.

CHAPTER 8

HILBERT SPACES

The trigonometric Hilbert basis of L^2

1. Let m denote the normalized Lebesgue measure on $[-\pi,\pi]$. (The integral of $f \in L^1(m)$ over this interval will be denoted by $\int f\,dm$.) Let $e_k(x) = e^{ikx}$ $(k \in \mathbb{Z})$, and denote

$$s_n = \sum_{|k|\le n} e_k \qquad (n = 0, 1, 2, ...)$$

$$\sigma_n = (1/n)\sum_{j=0}^{n-1} s_j \qquad (n \in \mathbb{N}).$$

Note that $\sigma_1 = s_0 = e_0 = 1$, $\{e_k; k \in \mathbb{Z}\}$ is orthonormal in the Hilbert space $L^2(m)$, and

$$\int s_n dm = (s_n, e_0) = \sum_{|k|\le n} (e_k, e_0) = 1 \tag{1}$$

$$\int \sigma_n dm = (1/n)\sum_{j=0}^{n-1} \int s_j dm = 1. \tag{2}$$

Prove

(a) $s_n(x) = \frac{\cos(nx) - \cos(n+1)x}{1-\cos x} = \frac{\sin(n+1/2)x}{\sin(x/2)}$.

Solution.

Let $t_n := \sum_{k=0}^n e_k$ $(n = 0, 1, \cdots)$. Then

$$s_n = t_n + \overline{t_n} - e_0. \tag{3}$$

Since $e_k = e_1^k$,

$$t_n = \sum_{k=0}^{n} e_1^k = \frac{e_1^{n+1} - 1}{e_1 - 1}. \tag{4}$$

By (3) and (4)

$$s_n = \frac{e_1^{n+1} - 1}{e_1 - 1} + \frac{\overline{e_1^{n+1} - 1}}{\overline{e_1} - 1} - 1$$

$$= \frac{e_1^n - e_1^{n+1} + \overline{e_1^n - e_1^{n+1}} + (1 - e_1) + \overline{1 - e_1}}{|e_1 - 1|^2} - 1$$

$$= \frac{\Re(e_1^n - e_1^{n+1})}{\Re(1 - e_1)} = \frac{\cos(nx) - \cos(n+1)x}{1 - \cos x}.$$

Using the addition identities for the cosine and sine functions and the identities $1 - \cos x = 2\sin^2(x/2)$ and $\sin x = 2\sin(x/2)\cos(x/2)$, we obtain

$$s_n = \frac{\cos(nx)(1 - \cos x) + \sin(nx)\sin x}{2\sin^2(x/2)}$$

$$= \frac{\cos(nx)\sin^2(x/2) + \sin(nx)\sin(x/2)\cos(x/2)}{\sin^2(x/2)}$$

$$= \frac{\cos(nx)\sin(x/2) + \sin(nx)\cos(x/2)}{\sin(x/2)} = \frac{\sin(n + 1/2)x}{\sin(x/2)}.$$

(b) $\sigma_n(x) = (1/n)\frac{1-\cos(nx)}{1-\cos x} = (1/n)\left(\frac{\sin(nx/2)}{\sin(x/2)}\right)^2$.

(This is an immediate consequence of Part (a) and the identity $1 - \cos t = 2\sin^2(t/2)$.)

(c) $\{\sigma_n\}$ is an "approximate identity" in the sense of Exercise 6, Chapter 4. Consequently $\sigma_n * f \to f$ uniformly on $[-\pi, \pi]$ if $f \in C([-\pi, \pi])$ and $f(\pi) = f(-\pi)$, and in L^p-metric if $f \in L^p := L^p(m)$ (for each $p \in [1, \infty)$). (Cf. Exercise 6, Chapter 4.)

(*Details*: $\sigma_n \geq 0$ by Part (b); $\int \sigma_n dm = 1$ by (2); finally, for any $0 < \delta \leq |x| \leq \pi$, $\sin^2(x/2) \geq \sin^2(\delta/2)$, and therefore

$$\sup_{\delta \leq |x| \leq \pi} \sigma_n(x) \leq \sin^{-2}(\delta/2)/n \to 0$$

as $n \to \infty$. This shows that $\{\sigma_n\}$ is an approximate identity in the sense of Exercise 6, Chapter 4, and the stated consequences follow from that exercise.)

(d) Consider the orthogonal projections P and P_n associated with the orthonormal sets $\{e_k; k \in \mathbb{Z}\}$ and $\{e_k; |k| \leq n\}$ respectively, and denote $Q_n = (1/n)\sum_{j=0}^{n-1} P_j$ for $n \in \mathbb{N}$.

Terminology: $Pf := \sum_{k\in\mathbb{Z}}(f, e_k)e_k$ is called the *(formal) Fourier series* of f for *any* integrable f (it converges in L^2 if $f \in L^2$); (f, e_k) is the k-th *Fourier coefficient* of f; $P_n f$ is the n-th *partial sum* of the Fourier series for f; $Q_n f$ is the n-th *Cesaro mean* of the Fourier series of f.

Observe that $P_n f = s_n * f$ and $Q_n f = \sigma_n * f$ for any integrable function f. Consequently $Q_n f \to f$ uniformly in $[-\pi, \pi]$ if $f \in C_T := \{f \in C([-\pi, \pi]); f(\pi) = f(-\pi)\}$, and in L^p-norm if $f \in L^p$ (for each $p \in [1, \infty)$). If $f \in L^\infty := L^\infty(-\pi, \pi)$, $Q_n f \to f$ in the *weak**-topology on L^∞. (Cf. Exercise 6, Chapter 4.)

(*Details*: we have

$$(f, e_k)e_k(x) = \Big(\int f(y)\overline{e_k(y)}\, dm(y)\Big) e_k(x) = \int f(y) e_k(x-y)\, dm(y) = (e_k * f)(x).$$

Hence

$$P_n f := \sum_{|k|\leq n} (f, e_k)e_k = \sum_{|k|\leq n} (e_k * f) = s_n * f \tag{5}$$

and

$$Q_n f := (1/n)\sum_{j=0}^{n-1} P_j f = (1/n)\sum_{j=0}^{n-1} s_j * f = \sigma_n * f. \tag{5'}$$

The remaining statements are a rewording of Part (c) (and Exercise 6(c), Chapter 4), using (5').)

(e) $\{e_k; k \in \mathbb{Z}\}$ is a Hilbert basis for $L^2(m)$. (Note that $Q_n f \in \operatorname{span}\{e_k\}$ and use Part (d).)

(*Details*: by Part (d), $Q_n f \to f$ in $L^2(m)$-norm, for every $f \in L^2(m)$. Hence $L^2(m) = \overline{\operatorname{span}\{e_k\}}$, since $Q_n f$ is clearly in the span of $\{e_k\}$, for each $n \in \mathbb{N}$. By Terminology 8.11, this shows that the orthonormal set $\{e_k\}$ is a Hilbert basis for $L^2(m)$.)

