Details.* By Part (b), the hypothesis $\int_K h d\mu = 0$ for all $h \in \mathcal{R}(K)$ is equivalent to the hypothesis $\tilde{\mu} = 0$ off K, which is in turn equivalent to the hypothesis $\tilde{\mu} = 0$ m-a.e. (when $m(K) = 0$). Hence the conclusion $\mu = 0$, and its corollary on the density of $\mathcal{R}(K)$ in $C(K)$.)

Arzela-Ascoli's theorem

3. Let X be a compact metric space. A set $\mathcal{F} \subset C(X)$ is *equicontinuous* if for each $\epsilon > 0$, there exists $\delta > 0$ such that $|f(x) - f(y)| < \epsilon$ for all $f \in \mathcal{F}$ and $x, y \in X$ such that $d(x,y) < \delta$. The set $\mathcal{F}$ is *equibounded* if $\sup_{f \in \mathcal{F}} ||f||_u < \infty$. Prove that if $\mathcal{F}$ is equicontinuous and equibounded, then it is relatively compact in $C(X)$. Sketch: X is necessarily separable. Let $\{a_k\}$ be a countable dense set in X. Let $\{f_n\} \subset \mathcal{F}$. $\{f_n(a_1)\}$ is a bounded complex sequence; therefore there is a subsequence $\{f_{n,1}\}$ of $\{f_n\}$ converging at a_1; $\{f_{n,1}(a_2)\}$ is a bounded complex sequence, and therefore there is a subsequence $\{f_{n,2}\}$ of $\{f_{n,1}\}$ converging at a_2 (and a_1). Continuing inductively, we get subsequences $\{f_{n,r}\}$ such that the $(r+1)$-th subsequence is a subsequence of the r-th subsequence, and the r-th subsequence converges at the points $a_1, ..., a_r$. The diagonal subsequence $\{f_{n,n}\}$ converges at *all* the points a_k. Use the compactness of X and an $\epsilon/5$ argument to show that $\{f_{n,n}\}$ is Cauchy in $C(X)$.

Details. For each $r \in \mathbb{N}$, the diagonal sequence $\{f'_n\} = \{f_{n,n}\}$ differs from a subsequence of $\{f_{n,r}\}$ by finitely many elements; therefore $\{f'_n\}$ converges at the points $a_1, \cdots, a_r$, and since r is arbitrary, it converges at all the points a_k.

Let $\epsilon > 0$ be given, and let $\delta > 0$ correspond to $\epsilon/5$ in the definition of equicontinuity, i.e., $|f(x) - f(y)| < \epsilon/5$ for all $f \in \mathcal{F}$ and $x, y \in X$ such that $d(x,y) < \delta$. By compactness of X, there exist points $x_1, \cdots, x_p \in X$ such that

$$X = \bigcup_{j=1}^{p} B(x_j, \delta). \tag{9}$$

Since $\{a_k\}$ is dense in X, for each $j = 1, \cdots, p$, there exists a point $a_{k_j} \in B(x_j, \delta)$. For each $j = 1, \cdots, p$, the sequence $\{f'_n(a_{k_j})\}$ converges. Therefore there exists $N \in \mathbb{N}$ such that for all $n, m > N$ and $j = 1, \cdots, p$,

$$|f'_n(a_{k_j}) - f'_m(a_{k_j})| < \epsilon/5. \tag{10}$$

For each $x \in X$, pick $j \in \{1, \cdots, p\}$ such that $x \in B(x_j, \delta)$ (cf. (9)). Then

$$|f'_n(x) - f'_m(x)| \leq |f'_n(x) - f'_n(x_j)| + |f'_n(x_j) - f'_n(a_{k_j})|$$

$$+|f'_n(a_{k_j}) - f'_m(a_{k_j})| + |f'_m(a_{k_j}) - f'_m(x_j)| + |f'_m(x_j) - f'_m(x)|.$$

The first and last summands on the right hand side are $< \epsilon/5$ for all n, m since $d(x, x_j) < \delta$. The second and fourth summand are $< \epsilon/5$ for all n, m because $d(a_{k_j}, x_j) < \delta$. The third summand is $< \epsilon/5$ for all $n, m > N$ by (10). Therefore $|f'_n(x) - f'_m(x)| < \epsilon$ for all $x \in X$ and $n, m > N$, i.e., $||f'_n - f'_m||_u \leq \epsilon$ for $n, m > N$. We thus proved that every sequence $\{f_n\}$ in $\mathcal{F}$ has a $C(X)$-convergent subsequence.

Compact normal operators

4. Let X be a Hilbert space, and $T \in K(X)$ be normal. Prove that there exist a sequence $\{\lambda_n\} \in c_0$ and a sequence $\{E_n\}$ of pairwise orthogonal finite rank projections such that $\sum_{n=1}^N \lambda_n E_n \to T$ in $B(X)$ as $N \to \infty$.

Solution.