Fourier coefficients

2. (Notation as in Exercise 1.) Given $k \in \mathbb{Z} \to c_k \in \mathbb{C}$, denote $g_n = \sum_{|k|\leq n} c_k e_k$ $(n = 0, 1, 2, \ldots)$ and $G_n = (1/n)\sum_{j=0}^{n-1} g_j$ $(n \in \mathbb{N})$. Note that $(g_n, e_m) = c_m$ for $n \geq |m|$ and $= 0$ for $n < |m|$, and consequently $(G_n, e_k) = (1 - |k|/n)c_k$ for $n > |k|$ and $= 0$ for $n \leq |k|$.

(*Details*: by orthonormality of $\{e_k\}$, $(g_j, e_k) = c_k$ if $j \geq |k|$ and $= 0$ otherwise. If $n > |k|$, there are $n - |k|$ indices j between $|k|$ and $n-1$; hence

$$(G_n, e_k) = (1/n) \sum_{j=0}^{n-1} (g_j, e_k) = (1/n)(n - |k|)c_k.$$

If $n \leq |k|$, $(g_j, e_k) = 0$ for all $j = 0, \cdots, n-1$, hence $(G_n, e_k) = 0$.)

(a) Let $p \in (1, \infty]$. Prove that if

$$M := \sup_n ||G_n||_p < \infty, \tag{6}$$

then there exists $f \in L^p$ such that $c_k = (f, e_k)$ for all $k \in \mathbb{Z}$, and conversely. (Hint: the ball $\overline{B}(0, M)$ in L^p is *weak**-compact, cf. Theorems 5.24 and 4.6. For the converse, see Exercise 1.)

Solution.

Let $q \in [1, \infty)$ be the conjugate exponent of the given $p \in (1, \infty]$; $L^p(m)$ is isometrically isomorphic to $L^q(m)^*$ (cf. Theorem 4.6). Hence, by Alaoglu's theorem (Theorem 5.24), the norm-closed ball $\overline{B}(0, M)$ in $L^p(m)$ is *weak**-compact. If (6) holds, the sequence $\{G_n\}$ is contained in $\overline{B}(0, M)$, and consequently there exists a subsequence $\{G_{n_l}\}$ converging *weak** to some $f \in L^p(m)$. For each $k \in \mathbb{Z}$, we have $e_k \in L^q(m)$; hence $(G_{n_l}, e_k) \to (f, e_k)$ as $l \to \infty$, and therefore, by Part (a),

$$(f, e_k) = \lim_{l \to \infty} (G_{n_l}, e_k) = \lim_l (1 - \frac{k}{n_l})c_k = c_k.$$

Conversely, if $c_k = (f, e_k)$ for some $f \in L^p(m)$ and all $k \in \mathbb{Z}$, then $g_n = \sum_{|k| \leq n} (f, e_k)e_k := P_n f$, hence $G_n = Q_n f$ for all $n \in \mathbb{N}$. Therefore, by Exercise 1 (Parts (d) and (c)) and Exercise 5, Chapter 4,

$$||G_n||_p = ||Q_n f||_p = ||\sigma_n * f||_p \leq ||\sigma_n||_1 ||f||_p = ||f||_p$$

for all $n \in \mathbb{N}$, that is $M \leq ||f||_p$.

(b) If $\{G_n\}$ converges in L^1-norm, then there exists $f \in L^1$ such that $c_k = (f, e_k)$ for all $k \in \mathbb{Z}$, and conversely.

Solution.

Let f be the limit of $\{G_n\}$ in $L^1(m)$-norm. Then for all $k \in \mathbb{Z}$, we have $(G_n, e_k) \to (f, e_k)$ as $n \to \infty$, since strong convergence implies weak convergence; hence

$$c_k = \lim_{n\to\infty}(1 - \frac{|k|}{n})c_k = \lim_n (G_n, e_k) = (f, e_k).$$

Conversely, if there exists $f \in L^1(m)$ such that $c_k = (f, e_k)$ for all $k \in \mathbb{Z}$, then as in the "converse" part in (a), $G_n = Q_n f \to f$ in $L^1(m)$-norm, by Exercise 1(d).

(c) If $\{G_n\}$ converges uniformly in $[-\pi, \pi]$, then there exists $f \in C_T$ such that $c_k = (f, e_k)$ for all $k \in \mathbb{Z}$, and conversely.

Solution.

Clearly $G_n \in C_T$. Let f be its uniform limit on $[-\pi, \pi]$ (according to the hypothesis). Then $f \in C_T$, and for all $k \in \mathbb{Z}$, we have

$$c_k = \lim_{n\to\infty}(1 - \frac{|k|}{n})c_k = \lim_n \int G_n e_{-k} dm = \int f e_{-k} dm = (f, e_k)$$

by uniform convergence.
Conversely, if $c_k = (f, e_k)$ for some $f \in C_T$ (for all $k \in \mathbb{Z}$), then $G_n = Q_n f \to f$ uniformly in $[-\pi, \pi]$, by Exercise 1(d).

(d) If $\sup_n ||G_n||_1 < \infty$, there exists a regular complex Borel measure μ on $[-\pi, \pi]$ with $\mu(\{\pi\}) = \mu(\{-\pi\})$ (briefly, $\mu \in M_T$) such that $c_k = \int e_k d\mu$ for all k, and conversely. (Hint: Consider the measures $d\mu_n = G_n dm$, and apply Theorems 5.24 and 4.9.)

Solution.

Let $M := \sup_n ||G_n||_1$ ($M < \infty$ by hypothesis), and define the measures $d\mu_n = G_n dm$. By Theorem 1.47,

$$||\mu_n|| := |\mu_n|([-\pi, \pi]) = \int |G_n| dm = ||G_n||_1 \le M. \tag{7}$$

By the Riesz Representation theorem (Theorem 4.9) and Alaoglu's theorem (Theorem 5.24), the strongly closed ball $\overline{B}(0, M)$ of $M_r([-\pi, \pi])$ (notation as in Exercise 2, Chapter 4) is *weak**- compact. Therefore, by (7), there exists a subsequence $\{\mu_{n_l}\}$ converging *weak** to some $\mu_0 \in M_r([-\pi, \pi])$. Since $e_k \in C([-\pi, \pi])$, we then have for all $k \in \mathbb{Z}$

$$\int e_{-k} d\mu_0 = \lim_{l\to\infty} \int e_{-k} G_{n_l} dm = \lim_{l\to\infty}(1 - \frac{|k|}{n_l})c_k = c_k.$$

Setting $\mu(E) = \mu_0(-E)$ for all Borel sets $E \subset [-\pi, \pi]$, we get $\int e_k d\mu = \int e_{-k} d\mu_0 = c_k$ for all $k \in \mathbb{Z}$. (μ "inherits" 2π-periodicity from G_n.)
Conversely, if $c_k = \int e_k d\mu$ for some $\mu \in M_T$ and all $k \in \mathbb{Z}$, then

$$g_j(x) := \sum_{|k| \le j} \int e_k(x+y) d\mu(y) = \int s_j(x+y) d\mu(y), \tag{8}$$

and therefore

$$G_n(x) = \int \sigma_n(x+y) d\mu(y). \tag{9}$$

Since σ_n are non-negative, continuous, 2π-periodic, with integral equal to 1, we obtain by an application of Tonelli's theorem

$$||G_n||_1 \le \int \int \sigma_n(x+y) d|\mu|(y)\, dm(x) = \int \int \sigma_n(x+y)\, dm(x) d|\mu|(y) = ||\mu||,$$

and the condition $\sup_n ||G_n|| < \infty$ is satisfied.