By the Riesz-Schauder theorem (Theorem 9.27), the compact operator T has either a finite spectrum $\sigma(T) = \{\lambda_1, \cdots, \lambda_N\}$, or $\sigma(T) = \{\lambda_n\} \in c_0$ (we may arrange the notation so that $||T|| \geq |\lambda_1| \geq |\lambda_2| \geq \cdots$); each $\lambda_n \neq 0$ is an isolated point of the spectrum and is an eigenvalue of T with finite dimensional eigenspace. If T is also normal, the eigenspace corresponding to $\lambda_n \neq 0$ coincides with the range of the projection $E_n := E(\delta_n)$, where $E(\cdot)$ is the resolution of the identity for T and δ_n denotes the singleton with element λ_n (cf. Theorem 9.8(5)). Furthermore, by the Spectral Theorem (Theorem 9.5), $T = \int_{\sigma(T)} \lambda \, dE(\lambda)$.

Let $f_N(\lambda) = \sum_{n=1}^N \lambda_n I_{\delta_n}$, $0 \neq \lambda_n \in \sigma(T)$, $N = 1, 2, \cdots$. Then $\int_{\sigma(T)} f_N(\lambda) dE(\lambda) = \sum_{n=1}^N \lambda_n E_n$, and therefore, by Theorem 9.6(1),

$$||(T - \sum_{n=1}^N \lambda_n E_n)x||^2 = \int_{\sigma(T)} |\lambda - f_N(\lambda)|^2 d(E(\lambda)x, x). \tag{11}$$

The integrands in (11) vanish on $\sigma(T)$ for N large enough if the spectrum is finite. In case the spectrum is infinite, the integrands are equal on $\sigma(T)$ to

$$|\sum_{n>N} \lambda_n I_{\delta_n}|^2 \leq |\lambda_{N+1}|^2.$$

Therefore

$$||T - \sum_{n=1}^N \lambda_n E_n|| \leq |\lambda_{N+1}| \to 0 \tag{12}$$

as $N \to \infty$ (when the spectrum is infinite; and trivially, as previously observed, when the spectrum is finite). As mentioned before, each projection E_n has finite dimensional range (which coincides with the eigenspace of $\lambda_n \neq 0$), and $E_n X \perp E_m X$ for $n \neq m$ by the properties of the selfadjoint spectral measure $E(\cdot)$.

Logarithms of Banach algebra elements

5. Let $\mathcal{A}$ be a (unital, complex) Banach algebra, and let $x \in \mathcal{A}$. Prove:

(a) If 0 belongs to the unbounded (path) component V of $\rho(x)$, then $x \in \exp \mathcal{A} (:= \{e^a;\, a \in \mathcal{A}\})$ (that is, x has a logarithm in $\mathcal{A}$). Hint: $\Omega := V^c$ is a simply connected open subset of $\mathbb{C}$ containing $\sigma(x)$, and the analytic function $f_1(\lambda) = \lambda$ does not vanish on Ω. Therefore there exists g analytic in Ω such that $e^g = f_1$.

(*Details.* Following up with the hint, we have by the Composite Function theorem (Theorem 9.21): $x = f_1(x) = e^{g(x)} = e^a$, where $a := g(x)$.)

(b) The group generated by $\exp \mathcal{A}$ is an open subset of $\mathcal{A}$.

Solution.

If $||e - x|| < 1$, the series $-\sum_{n=1}^{\infty}(e-x)^n/n$ converges (absolutely) in $\mathcal{A}$. Its sum u corresponds through the analytic operational calculus to the function $-\sum(1-\lambda)^n/n = \log \lambda$. Therefore, by the Composite Function theorem (Theorem 9.21), $x = \exp u$. This shows that $\exp \mathcal{A}$ contains the open ball $B(e, 1)$. Consequently, if y belongs to the (multiplicative) group H generated by $\exp \mathcal{A}$, then $yB(e, 1)$ is an open neighbourhood of y contained in H; this proves that H is open in $\mathcal{A}$.

6. Let $\mathcal{A}$ be a (unital, complex) Banach algebra, and let G_e denote the (path) component of $G := G(\mathcal{A})$ containing the identity e. Prove:

(a) G_e is open.

(*Details.* Let $a \in G_e$. There exists a continuous path γ in G such that $\gamma(0) = e$ and $\gamma(1) = a$. Since $G(\mathcal{A})$ is open in $\mathcal{A}$ (cf. Theorem 7.3), there exists $r > 0$ such that $B(a, r) \subset G$. Let $b \in B(a, r)$. The continuous path $\gamma'(t) = a + t(b-a)$ $(0 \le t \le 1)$ lies in $B(a, r) \subset G$ and connects a to b. The continuous path $\gamma + \gamma'$ lies in G and connects e to b, hence $b \in G_e$. This shows that $B(a, r) \subset G_e$, and we conclude that G_e is open.)

(b) G_e is a normal subgroup of G.

(*Details.* If γ_k are continuous paths in G such that $\gamma_k(0) = e$ and $\gamma_k(1) = a_k \in G_e$ $(k = 1, 2)$, then the continuous path $\gamma := \gamma_1\gamma_2$ in G satisfies $\gamma(0) = e$ and $\gamma(1) = a_1 a_2$; also the continuous path γ_1^{-1} in G connects e with a_1^{-1}. This shows that G_e is closed under multiplication and under the inverse operation, that is, G_e is a subgroup of G. If $a \in G_e$ and γ is a continuous path in G connecting e and a, then for any $x \in G$, the continuous path $x\gamma x^{-1}$ lies in G and connects e with xax^{-1}. Hence $xG_e x^{-1} \subset G_e$ for all $x \in G$, that is, G_e is a *normal* subgroup of G.)