(e) If $G_n \ge 0$ for all $n \in \mathbb{N}$, there exists a finite *positive* Borel measure μ as in Part (d), and conversely.

Solution.

If $G_n \ge 0$ for all $n \in \mathbb{N}$, $||G_n||_1 = \int G_n dm = (G_n, e_0) = c_0$, and the condition in Part (d) is trivially satisfied. In the present case, the measure $\mu \in M_T$ obtained in Part (d) is the *weak**-limit of *positive* measures μ_{n_l}, and therefore $\int f\, d\mu \ge 0$ for all $f \in C([-\pi, \pi])^+$. Since μ is regular, the latter space is dense in $L^1(|\mu|)^+$ (cf. Corollary 3.21), and it follows that $\int f\, d\mu$ is non-negative for all $f \in L^1(|\mu|)^+$, hence in particular for indicators I_E of Borel sets $E \subset [-\pi, \pi]$, i.e., $\mu(E) \ge 0$.
The converse follows from (9), since $\sigma_n \ge 0$ for all $n \in \mathbb{N}$.

Poisson integrals

3. (Notation as in Exercise 1.) Let D be the open unit disc in $\mathbb{C}$.

(a) Verify that $\frac{e_1+z}{e_1-z} = 1 + 2\sum_{k\in\mathbb{N}} e_{-k} z^k$ for all $z \in D$, where the series converges absolutely and uniformly in z in any compact subset of D. Conclude that for any complex Borel measure μ on $[-\pi, \pi]$,

$$g(z) := \int \frac{e_1+z}{e_1-z} d\mu = \mu([-\pi, \pi]) + 2\sum_{k\in\mathbb{N}} c_k z^k, \tag{10}$$

where $c_k = \int e_{-k} d\mu$ and integration is over $[-\pi, \pi]$. In particular, g is analytic in D, and if μ is *real*, $\Re g(z) = \int \Re \frac{e_1+z}{e_1-z} d\mu$ is (real) harmonic in D. Verify that the "kernel" in the last integral has the form $P_r(\theta - t)$, where $z = re^{i\theta}$ and

$$P_r(\theta) := \frac{1 - r^2}{1 - 2r\cos\theta + r^2}$$

is the classical *Poisson kernel* (for the disc). Thus

$$(\Re g)(re^{i\theta}) = (P_r * \mu)(\theta) := \int P_r(\theta - t) d\mu(t). \tag{11}$$

(*Details.* As a geometric series $\sum e_{-k} z^k = \sum (e_{-1} z)^k$ converges absolutely and uniformly in z on any compact subset of D (since $|e_{-1} z| = |z|$). Also

$$1 + 2\sum_{k=1}^{\infty} e_{-k} z^k = 1 + \frac{2e_{-1} z}{1 - e_{-1} z} = \frac{e_1 + z}{e_1 - z}.$$

For each fixed $z \in D$, the series of absolute values is a constant series with respect to t on $[-\pi, \pi]$; by the Weierstrass test, the series converges uniformly with respect to t on $[-\pi, \pi]$, hence can be integrated term-by-term on this interval. This yields (10). The formula $\Re \frac{e_1(t)+z}{e_1(t)-z} = P_r(\theta - t)$ (where $z = re^{i\theta}$) is verified by direct calculation.)

(b) Let μ be a complex Borel measure on $[-\pi, \pi]$. Then $P_r * \mu$ is a complex harmonic function in D (as a function of $z = re^{i\theta}$). (This is true in particular for $P_r * f$, for any $f \in L^1(m)$.)

(*Details.* It was observed in Part (a) that g (defined by (10)) is analytic in D, and if μ is real, then $P_r * \mu = \Re g$ is (real) harmonic in D (cf. (11)). In the general case, write $\mu = \lambda + i\nu$ with λ, ν *real* Borel measures. Then $P_r * \mu = P_r * \lambda + iP_r * \nu$ is a complex harmonic function in D as a function of $z = re^{i\theta}$.)

(c) Verify that $\{P_r; 0 < r < 1\}$ is an "approximate identity" for L^1 in the sense of Exercise 6, Chapter 4 (with the continuous parameter r instead of the discrete n). Consequently, as $r \to 1$,

(i) if $f \in L^p$ for some $p < \infty$, then $P_r * f \to f$ in L^p-norm;

(ii) if $f \in C_T$, then $P_r * f \to f$ uniformly in $[-\pi, \pi]$;

(iii) if $f \in L^\infty$, then $P_r * f \to f$ in the *weak**-topology on L^∞;

(iv) if $\mu \in M_T$, then $(P_r * \mu) dm \to d\mu$ in the *weak**-topology.

Solution.

Since $1-2r\cos\theta+r^2 \geq 1-2r+r^2=(1-r)^2>0$, the Poisson kernel P_r is positive and continuous (hence Lebesgue measurable) on $[-\pi,\pi]$. By (11) and (10) with $\mu=m$, the periodicity of P_r, and orthonormality of $\{e_k\}$

$$\int P_r d\theta = P_r * 1 = \Re g = \Re\Big(m([-\pi,\pi]) + 2\sum_{k\in\mathbb{N}}(e_0,e_k)z^k\Big) = 1.$$

Since $\cos\theta \leq \cos\delta$ when $\delta \leq |\theta| \leq \pi$, we have for any $\delta>0$

$$\sup_{\delta\leq\theta\leq\pi} P_r(\theta) \leq \frac{1-r^2}{1-2r\cos\delta+r^2} \to 0$$

as $r\to 1-$. Collecting, this verifies that $\{P_r;\, 0<r<1\}$ is an approximate identity for $L^1(m)$ in the senses of Exercise 6, Chapter 4. Statements (i), (ii), (iii) follow then from (b), (a), (c) in that exercise (respectively).
For any $f:[-\pi,\pi]\to\mathbb{C}$, denote $\tilde f(t)=f(-t)$. Let $f\in C_T$. By (ii) (and commutativity of the convolution), $\tilde f * P_r \to \tilde f$ uniformly on $[-\pi,\pi]$ as $r\to 1-$; hence for any $\mu\in M_T$,

$$[(\tilde f * P_r)*\mu](0) \to [\tilde f * \mu](0) = \int f\, d\mu.$$

By associativity of the convolution, this means that

$$\int f\,(P_r*\mu)\,dm = [\tilde f * (P_r*\mu)](0) \to \int f\, d\mu$$

as $r\to 1-$, for all $f\in C_T$. Equivalently, $(P_r*\mu)\,dm \to d\mu$ in the *weak**-topology of M_T.

(d) For $\mu\in M_T$, denote $F(t)=\mu([-\pi,t))$. If $\mu([-\pi,\pi])=0$, verify the identity

$$(P_r*\mu)(\theta) = r\int K_r(t)\frac{F(\theta+t)-F(\theta-t)}{2\sin t}dt, \tag{12}$$

where

$$K_r(t) = \frac{2(1-r^2)\sin^2 t}{(1-2r\cos t+r^2)^2}. \tag{13}$$

Verify that $\{K_r; 0<r<1\}$ is an approximate identity for $L^1(m)$ in the sense of Exercise 6, Chapter 4. (Hint: integration by parts.)

Solution.

Routine calculation shows that

$$P_r'(t) = -r\frac{K_r(t)}{\sin t}. \tag{14}$$

Therefore the right hand side of (12) is equal to

$$(1/2)\int P_r'(t)[F(\theta - t) - F(\theta + t)]\,dt. \tag{15}$$

We now integrate by parts. Since $P_r(\pi) = P_r(-\pi)$, and $\mu \in M_T$ is extended beyond $[-\pi, \pi]$ in a 2π-periodic manner, the integrated part is equal to

$$P_r(\pi)[F(\theta - \pi) - F(\theta + \pi) - F(\theta + \pi) + F(\theta - \pi)]$$

$$= -2P_r(\pi)\mu([-\pi + \theta, \pi + \theta)) = -2P_r(\pi)\mu([-\pi, \pi]) = 0 \tag{16}$$

by the hypothesis $\mu([-\pi, \pi]) = 0$. The integral part is equal to

$$\int P_r(t)d\mu(\theta - t) + \int P_r(t)d\mu(\theta + t). \tag{17}$$

Since P_r is even, this last expression is equal to $2\int P_r(t)d\mu(\theta - t) = 2(P_r * \mu)(\theta)$, and (12) follows now from (15), (16), and (17).

The functions K_r are non-negative elements of C_T. By (14) and integration by parts

$$\int K_r dm = -(1/r)\int P_r'(t)\sin t\,dm = (1/r)\int P_r(t)\cos t\,dm = (1/r)(P_r * \cos t)(0). \tag{18}$$

Let g correspond to the measure $d\mu = \cos t\,dm$ as in Part (a). Then $\mu([-\pi, \pi]) = \int \cos t\,dm = 0$, and

$$c_k = \int \cos t\, e_{-k} dm = (1/2)(e_1 + e_{-1},\, e_k) = (1/2)\delta_{1,k}$$

for all $k \in \mathbb{N}$. By (10) we then have $g(z) = z$, and therefore $(P_r * \cos t)(\theta) = \Re g(re^{i\theta}) = r\cos\theta$. Hence by (18), $\int K_r dm = 1$. Finally, for any $0 < \delta \leq \pi$,

$$\sup_{\delta \leq t \leq \pi} K_r(t) \leq \frac{2(1 - r^2)}{(1 - 2r\cos\delta + r^2)^2} \to 0$$

as $r \to 1-$.

(e) Let $G_\theta(t)$ denote the function integrated against $K_r(t)$ in (12). If F is differentiable at the point θ, $G_\theta(\cdot)$ is continuous at 0 and $G_\theta(0) = F'(\theta)$. Conclude from Part (d) that $P_r * \mu \to 2\pi F'(= 2\pi D\mu = \frac{d\mu_a}{dm})$ as $r \to 1$ at all points θ where F is differentiable, that is, m-almost everywhere in $[-\pi, \pi]$. (Cf. Theorem 3.28 with $k = 1$ and Exercise 4(e), Chapter 3; note that here m is *normalized* Lebesgue measure on $[-\pi, \pi]$.) This is the "radial limit" version of *Fatou's theorem* on "Poisson integrals". (The same conclusion is true with "non-tangential convergence" of $re^{i\theta}$ to points of the unit circle.)

Solution.

If F is differentiable at the point θ, $\frac{F(\theta+t)-F(\theta-t)}{2t} \to F'(\theta)$ as $t \to 0$. Therefore $G_\theta(t) = \frac{F(\theta+t)-F(\theta-t)}{2t}(t/\sin t) \to F'(\theta)$, and if we extend the definition of G_θ to $t = 0$ by letting $G_\theta(0) := F'(\theta)$ (and 2π-periodically outside $[-\pi, \pi]$), we obtain $G_\theta \in C_T$. Suppose $\mu([-\pi, \pi]) = 0$. By (12) and Exercise 6(a), Chapter 4, since $K_r(-t) = K_r(t)$, we have for all θ as above

$$(P_r * \mu)(\theta) = 2\pi r \int K_r(0-t)G_\theta(t)\, dm \to 2\pi G_\theta(0) = 2\pi F'(\theta) = 2\pi(D\mu)(\theta) \qquad (19)$$

as $r \to 1-$, where $D\mu$ is the derivative of μ with respect to the Lebesgue measure dt (cf. Exercise 4(e), Chapter 3). Furthermore, these points θ are precisely the points where $(D\mu)(\theta)$ exist, that is, m-almost every point in $[-\pi, \pi]$, by Theorem 3.28. Also, by that theorem, $2\pi D\mu = 2\pi \frac{d\mu_a}{dt} = \frac{d\mu_a}{dm}$ (the Radon-Nikodym derivative of the absolutely continuous part μ_a of μ).
The restriction $\mu([-\pi, \pi]) = 0$ can be removed as follows. Denote (for any $\mu \in M_T$) $\mu([-\pi, \pi]) = c$ and $F_\mu(t) = \mu([-\pi, t))$. Since $F_{cm}(t) = (c/2\pi)(t+\pi)$ is differentiable for all t (with derivative $c/2\pi$), and $F_\mu = F_{\mu-cm} + F_{cm}$, it follows that F_μ is differentiable at θ iff $F_{\mu-cm}$ is, and in that case

$$2\pi F'_{\mu-cm}(\theta) = 2\pi F'_\mu(\theta) - c. \qquad (20)$$

The measure $\mu - cm$ satisfies the condition $(\mu - cm)([-\pi, \pi]) = 0$, and therefore, if F_μ is differentiable at θ (i.e., $F_{\mu-cm}$ is differentiable at θ!), then as $r \to 1-$,

$$(P_r * (\mu - cm))(\theta) \to 2\pi F'_{\mu-cm}(\theta). \qquad (21)$$

By Part (c),

$$(P_r * cm)(\theta) = c \int P_r(\theta - t) dm = c \qquad (\theta \in [-\pi, \pi]). \qquad (22)$$

Therefore, by (20)-(22),

$$(P_r * \mu)(\theta) = (P_r * (\mu - cm))(\theta) + (P_r * cm)(\theta) \to 2\pi F'_\mu(\theta).$$

The verification of the remaining assertions does not depend on the extraneous condition $\mu([-\pi,\pi]) = 0$.

(f) State and prove the analog of Exercise 2 for the representation of harmonic functions in D as Poisson integrals.

Solution.

Statement. Let f be a complex harmonic function in D, and let $f_r(\theta) = f(re^{i\theta})$, $0 \le r < 1$, $\theta \in [-\pi,\pi]$.

(i) If $1 < p \le \infty$, then f is the Poisson integral of an L^p- function iff $\sup_r ||f_r||_p < \infty$.

(ii) f is the Poisson integral of an L^1-function iff f_r converge in L^1 as $r \to 1-$.

(iii) f is the Poisson integral of a continuous function iff f_r converge uniformly as $r \to 1-$.

(iv) f is the Poisson integral of a (regular) complex Borel measure iff $\sup_r ||f_r||_1 < \infty$.

(v) f is the Poisson integral of a finite positive (regular) Borel measure iff $f \ge 0$ (Herglotz' theorem).