(c) $\exp \mathcal{A} \subset G_e$.

(*Details.* Let $e^a \in \exp \mathcal{A}$. The continuous path $\gamma(t) = e^{ta}$ $(0 \le t \le 1)$ lies in G (since e^{ta} has the inverse e^{-ta}), $\gamma(0) = e$, and $\gamma(1) = e^a$. Hence $e^a \in G_e$.)

(d) $\bigcup \exp \mathcal{A} \dots \exp \mathcal{A}$ (union of all finite products) is an open subset of G_e (cf. Exercise 5(b)).

(*Details.* $\exp \mathcal{A}$ is closed under the inverse operation (because $(\exp a)^{-1} = \exp(-a)$), and consequently the group H generated by $\exp \mathcal{A}$ coincides with the union of all finite products $\exp \mathcal{A} \cdots \exp \mathcal{A}$; the latter is then an open set in $\mathcal{A}$, by Exercise 5(b). By Part (c) and the continuity of multiplication, each product is (path) connected, and contains e; therefore the union H is (path) connected (and contains e), hence $H \subset G_e$.)

(e) Let H be the group generated by $\exp \mathcal{A}$. Then H is an open and closed subset of G_e. Conclude that $H = G_e$.

(*Details.* H is an open subgroup of G_e by Part (d). Since the right translations are homeomorphisms of the group G onto itself, the right cosets Hx are open for all $x \in G_e$, and their union is G_e. Therefore the complement H^c of $H = He$ in G_e is the open set $\bigcup_{x \in G_e; x \neq e} Hx$, and we conclude that H is closed. Now the connected set G_e is the union of the disjoint open sets H and H^c, with $H \neq \emptyset$ (since $e \in H$). Therefore $H = G_e$.)

(f) If $\mathcal{A}$ is commutative, then $G_e = \exp \mathcal{A}$.

(Indeed, in that case, $\exp \mathcal{A}$ is a subgroup, because $e^a (e^b)^{-1} = e^{a-b} \in \exp \mathcal{A}$ for all $a, b \in \mathcal{A}$, and therefore $G_e = \exp \mathcal{A}$ by Part (e).)

Non-commutative Taylor theorem

7. (Notation as in Exercise 10, Chapter 7.) Let $\mathcal{A}$ be a (unital, complex) Banach algebra, and let $a, b \in \mathcal{A}$. Prove the following *non-commutative Taylor theorem* for each $f \in H(\sigma(a,b))$:

$$f(b) = \sum_{j=0}^{\infty} (-1)^j \frac{f^{(j)}(a)}{j!} [C(a,b)^j e]$$

$$= \sum_{j=0}^{\infty} [C(b,a)^j e] \frac{f^{(j)}(a)}{j!}. \tag{13}$$

In particular, if a, b commute,

$$f(b) = \sum \frac{f^{(j)}(a)}{j!} (b-a)^j \tag{14}$$

for all $f \in H(\sigma(a,b))$, where (in this special case)

$$\sigma(a,b) = \{\lambda \in \mathbb{C};\ d(\lambda, \sigma(a)) \leq r(b-a)\}.$$

If $b-a$ is quasi-nilpotent, (14) is valid for all $f \in H(\sigma(a))$.

Solution.

Let $f \in H(\sigma(a,b))$. Thus f is analytic in an open set Ω containing the compact set $\sigma(a,b)$. Let $\Gamma \in \Gamma(\sigma(a,b),\Omega)$ (cf. Notation 9.18). Since $\sigma(b) \subset \sigma(a,b)$ (cf. Exercise 10, Chapter 7), $\Gamma \in \Gamma(\sigma(b),\Omega)$, and therefore (cf. definition preceding Theorem 9.19 and notation preceding Theorem 9.20) $f(b) = (1/2\pi i)\int_\Gamma f(\lambda)R(\lambda;b)d\lambda$. We use the series expansions $b_L(\lambda)$ and $b_R(\lambda)$ of $R(\lambda;b)$ as in Exercise 10, Chapter 7, which converge uniformly on the compact subset Γ of $\sigma(a,b)^c$. Integrating term by term, we obtain the formulae (13) by observing that

$$f^{(j)}(a) = j!/(2\pi i)\int_\Gamma f(\lambda)R(\lambda;a)^{j+1}d\lambda \tag{15}$$

for all $j \in \mathbb{N}$, since $\Gamma \in \Gamma(\sigma(a),\Omega)$.

By the binomial formula (since L_a commutes with R_b),

$$C(a,b)^j e = \sum_{k=0}^{j}\binom{j}{k}(-1)^{j-k}L_a^k R_b^{j-k}e = \sum_{k=0}^{j}\binom{j}{k}(-1)^{j-k}a^k b^{j-k}. \tag{16}$$

In particular, when a,b commute, $C(a,b)^j e = (a-b)^j$, and (14) follows from (13). Furthermore, in that case, $r(a,b) = r(a-b)$, and the remaining observations follow trivially.

Positive operators

8. Let X be a Hilbert space. Recall that $T \in B(X)$ is *positive* (in symbols, $T \geq 0$) iff $(Tx,x) \geq 0$ for all $x \in X$. Prove:

(a) The positive operator T is non-singular (i.e., invertible in $B(X)$) iff $T - \epsilon I \geq 0$ for some $\epsilon > 0$ (one writes also $T \geq \epsilon I$ to express the last relation).

Solution.

The operator T is non-singular iff $0 \in \rho(T)$. Since $\rho(T)$ is open, this is equivalent to the existence of $\epsilon > 0$ such that the disc $B(0,\epsilon)$ is contained in $\rho(T)$. Since T is positive, $\sigma(T) \subset [0,\infty)$, and therefore, in that case, T is non-singular iff $\sigma(T) \subset [\epsilon,\infty)$, that is, iff $\sigma(T) - \epsilon \subset [0,\infty)$, i.e., iff $\sigma(T-\epsilon I) \subset [0,\infty)$. Since $T - \epsilon I$ is selfadjoint (because T is selfadjoint!), the latest condition is equivalent to the positivity of $T - \epsilon I$ (cf. Section 7.5).

(b) The (arbitrary) operator T is non-singular iff *both* $TT^* \geq \epsilon I$ and $T^*T \geq \epsilon I$ for some $\epsilon > 0$.

Solution.

For $T \in B(X)$ arbitrary, TT^* and T^*T are positive operators (they are selfadjoint, and for all x $(TT^*x, x) = (T^*x, T^*x) \geq 0$ and $(T^*Tx, x) = (Tx, Tx) \geq 0$). If T is non-singular, so is T^* (because $T^*(T^{-1})^* = (T^{-1}T)^* = I^* = I$, etc.); hence both TT^* and T^*T are positive non-singular operators. Therefore, by Part (a), there exists $\epsilon > 0$ such that $TT^* \geq \epsilon I$ and $T^*T \geq \epsilon I$. Conversely, if the "ϵ-condition" is satisfied, then by Part (a) for the positive operators TT^* and T^*T, both of these operators are non-singular. Therefore

$$T\,[T^*(TT^*)^{-1}] = I \quad \text{and} \quad [(T^*T)^{-1}T^*]\,T = I,$$

that is, T has both right and left inverses in $B(X)$, i.e., T is non-singular.

9. Let X be a Hilbert space, $T \in B(X)$. Prove:

(a) If T is positive, then

$$|(Tx, y)|^2 \leq (Tx, x)\,(Ty, y) \quad \text{for all} \quad x, y \in X. \tag{17}$$

Solution.

The form $[x, y] := (Tx, y)$ on $X \times X$ is a semi-inner product (s.i.p.): indeed, $[x, x] \geq 0$ for all x by the positivity of T; $[\cdot, y]$ is linear for each fixed y by linearity of T; and since T is selfadjoint, $[y, x] = (Ty, x) = (y, Tx) = \overline{[x, y]}$. Therefore (17) is precisely the Cauchy-Schwarz inequality for the s.i.p. $[\cdot,\cdot]$ (cf. (13), page 31).

(b) Let $\{T_k\} \subset B(X)$ be a sequence of positive operators. Then $T_k \to 0$ in the s.o.t. iff it does so in the w.o.t.

Solution. (The non-trivial direction.)

Suppose T_k are positive operators converging to 0 in the w.o.t. Let $T_k^{1/2}$ denote the positive square root of T_k (corresponding to the continuous non-negative function $t^{1/2}$ on $\sigma(T_k) \subset [0, \infty)$ in the operational calculus for the seladjoint operator T_k). Then for all $x \in X$, $||T_k^{1/2}x||^2 = (T_k x, x) \to 0$ as $k \to \infty$, that is, $T_k^{1/2} \to 0$ in the s.o.t. Also $\sup_k ||T_k^{1/2}x|| < \infty$ for all x, and therefore $M := \sup_k ||T_k^{1/2}|| < \infty$ by the Uniform Boundedness theorem (Corollary 6.5). Since multiplication is continuous on the closed ball $\{T \in B(X);\ ||T|| \leq M\}$ in $B(X)$ with respect to the relative s.o.t. (cf. Exercise 17, Chapter 8), it follows that $T_k(= (T_k^{1/2})^2)$ converge to 0 in the s.o.t.

(c) If $0 \le T_k \le T_{k+1} \le KI$ for all k (for some positive constant K), then $\{T_k\}$ converges in $B(X)$ in the s.o.t.

Solution.

For each $x \in X$, the non-negative sequence $(T_k x, x)$ is non-decreasing and bounded above by $K||x||^2$, and is therefore convergent. Hence $0 \le ((T_n - T_m)x, x) \to 0$ for all x when $n > m \to \infty$. Applying (17) to the positive operators $T_n - T_m$ $(n > m)$, we get $((T_n - T_m)x, y) \to 0$ as $n > m \to \infty$, for all $x, y \in X$, that is, the positive operators $T_n - T_m$ $(n > m)$ converge to 0 in the w.o.t., hence in the s.o.t., by Part (b). By Exercise 19, Chapter 6, it follows that $\{T_n\}$ converges in $B(X)$ in the s.o.t.