Proof.

(i) If f is the Poisson integral of $h \in L^p$, then $f_r = P_r * h$, and therefore, by Exercise 5, Chapter 4 (adapted to periodic functions on $[-\pi,\pi]$), $\sup_r ||f_r||_p \le \sup_r ||P_r||_1 ||h||_p = ||h||_p < \infty$. Conversely, suppose $M := \sup_r ||f_r||_p < \infty$. Then the set $\{f_r;\ 0 \le r < 1\}$ is contained in the closed M-ball of L^p, which is *weak**-compact by Theorems 4.6 and 5.24. Let $h \in L^p$ be a *weak**-limit point of f_r as $r \to 1-$, say $f_{r_k} \to h$ *weak** and $r_k \to 1-$ as $k \to \infty$.
If $f = f(re^{i\theta})$ is *real* harmonic in D, it is the real part of an analytic function in D, and has therefore an (absolutely convergent) expansion of the form $\sum_{n\in\mathbb{Z}} c_n r^{|n|} e^{in\theta}$ in D. The same is true for f complex harmonic (consider $\Re f$ and $\Im f$, etc.). Since the series converges uniformly on any circle $\{z;\ |z| = r\}$, it follows that $(f_r, e_n) = c_n r^{|n|}$ for all $n \in \mathbb{Z}$. Since $e_n \in L^q$ (for q the conjugate exponent of p),

$$(h, e_n) = \lim_k (f_{r_k}, e_n) = \lim_k c_n r_k^{|n|} = c_n$$

for all $n \in \mathbb{Z}$. We observe that $r^{|n|}$ is the n-th Fourier coefficient of P_r; since we just found that c_n is the n-th Fourier coefficient of h, it follows (cf. Exercise 7(f), Chapter 2, adapted to the periodic case) that $c_n r^{|n|}$ is the n-th Fourier coefficient of $P_r * h$, for all $n \in \mathbb{Z}$. Since the Fourier coefficients determine an integrable function uniquely (a.e.) on $[-\pi, \pi]$, we conclude that $f_r = P_r * h$ for $r < 1$ and $\theta \in [-\pi, \pi]$ (everywhere, by continuity!).

(ii) If $f_r = P_r * h$ with $h \in L^1$, then $f_r \to h$ in L^1 as $r \to 1-$, by Part (c)(i). Conversely, if $f_r \to h$ in L^1, then for all $n \in \mathbb{Z}$ (by our previous calculation)

$$c_n = \lim_{r \to 1-} c_n r^{|n|} = \lim_r (f_r, e_n) = (h, e_n),$$

and we conclude as before that $f_r = P_r * h$ (because they have the same Fourier coefficients $c_n r^{|n|}$, $n \in \mathbb{Z}$).

(iii) If $f_r = P_r * h$ for some $h \in C_T$, then $f_r \to h$ uniformly on $[-\pi, \pi]$ as $r \to 1-$, by Part (c)(ii). Conversely, if f_r converges uniformly as $r \to 1-$ (to some h), then necessarily $h \in C_T$ and $c_n = (h, e_n)$ for all n (same calculation as above). It then follows as before that $f_r = P_r * h$.

(iv) If $f_r = P_r * \mu$ for some complex Borel measure μ , then by Tonelli's theorem (Theorem 2.18)

$$||f_r||_1 \leq \int_{-\pi}^{\pi} \int_{-\pi}^{\pi} P_r(\theta - t) d|\mu|(t) dm = \int \int P_r(\theta - t) dm \; d|\mu|(t) = ||\mu||.$$

Conversely, assume $M := \sup_r ||f_r||_1 < \infty$. The set of (regular) complex Borel measures $\{f_r dm;\ 0 < r < 1\}$ is contained in the closed M-ball of M_T, which is *weak**-compact, by Theorems 4.9 and 5.24. If $\mu \in M_T$ is the *weak**-limit point of the measures $f_{r_k} dm$ (where $r_k \to 1-$), then (since $e_n \in C_T$ and M_T is identified with C_T^* through Theorem 4.9)

$$\int_{-\pi}^{\pi} e_n d\mu = \lim_k \int e_n f_{r_k} dm = \lim_k c_n r_k^{|n|} = c_n$$

for all $n \in \mathbb{Z}$. Thus $c_n r^{|n|}$ are the Fourier coefficients of $P_r * \mu$ (cf. solution of Part (i)), and we conclude that $f_r = P_r * \mu$.

(v) If $f_r = P_r * \mu$ for some non-negative finite Borel measure μ, then $f_r \geq 0$ trivially, because $P_r \geq 0$. Suppose conversely that f is a non-negative harmonic function in D. Then (cf. proof of Part (i))

$$||f_r||_1 = (f_r, e_0) = c_0$$

for all $0 < r < 1$, that is, f_r satisfies trivially the condition of Part (iv). Hence $f_r = P_r * \mu$ for some $\mu \in M_T$. By Part (c)(iv), it follows that $f_r dm \to d\mu$ *weak** as $r \to 1-$, that is,

$$\int_{-\pi}^{\pi} h d\mu = \lim_{r \to 1-} \int_{-\pi}^{\pi} h f_r dm$$

for all $h \in C_T$. In particular, $\int h d\mu \geq 0$ for $h \geq 0$ (because $f_r \geq 0$), that is, the map $h \to \int h d\mu$ is a positive linear functional on C_T. By the Riesz-Markov theorem (Theorem 3.18, including the uniqueness of the representation), the measure μ is positive (and finite, since it belongs to M_T).

4. *Poisson integrals in the right half-plane.* Let $\mathbb{C}^+$ denote the right half-plane, and

$$P_x(y) := \pi^{-1}\frac{x}{x^2+y^2} \qquad (x>0; y \in \mathbb{R}).$$

(This is the so-called *Poisson kernel of the right half-plane.*) Prove:

(a) $\{P_x; x>0\}$ is an approximate identity for $L^1(\mathbb{R})$ (as $x \to 0+$). (Cf. Exercise 6, Chapter 4.) Consequently, as $x \to 0+$, $P_x * f \to f$ uniformly on $\mathbb{R}$ if $f \in C_c(\mathbb{R})$, and in L^p-norm if $f \in L^p(\mathbb{R})$ $(1 \leq p < \infty)$.

Solution.