Analytic functions operate on $\hat{\mathcal{A}}$

10. Let $\mathcal{A}$ be a complex unital commutative Banach algebra, and $a \in \mathcal{A}$. Let $f \in H(\sigma(a))$. Prove that there exists $b \in \mathcal{A}$ such that $\hat{b} = f \circ \hat{a}$. ($\hat{a}$ denotes the Gelfand transform of a.) In particular, if $\hat{a} \neq 0$, there exists $b \in \mathcal{A}$ such that $\hat{b} = 1/\hat{a}$. (This is *Wiener's theorem.*) Hint: Use the analytic operational calculus.

Solution.

Since $f \in H(\sigma(a))$, $b := f(a)$ is well-defined by means of the Riesz-Dunford integral. Let $\Omega \subset \mathbb{C}$ be an open set containing $\sigma(a)$ in which f is analytic, and let $\Gamma \in \Gamma(\sigma(a), \Omega)$ (cf. Notation 9.18). Then

$$b := f(a) := (1/2\pi i) \int_\Gamma f(\lambda) R(\lambda; a) d\lambda.$$

Let $\Phi = \Phi(\mathcal{A})$ be the space of homomorphisms of $\mathcal{A}$ onto $\mathbb{C}$ with the Gelfand topology. For each $\phi \in \Phi$, by linearity and continuity of the homomorphism,

$$\hat{b}(\phi) = \phi(b) = (1/2\pi i) \int_\Gamma f(\lambda) \phi(R(\lambda; a))\, d\lambda.$$

Since ϕ is multiplicative and $\phi(e) = 1$, we have for all $\lambda \in \rho(a)$

$$1 = \phi(e) = \phi(R(\lambda; a)(\lambda e - a)) = \phi(R(\lambda; a))(\lambda - \phi(a)),$$

hence $\phi(R(\lambda; a)) = (\lambda - \phi(a))^{-1}$, and therefore

$$\hat{b}(\phi) = (1/2\pi i) \int_\Gamma \frac{f(\lambda)}{\lambda - \phi(a)} d\lambda.$$

Since $\phi(a) \in \sigma(a)$ and $\Gamma \in \Gamma(\sigma(a), \Omega)$, it follows from Cauchy's integral formula that the right hand side is equal to $f(\phi(a)) = f(\hat{a}(\phi)) = (f \circ \hat{a})(\phi)$, that is, $\hat{b} = f \circ \hat{a}$.

If $\hat{a} \neq 0$, $0 \notin \hat{a}(\Phi) = \sigma(a)$ (cf. Section 7.2(4)), and therefore $f(\lambda) = 1/\lambda$ belongs to $H(\sigma(a))$. By the preceding result, there exists $b \in \mathcal{A}$ such that $\hat{b} = 1/\hat{a}$.

Polar decomposition

11. Let X be a Hilbert space, and let $T \in B(X)$ be non-singular. Prove that there exist a unique pair of operators S, U such that S is non-singular and positive, U is unitary, and $T = US$. If T is normal, the operators S, U commute with each other and with T. Hint: assuming the result, find out how to define S and $U|_{SX}$; verify that U is isometric on SX, etc.

Solution.

We first prove uniqueness. Suppose $T = US$ with U, S as in the statement of the exercise. Then $T^*T = SU^*US = S^2$, and therefore S is the unique positive square root $S = (T^*T)^{1/2}$ (cf. paragraph preceding Theorem 7.23), and U is uniquely determined as $U = TS^{-1}$.

The above argument indicates how to construct the factors S, U of the "polar decomposition" for T. We *define* $S = (T^*T)^{1/2}$. Since $S^2 = T^*T$ is non-singular, $0 \notin \sigma(S^2) = \sigma(S)^2$, and therefore S is a *non-singular* (positive) operator. We then *define* $U = TS^{-1}$. Then U is non-singular and isometric ($||Ux||^2 = (TS^{-1}x, TS^{-1}x) = (T^*TS^{-1}x, S^{-1}x) = (S^2S^{-1}x, S^{-1}x) = ||x||^2$), hence unitary.

If T is normal, $S^2 = T^*T = TT^* = (US)(SU^{-1})$, hence $S^2U = US^2$. It follows that U commutes with the unique positive square root of S^2, which is S (since the square root S is the norm-limit in $B(X)$ of polynomials in S^2, cf. paragraph preceding Theorem 7.23). Consequently $UT = U(US) = U(SU) = (US)U = TU$, and $ST = S(US) = (SU)S = (US)S = TS$. (Conversely, if U and S commute, T is necessarily normal, because $TT^* = (US)(SU^*) = US^2U^* = S^2UU^* = S^2 = T^*T$.)

Cayley transform

12. Let X be a Hilbert space, and let $T \in B(X)$ be selfadjoint. Prove:

(a) The operator $V := (T + iI)(T - iI)^{-1}$ (called the *Cayley transform* of T) is unitary and $1 \notin \sigma(V)$.

Solution.

Since T is selfadjoint, its spectrum is real (cf. Lemma 7.17) and therefore V is a well-defined non-singular operator with the inverse

$$V^{-1} = (T - iI)(T + iI)^{-1} = (T + iI)^{-1}(T - iI)$$

$$= [(T - iI)^{-1}]^*(T + iI)^* = \left[(T + iI)(T - iI)^{-1}\right]^* = V^*,$$

where we used various properties of the adjoint operation and the commutativity of the factors in the definition of V. Furthermore, by the spectral mapping theorem, $\mu \in \sigma(V)$ iff $\mu = \frac{\lambda+i}{\lambda-i}$ with $\lambda \in \sigma(T)$, hence clearly $\mu \neq 1$.

(b) Conversely, every unitary operator V such that $1 \notin \sigma(V)$ is the Cayley transform of some selfadjoint operator T.

Solution.

Since $1 \in \rho(V)$, the operator

$$T = -i(I - V)^{-1}(I + V)$$

is a well-defined element of $B(X)$. We have

$$(I - V)T = -i(I + V). \tag{18}$$

Taking adjoints, we get

$$T^*(I - V^*) = i(I + V^*).$$

Multiplying this equation on the right by V and using the relation $V^*V = I$, we obtain

$$T^*(V - I) = i(V + I),$$

Hence

$$T^* = -i(I + V)(I - V)^{-1} = T,$$

by commutativity of the factors. Thus T is selfadjoint. Rearranging Equation (18), we have $T + iI = V(T - iI)$, and since $T - iI$ is non-singular (for T selfadjoint!), it follows that $V = (T + iI)(T - iI)^{-1}$, as desired.

Riemann integrals of operator functions

13. Let X be a Banach space, and let $T(\cdot) : [a, b] \to B(X)$ be *strongly continuous* (that is, continuous with respect to the s.o.t. on $B(X)$). Prove:

(a) $||T(\cdot)||$ is bounded and lower semi-continuous (cf. Exercise 6, Chapter 3).

(*Details.* For each $x \in X$, the function $||T(\cdot)x||$ is continuous on the closed interval $[a,b]$ (by continuity of $T(\cdot)$ in the s.o.t. and continuity of the norm). Therefore

$$\sup_{t\in[a,b]} ||T(t)x|| < \infty \qquad (x \in X),$$

and consequently

$$\sup_{t\in[a,b]} ||T(t)|| < \infty$$

by the Uniform Boundedness theorem (Corollary 6.5). For each $t \in [a,b]$, $||T(t)|| = \sup_{x\in X;\, ||x||=1} ||T(t)x||$. Since each function $||T(\cdot)x||$ (for unit vectors x in X) is continuous on $[a,b]$, their (pointwise defined) supremum $||T(\cdot)||$ is lower semi-continuous, by Exercise 6(d), Chapter 3.)

(b) For each $x \in X$, the Riemann integral $\int_a^b T(t)x\,dt$ is a well-defined element of X with norm $\leq \int_a^b ||T(t)||\,dt\,||x||$. Therefore the operator $\int_a^b T(t)dt$ *defined by* $(\int_a^b T(t)dt)x = \int_a^b T(t)x\,dt$ has norm $\leq \int_a^b ||T(t)||\,dt$. For each $S \in B(X)$, $ST(\cdot)$ and $T(\cdot)S$ are strongly continuous on $[a,b]$, and $S\int_a^b T(t)dt = \int_a^b ST(t)dt$; $(\int_a^b T(t)dt)S = \int_a^b T(t)Sdt$.

(*Details.* The existence of the Riemann integral of the *continuous* function $T(\cdot)x$ is proved in the same manner as in the scalar case, through Riemann sums, etc. The Riemann sums have norms $\leq \sum_k ||T(\tau_k)x||(t_k - t_{k-1})$ for any partition $a = t_0 < x_1 < \cdots < x_n = b$ and $\tau_k \in [t_{k-1}, t_k]$, $k = 1, \cdots, n$. Since $||T(\cdot)x||$ is continuous, these latest Riemann sums converge to $\int_a^b ||T(t)x||\,dt$ when the "norm" of the partition tends to zero. Hence

$$|| \int_a^b T(t)x\,dt|| \leq \int_a^b ||T(t)x||\,dt \tag{19}$$

for all $x \in X$. The integrand on the right hand side of (19) is $\leq ||T(t)||\,||x||$, which is l.s.c. and bounded (by Part (a)), and is therefore a Borel integrable function. We then conclude from (19) that $|| \int_a^b T(t)\,dt|| \leq \int_{[a,b]} ||T(t)||\,dt$.
If $S \in B(X)$, the strong continuity in $[a,b]$ of $ST(\cdot)$ and $T(\cdot)S$ are obvious, and therefore their integrals over $[a,b]$ make sense as described in the statement of the exercise. For each $x \in X$ and each partition $\{t_k\}$ and τ_k as above, $S\sum_k T(\tau_k)x\,(t_k - t_{k-1}) = \sum_k ST(\tau_k)x\,(t_k - t_{k-1})$, and therefore, letting the norm of the partition tend to zero, we obtain (by continuity of S)

$$S\int_a^b T(t)x\,dt = \int_a^b ST(t)x\,dt, \tag{20}$$

that is, $S\int_a^b T(t)\,dt = \int_a^b ST(t)\,dt$. Multiplication on the right by S is dealt in the same manner. Note that (20) is also valid for $S \in B(X,Y)$, for any Banach space Y. Taking in particular $Y = \mathbb{C}$, we have $x^* \int_a^b T(t)x\,dt = \int_a^b x^*T(t)x\,dt$ for all $x^* \in X^*$.)