The functions P_x $(x>0)$ are positive and continuous on $\mathbb{R}$, and the change of variables $y = tx$ ($x>0$ fixed) shows that $||P_x||_1 = (1/\pi)\int_{\mathbb{R}} dt/(1+t^2) = 1$. For any $\delta > 0$,

$$\int_{|y|\geq\delta} P_x(y)\,dy = (2/\pi)\int_\delta^\infty \frac{x\,dy}{x^2+y^2} = (2/\pi)[\pi/2 - \arctan(\delta/x)] \to 0 \qquad (23)$$

as $x \to 0+$. These properties of $\{P_x; x>0\}$ are the $L^1(\mathbb{R})$ version of the properties of an "approximate identity". (cf. Exercise 6, Chapter 4 for the $L^1([-\pi,\pi])$-version).
If $f \in C_c(\mathbb{R})$, it is uniformly continuous on $\mathbb{R}$. Given $\epsilon > 0$, there exists $\delta > 0$ such that $|f(t-y) - f(t)| < \epsilon/2$ for $|y| < \delta$. We may consider only f for which $||f||_u \neq 0$ (since the following result is trivial for f identically 0). By the properties of the approximate identity, there exists $\eta > 0$ such that $\int_{|y|\geq\delta} P_x(y)\,dy < \epsilon/(4||f||_u)$ for $0 < x < \eta$. Hence for $0 < x < \eta$ and all $t \in \mathbb{R}$,

$$|(P_x * f)(t) - f(t)| = |\int [f(t-y) - f(t)]P_x(y)dy| \qquad (24)$$

$$\leq \int |f(t-y) - f(t)|P_x(y)dy = \int_{|y|<\delta} + \int_{|y|\geq\delta}.$$

The first integral is $< \epsilon/2$ by the choice of δ (since $P_x > 0$ and $\int P_x(y)\,dy = 1$). The second integral is $\leq 2||f||_u \int_{|y|\geq\delta} P_x(y)dy < \epsilon/2$ for $0 < x < \eta$. Hence $||P_x * f - f||_u < \epsilon$ for $0 < x < \eta$, that is, $P_x * f \to f$ uniformly on $\mathbb{R}$.
Moreover, for any $p \in [1,\infty)$ (still for $f \in C_c(\mathbb{R})$), $P_x * f \to f$ in $L^p(\mathbb{R})$-norm. Indeed

by (24) and the integral form of Minkowski's inequality (which is easily verified, using Riemann sums approximations of the integral in (24)),

$$||P_x * f - f||_p \leq \int ||f_y - f||_p P_x(y)\, dy, \tag{25}$$

where $f_y(t) = f(t-y)$. We break the integral as before, with $\delta > 0$ chosen such that $||f_y - f||_p < \epsilon/2$ for $|y| < \delta$ (cf. Exercise 1, Chapter 3). We then choose η such that $\int_{|y|\geq\delta} P_x(y)dy < \epsilon/(4||f||_p)$ for $0 < x < \eta$(cf. (23)). Since $||f_y||_p = ||f||_p$ (by invariance of the Lebesgue measure), we conclude from (25) that $||P_x * f - f||_p < \epsilon$ for $0 < x < \eta$, that is, $P_x * f \to f$ in $L^p(\mathbb{R})$-norm (for $f \in C_c(\mathbb{R})$). An $\epsilon/3$-argument as in Exercise 6, Chapter 4 extends this fact to all $f \in L^p(\mathbb{R})$. Let $f \in L^p(\mathbb{R})$ and $\epsilon > 0$. By density of $C_c(\mathbb{R})$ in $L^p(\mathbb{R})$ (for $1 \leq p < \infty$; cf. Corollary 3.21), there exists $g \in C_c(\mathbb{R})$ such that $||f - g||_p < \epsilon/3$. By the preceding result for $C_c(\mathbb{R})$, $||P_x * g - g||_p < \epsilon/3$ for $0 < x < \eta$, for suitable η. Then

$$||P_x * f - f||_p \leq ||P_x * (f-g)||_p + ||P_x * g - g||_p + ||g - f||_p < \epsilon$$

for $0 < x < \eta$ (the first summand above is $\leq ||P_x||_1 ||f-g||_p = ||f-g||_p < \epsilon/3$, by Exercise 5, Chapter 4).

(b) $(P_x * f)(y)$ is a harmonic function of (x, y) in $\mathbb{C}^+$ (for any $f \in L^p(\mathbb{R})$, $1 \leq p < \infty$).

Solution.

Clearly $P_x(y-t) = \Re g(z - it)$ $(z = x + iy)$, where $g(z) = 1/(\pi z)$. Therefore for any *real* measurable f such that $g(z - i\cdot)f$ is integrable over $\mathbb{R}$,

$$(P_x * f)(y) = \Re \int_{\mathbb{R}} g(z - it) f(t)\, dt \qquad (z = x + iy). \tag{26}$$

Since $|g(z - i\cdot)| = (1/\pi)(x^2 + (y - \cdot)^2)^{-1/2} \in L^q(\mathbb{R})$ for each fixed $z \in \mathbb{C}^+$ and $1 < q \leq \infty$, $fg(z - i\cdot) \in L^1(\mathbb{R})$ for $f \in L^p(\mathbb{R})$ $(1 \leq p < \infty)$, so that (26) is valid in that case (for f *real-valued*). Denote the integral in (26) by $G(z)$. One verifies easily that G is analytic in $\mathbb{C}^+$, and therefore, by (26), $(P_x * f)(x)$ is a real harmonic function for (x, y) in the right half-plane (for each real $f \in L^p(\mathbb{R})$, $1 \leq p < \infty$). If $f \in L^p(\mathbb{R})$ is complex-valued, we apply the preceding conclusion to its real and imaginary parts, and obtain (by linearity of convolution with P_x) that $(P_x * f)(y)$ is a complex harmonic function for $(x, y) \in \mathbb{C}^+$).
(*Detail*: the analyticity of G in $\mathbb{C}^+$. Fix $z = x + iy \in \mathbb{C}^+$. Consider $w = u + iv$ in the $(x/2)$-neighbourhood of z. We have

$$\frac{G(w) - G(z)}{w - z} = -(1/\pi) \int f(t)[(z - it)(w - it)]^{-1} dt. \tag{27}$$

The integrand is dominated by $(2/x)|f(t)||z-it|^{-1} \in L^1(\mathbb{R})$ (for $f \in L^p(\mathbb{R})$, $1 \le p < \infty$), and converges pointwise to $f(t)(z-it)^{-2}$ as $w \to x$. Hence, by the Dominated Convergence theorem (Theorem 1.20), the left-hand side of (27) converges to $(1/\pi)\int f(t)(z-it)^{-2}dt$ as $w \to z$.)

(c) If $f \in L^p(\mathbb{R})$, then for each $\delta > 0$, $(P_x * f)(y) \to 0$ uniformly for $x \ge \delta$ as $x^2+y^2 \to \infty$. (Hint: use Holder's inequality with the probability measure $d\mu = P_x(y-t)dt$ for x, y fixed.)

Solution.

Fix $\delta > 0$. Let q be the conjugate exponent of p, $1 \le p < \infty$, and let μ be the probability measure in the hint. By Holder's inequality with the functions $1 \in L^q(\mu)$ and $f \in L^p(\mu)$ ($||f||_p > 0$ without loss of generality),

$$|(P_x * f)(y)| \le ||f||_{L^p(\mu)} = (P_x * |f|^p)(y)^{1/p}. \tag{28}$$

However $P_x(y-t) \le 1/(\pi x)$. Hence by (28) $|(P_x * f)(y)| \le ||f||_p(\pi x)^{-1/p}$.
Given $\epsilon > 0$, fix $M > \delta$ such that $M > (1/\pi)(||f||_p/\epsilon)^p$. Then for $x > M$, $|(P_x*f)(y)| < \epsilon$ for all $y \in \mathbb{R}$.
For $\delta \le x \le M$,

$$\int P_x(y-t)|f(t)|^p dt \le (M/\pi)\int [\delta^2 + (y-t)^2]^{-1}|f(t)|^p dt. \tag{29}$$

The integrand on the right hand side of (29) is dominated by $\delta^{-2}|f(t)|^p \in L^1(\mathbb{R})$ and converges to 0 pointwise as $y \to \infty$. Hence by the Dominated Convergence theorem, the right hand side of (29) converges to 0 as $y \to \infty$. There exists then $R > 0$ such that the right hand side of (29) is $< \epsilon^p$ for $|y| > R$. If $x^2+y^2 > K := M^2+R^2$, then $|y| > R$ for all $x \le M$, hence by (28) and (29) (and the choice of R), $\sup_{\delta \le x \le M} |(P_x*f)(y)| < \epsilon$. By the choice of M, it follows that $\sup_{\delta \le x} |(P_x * f)(y)| < \epsilon$ for all y, when $x^2+y^2 > K$.