(c) $(\int_a^t T(s)ds)'(c) = T(c)$ (derivative in the s.o.t.).

(*Details.* Denote $V(t) = \int_a^t T(s)\,ds$. Then for $t \neq c$ in $[a,b]$

$$||[\frac{V(t)-V(c)}{t-c} - T(c)]x|| = ||\int_c^t [T(s)x - T(c)x]ds/(t-c)||. \tag{21}$$

Let $\epsilon > 0$. By continuity of $T(\cdot)x$ at c, there exists $\delta > 0$ such that $||T(s)x - T(c)x|| < \epsilon$ if $|s-c| < \delta$. Hence for $|t-c| < \delta$, it follows that the left hand side of (21) is $< \epsilon$.)

(d) If $T(\cdot) = V'(\cdot)$ (derivative in the s.o.t.) for some operator function V, then $\int_a^b T(t)dt = V(b) - V(a)$.

(*Details.* For all $x \in X$ and $x^* \in X^*$, $x^*T(\cdot)x$ is the derivative of $x^*V(\cdot)x$, and therefore, by the classical fundamental theorem of calculus (cf. observation following (20))

$$x^* \int_a^b T(s)x\,ds = \int_a^b x^*T(s)x\,ds = x^*V(b)x - x^*V(a)x,$$

and (d) results from Corollary 5.7.)

(e) If $T(\cdot) : [a,\infty) \to B(X)$ is strongly continuous and $\int_a^\infty ||T(t)||\,dt < \infty$, then $\lim_{b\to\infty} \int_a^b T(t)dt := \int_a^\infty T(t)dt$ exists in the norm topology of $B(X)$, and $||\int_a^\infty T(t)dt|| \leq \int_a^\infty ||T(t)||\,dt$. (Note that $||T(\cdot)||$ is l.s.c. by Part (a), and the integral on the right makes sense as the integral of a non-negative Borel function.)

(*Details.* For $c > b \geq a$ we have by Part (b)

$$||\int_a^c T(t)\,dt - \int_a^b T(t)\,dt|| = ||\int_b^c T(t)\,dt|| \leq \int_{(b,c]} ||T(t)||\,dt. \tag{22}$$

By hypothesis (and Part (a)), $||T(\cdot)|| \in L^1([a,\infty))$. Since $||T(\cdot)||I_{(b,c]}$ are dominated by $||T(\cdot)||$ and converge to 0 as $c > b \to \infty$, it follows from the Dominated Convergence theorem that the right hand side of (22) converges to 0 as $c > b \to \infty$. Since $B(X)$ is complete, we conclude from (22) that $\lim_{b\to\infty} \int_a^b T(t)\,dt := \int_a^\infty T(t)\,dt$ exists in the norm topology of $B(X)$. Furthermore, taking $b = a$ in (22) and letting $c \to \infty$, we obtain the inequality $||\int_a^\infty T(t)\,dt|| \leq \int_a^\infty ||T(t)||\,dt$.)

Semigroups of operators

14. Let X be a Banach space, and let $T(\cdot) : [0,\infty) \to B(X)$ be such that $T(t+s) = T(t)T(s)$ for all $t,s \geq 0$ and $T(0) = I$. (Such a function is called a *semigroup of*

operators.) Assume $T(\cdot)$ is (right) continuous at 0 in the s.o.t. (briefly, $T(\cdot)$ is a *C_0-semigroup*). Prove:

(a) $T(\cdot)$ is right continuous on $[0, \infty)$, in the s.o.t.

(*Details.* Fix $t \geq 0$ and $x \in X$. For $h > 0$,

$$||T(t+h)x - T(t)x|| = ||T(h)[T(t)x] - [T(t)x]|| \to 0$$

as $h \to 0$, by the C_0 condition with the fixed vector $T(t)x$. Since $x \in X$ and $t \geq 0$ are arbitrary, this proves the right continuity of $T(\cdot)$ on $[0, \infty)$ in the s.o.t.)

(b) Let $c_n := \sup\{||T(t)||;\ 0 \leq t \leq 1/n\}$. Then there exists n such that $c_n < \infty$. (Fix such an n and let $c := c_n (\geq 1)$.) Hint: the Uniform Boundedness theorem.

(*Details.* If $c_n = \infty$ for all n, there exists $t_n \in [0, 1/n]$ such that $||T(t_n)|| > n$ (for $n = 1, 2, \cdots$). Hence $\sup_n ||T(t_n)|| = \infty$. By the Uniform Boundedness theorem (Corollary 6.5), there exists $x \in X$ such that $\sup_n ||T(t_n)x|| = \infty$. However, since $t_n \to 0$, $||T(t_n)x|| \to ||x||$ by the C_0 condition, and therefore the sequence $\{||T(t_n)x||\}$ is bounded, contradiction! Note that $c \geq ||T(0)|| = 1$.)