(d) If $f \in L^1(dt/(1+t^2))$, then $P_x * f \to f$ as $x \to 0+$ pointwise a.e. on $\mathbb{R}$. (Hint: imitate the argument in Parts (d)-(e) of the preceding exercise, or transform the disc onto the half-plane and use Fatou's theorem for the disc).

(*Details.* Let $h : w \to z = \frac{w-1}{w+1}$ be the fractional linear map of $\mathbb{C}^+$ onto D; since it maps the boundary $i\mathbb{R}$ onto the unit circle Γ, we write $h(it) = e^{-i\theta}$. Then $\frac{dm}{dt} = [\pi(1+t^2)]^{-1}$. If g is a measurable function on Γ and F is defined on $i\mathbb{R}$ by $F(it) = g(e^{-i\theta}) = g(h(it))$, then $g(e^{-i\theta}) \in L^1(m)$ iff $F(it) \in L^1(dt/(1+t^2))$, and in that case

$$\int_{-\pi}^{\pi} g(e^{-i\theta})\, dm = (1/\pi)\int_{\mathbb{R}} F(it)\, dt/(1+t^2).$$

For $z := h(w) \in D$, writing $w = x + iy$, we calculate

$$P_r(\theta)(= P_r(-\theta)) = \Re\frac{e^{-i\theta}+z}{e^{-i\theta}-z} = \Re\frac{h(it)+h(w)}{h(it)-h(w)}$$

$$= \Re\frac{itw-1}{it-w} = \frac{x}{x^2+(y-t)^2}(1+t^2).$$

Given $f \in L^1(dt/(1+t^2))$ and setting $F(iy) := f(y)$, we have by Exercise 3(e) (with the proper change of variables and the correspondence $iy \to e^{-i\phi}$)

$$(P_x * f)(y) = (P_r * g)(e^{-i\phi}) \to g(e^{-i\phi}) = F(iy) = f(y)$$

for almost all y, as $x \to 0+$.)

5. Let μ be a complex Borel measure on $[-\pi, \pi]$. Show that

$$\lim_{n\to\infty} \int e_{-n} d\mu = 0 \tag{30}$$

if and only if

$$\lim_{n\to\infty} \int e_{-n} d|\mu| = 0. \tag{31}$$

(Hint: (30) for the measure μ implies (30) for the measure $d\nu = h d\mu$ for any trigonometric polynomial h; use a density argument and the relation $d|\mu| = h d\mu$ for an appropriate h.)

Solution.

Assume $||\mu|| \neq 0$ without loss of generality. All integrals in the following are over $[-\pi, \pi]$.
By linearity, if μ satisfies (30), then for any trigonometric polynomial $h = \sum c_j e_j$, the measure $d\nu := h\, d\mu$ satisfies (30) as well ($\int e_{-n} d\nu = \sum_j c_j \int e_{j-n} d\mu \to 0$ as $n \to \infty$).
Let $h \in C_T$ and $\epsilon > 0$ be given. By Exercise 1(d), there exists a trigonometric polynomial h_0 such that $||h - h_0||_u < \epsilon/(2||\mu||)$. Since $\int e_{-n} h_0 d\mu \to 0$ as $n \to \infty$, there exists $n_0 \in \mathbb{N}$ such that $|\int e_{-n} h_0 d\mu| < \epsilon/2$ for all $n > n_0$. Then for all such n,

$$|\int e_{-n} h\, d\mu| \le |\int e_{-n}(h - h_0)\, d\mu| + |\int e_{-n} h_0 d\mu|$$

$$< ||h - h_0||_u ||\mu|| + \epsilon/2 < \epsilon.$$

This shows that (30) is valid with $d\mu$ replaced by $h\, d\mu$ for any $h \in C_T$.
By Theorem 1.46, there exists a complex Borel function k such that $|k| = 1$ (identically)

and $d|\mu| = k\,d\mu$. By Exercise 2, Chapter 3, $|\mu|$ is a *regular* finite positive Borel measure on $[-\pi, \pi]$. By Exercise 9, Chapter 3, there exists a sequence $g_j \in C_T$ such that $g_j \to k$ $|\mu|$-almost everywhere (as $j \to \infty$) and $||g_j||_u \le ||k||_u = 1$. Then for each $n \in \mathbb{N}$

$$\Big|\int e_{-n} d|\mu|\Big| = |\int e_{-n} k\, d\mu| \le |\int e_{-n}(k - g_j)\, d\mu| + |\int e_{-n} g_j d\mu|. \qquad (32)$$

The first summand on the right hand side of (32) is $\le \int |k - g_j| d|\mu|$. Since $|k - g_j| \to 0$ $|\mu|$-a.e. and $|k - g_j| \le 2 \in L^1(|\mu|)$, this integral converges to 0 as $j \to \infty$, by the Dominated Convergence theorem. Given $\epsilon > 0$, fix then j such that $\int |k - g_j| d|\mu| < \epsilon/2$. For this j, the second summand in (32) converges to 0 as $n \to \infty$ (since $g_j \in C_T$). There exists therefore n_0 such that this summand is $< \epsilon/2$ for all $n > n_0$. By (32), we conclude that $\Big|\int e_{-n} d|\mu|\Big| < \epsilon$ for all $n > n_0$, and (31) is satisfied.
One shows that (31) implies (30) in the same manner.

Divergence of Fourier series

6. (Notation as in Exercise 1.) Consider the partial sums $P_n f$ of the Fourier series of $f \in C_T$. Let

$$\phi_n(f) := (P_n f)(0) = (s_n * f)(0) \qquad (f \in C_T). \qquad (32)$$

Prove:

(a) For each $n \in \mathbb{N}$, ϕ_n is a bounded linear functional on C_T with norm $||s_n||_1$ (the $L^1(m)$-norm of s_n). Hint: $||\phi_n|| \le ||s_n||_1$ trivially. Consider real functions $f_j \in C_T$ such that $||f_j||_u \le 1$ and $f_j \to \operatorname{sgn} s_n$ a.e., cf. Exercise 9, Chapter 3. Then $\phi_n(f_j) \to ||s_n||_1$.

Solution.