(c) With n and c as in Part (b), $||T(t)|| \leq Me^{at}$ on $[0, \infty)$, where $M := c^n (\geq 1)$ and $a := \log M (\geq 0)$.

Solution.

Fix $t > 0$, let $[t]$ be its entire part and $\{t\}$ its fractional part. By the semigroup property,

$$T(t) = T(1/n)^{n[t]} T(\{t\}/n)^n.$$

Since $1/n$ and $\{t\}/n$ belong to $[1, 1/n]$, we have $||T(1/n)|| \leq c$ and $||T(\{t\}/n)|| \leq c$, and therefore (since $c \geq 1$)

$$||T(t)|| \leq (c^n)^{[t]+1} \leq (c^n)^{t+1} = M\, e^{at}$$

for M and a as in the statement of the exercise. Note that $M \geq 1$ (since $c \geq 1$), and therefore the desired inequality is also true for $t = 0$.

(d) $T(\cdot)$ is strongly continuous on $[0, \infty)$.

Solution.

By Part (a), it suffices to prove left continuity on $(0, \infty)$ in the s.o.t. Fix $x \in X$ and $t > 0$. For $0 < h \leq t$, we have by the semigroup property and Part (c)

$$||T(t-h)x - T(t)x|| = ||T(t-h)[x - T(h)x]|| \leq ||T(t-h)||\ ||T(h)x - x||$$

$$\leq M\, e^{a(t-h)}||T(h)x - x|| \to 0$$

as $h \to 0+$, by the C_0 property.

(e) Let $V(t) := \int_0^t T(s)ds$. Then

$$T(h)V(t) = V(t+h) - V(h) \qquad (h, t > 0).$$

Conclude that $(1/h)(T(h)-I)V(t) \to T(t)-I$ in the s.o.t., as $h \to 0+$ (i.e., the strong right derivative of $T(\cdot)V(t)$ at 0 exists and equals $T(t) - I$, for each $t > 0$). Hint: Exercise 13, Part (c).

Solution.

By Exercise 13(b) and the semigroup property

$$T(h)V(t) = \int_0^t T(h)T(s)\,ds = \int_0^t T(s+h)\,ds = \int_h^{t+h} T(s)\,ds = V(t+h) - V(h).$$

Hence

$$(1/h)[T(h) - I]V(t) = (1/h)[V(t+h) - V(h) - V(t)]$$
$$= (1/h)[V(t+h) - V(t)] - (1/h)[V(h) - V(0)].$$

By Exercise 13(c), it follows that (in the s.o.t.)

$$\lim_{h\to 0+} \frac{T(h) - I}{h} V(t) = V'(t) - V'(0) = T(t) - I.$$

(f) Let $\omega := \inf_{t>0} t^{-1} \log ||T(t)||$. Then $\omega = \lim_{t\to\infty} t^{-1} \log ||T(t)||(< \infty)$ (cf. Part (c)). Hint: fix $s > 0$ and $r > s^{-1} \log ||T(s)||$. Given $t > 0$, let $n = [t/s]$. Then $t^{-1} \log ||T(t)|| < rns/t + t^{-1} \sup_{[0,s]} \log ||T(\cdot)||$. ($\omega$ is called the *type* of the semigroup $T(\cdot)$.)

Solution.

Fix $s > 0$ and set $K_s = \sup_{[0,s]} \log ||T(\cdot)||$ (cf. Exercise 13(a)). Let r be as in the hint. Fix $t > 0$ arbitrary, and let $n = [t/s]$. Then $t = sn + s'$, where $s' = s\{t/s\} < s$, and therefore by the semigroup property

$$||T(t)|| = ||T(s)^n T(s')|| \leq ||T(s)||^n ||T(s')||,$$

and consequently

$$(1/t) \log ||T(t)|| \leq (n/t) \log ||T(s)|| + (1/t)K_s \leq nrs/t + K_s/t \leq r + K_s/t.$$

Hence

$$\limsup_{t\to\infty}(1/t)\log||T(t)|| \le r$$

for any $r > (1/s)\log||T(s)||$, and any $s > 0$. It then follows that

$$\limsup_{t\to\infty}(1/t)\log||T(t)|| \le \omega.$$

Since we have trivially $\omega \le \liminf_{t\to\infty}(1/t)\log||T(t)||$, we conclude that the limit $\lim_{t\to\infty}(1/t)\log||T(t)||$ exists and is equal to ω. By Part (c), the limit is $\le a < \infty$.

(g) Let ω be the type of $T(\cdot)$. Then the spectral radius of $T(t)$ is $e^{\omega t}$, for each $t \ge 0$.

Solution.

This is trivial for $t = 0$. Fix then $t > 0$. By the Beurling-Gelfand formula (Theorem 7.9) and Part (f), we have

$$r(T(t)) = \lim_n ||T(t)^n||^{1/n} = \lim_n \exp\{t[(1/nt)\log||T(nt)||]\}$$

$$= \exp\{t\lim_n(1/nt)\log||T(nt)||\} = e^{\omega t}.$$