For all $f \in C_T$, $|\phi_n(f)| = |\int s_n f\, dm| \le ||f||_u ||s_n||_1$, hence $||\phi_n|| \le ||s_n||_1$ for $n = 1, 2, \cdots$.
Since $\operatorname{sgn}(z) : \mathbb{C} \to \mathbb{C}$ is a Borel function, the function $\operatorname{sgn}(s_n)$ (for each fixed n) is Borel on $[-\pi, \pi]$, and has absolute value identically equal to 1. By Exercise 9, Chapter 3, there exist $f_j \in C_T$ such that $f_j \to \operatorname{sgn}(s_n)$ m-a.e. and $||f_j||_u \le ||\operatorname{sgn}(s_n)||_u = 1$. Hence $\lim_j s_n f_j = |s_n|$ a.e., and $|s_n f_j| \le |s_n| \in L^1(m)$. By the Dominated Convergence theorem,

$$\lim_j \phi_n(f_j) = \lim_j \int s_n f_j dm = \int |s_n| dm = ||s_n||_1.$$

Since $||\phi_n|| \ge |\phi_n(f_j)|$ for all j (by definition of the functional's norm), it follows that $||\phi_n|| \ge ||s_n||_1$, and therefore $||\phi_n|| = ||s_n||_1$ for all $n \in \mathbb{N}$.

(b) $\lim_n ||s_n||_1 = \infty$. (Use the fact that the *Dirichlet integral* $\int_0^\infty \frac{\sin t}{t} dt$ does *not* converge *absolutely*.)

Details. Since $|\sin x| \le |x|$ on $[-\pi, \pi]$, we have by Exercise 1(a)

$$||s_n||_1 \ge 2\int_{-\pi}^{\pi} |\frac{\sin(n+1/2)x}{x}|\, dm = (2/\pi)\int_0^{(n+1/2)\pi} |\frac{\sin y}{y}|\, dy \to \infty$$

as $n \to \infty$, because the Dirichlet integral $\int_0^\infty \frac{\sin y}{y} dy$ does not converge absolutely. Hence $||s_n||_1 \to \infty$ as $n \to \infty$.

(c) The subspace

$$Z := \{f \in C_T;\ \sup_n |\phi_n(f)| < \infty\}$$

is of Baire's first category in the Banach space C_T. Conclude that the subspace of C_T consisting of all $f \in C_T$ with convergent Fourier series at 0 is of Baire's first category in C_T. (Hint: assume Z is of Baire's second category in C_T and apply Theorem 6.4 and Parts (a) and (b).)

Solution.

Suppose Z is of Baire's second category in C_T. By Theorem 6.4, it follows that $\sup_n ||\phi_n|| < \infty$, that is, $\sup_n ||s_n||_1 < \infty$ (by Part (a)). This contradicts Part (b). Hence Z is of Baire's first category in C_T.
By Exercise 1(d), $\phi_n(f) := (s_n * f)(0)$ is the n-th partial sum of the Fourier series of f at 0. Hence the set Z_0 of all $f \in C_T$ whose Fourier series converges ot 0 is equal to the set $\{f \in C_T;\ \lim_n \phi_n(f) \text{ exists}\}$. Since $Z_0 \subset Z$, Z_0 is of Baire's first category in C_T.

Fourier coefficients of L^1 functions

7. (Notation as in Exercise 1.)

(a) If $f \in L^1 := L^1(m)$, prove that

$$\lim_{|k|\to\infty} (f, e_k) = 0. \tag{33}$$

Hint: if $f = e_n$ for some $n \in \mathbb{Z}$, $(f, e_k) = 0$ for all k such that $|k| > |n|$, and (33) is trivial. Hence (33) is true for $f \in \text{span}\{e_n; n \in \mathbb{Z}\}$, and the general case follows by density of this span in L^1, cf. Exercise 1, Part (d).

(*Details.* Let $\epsilon > 0$ and $f \in L^1$ be given. By Exercise 1(d), there exists $g \in \text{span}\{e_n\}$ such that $||f-g||_1 < \epsilon/2$. Then $|(f-g,e_k)| \leq \int |f-g| dm < \epsilon/2$. Since (33) is valid for g, there exists $n_0 \in \mathbb{N}$ such that $|(g,e_k)| < \epsilon/2$ for $|k| > n_0$. Hence for $|k| > n_0$,

$$|(f,e_k)| \leq |(f-g,e_k)| + |(g,e_k)| < \epsilon,$$

which shows that (33) is valid for all $f \in L^1$.)

(b) Consider $\mathbb{Z}$ with the discrete topology (this is a locally compact Hausdorff space!), and let $c_0 := C_0(\mathbb{Z})$; cf. Definition 3.23. Consider the map

$$F : f \in L^1 \to \{(f,e_k)\} \in c_0.$$

(Cf. Part (a).) Then $F \in B(L^1, c_0)$ is one-to-one. Hint: if $(f,e_k) = 0$ for all $k \in \mathbb{Z}$, then $(f,g) = 0$ for all $g \in \text{span}\{e_k\}$, hence for all $g \in C_T$ by Exercise 1(d), hence for $g = I_E$ for any measurable subset E of $[-\pi,\pi]$; Cf. Exercise 9, Chapter 3, and Proposition 1.22.

Solution.

Suppose $f \in L^1$ is such that $(f,e_k) = 0$ for all $k \in \mathbb{Z}$. By linearity, $(f,h) = 0$ for all $h \in \text{span}\{e_k\}$. If $g \in C_T$, we have for any $h \in \text{span}\{e_k\}$

$$|(f,g)| = |(f,g-h)| \leq ||f||_1 ||g-h||_u \to 0$$

as $h \to g$ in C_T (cf. Exercise 1(d)). Hence $(f,g) = 0$ for all $g \in C_T$.
Let E be any (Lebesgue) measurable subset of $[-\pi,\pi]$. By Exercise 9, Chapter 3, there exist $g_n \in C_T$ such that $g_n \to I_E$ almost everywhere and $|g_n| \leq 1$. Then $fg_n \to fI_E$ a.e. and $|fg_n| \leq |f| \in L^1$. By the Dominated Convergence theorem, $\int_E f\, dm = \lim_n \int fg_n dm = 0$. By Proposition 1.22, it follows that $f = 0$ a.e.

(c) Prove that the range of F is of Baire's first category in the Banach space c_0 (in particular, F is *not onto*). Hint: if the range of F is of Baire's second category in c_0, F^{-1} (with domain equal to the range of F) is continuous, by Theorem 6.9. Therefore there exists $c > 0$ such that $||Ff||_u \geq c||f||_1$ for all $f \in L^1$. Get a contradiction by choosing $f = s_n$; cf. Exercise 6(b).

Solution.

Suppose the range of F is of Baire's second category in c_0. By the Open Mapping theorem (Theorem 6.9), the continuous linear map F is open, and it is one-to-one by Part (b). It follows that the inverse map F^{-1} : range$(F) \to L^1$ is continuous (and obviously $\neq 0$). Let $c := ||F^{-1}||^{-1}$. Then for all $f \in L^1$

$$||Ff||_u \geq c||F^{-1}(Ff)||_1 = c||f||_1. \tag{34}$$

Take $f = s_n$ in (34). Since $(s_n, e_k) = \sum_{|j|\le n} \delta_{jk} = 0$ for $|k| > n$ and $= 2$ for $|k| \le n$, $||Fs_n||_u = 2$, and (34) implies that $||s_n||_1 \le 2/c$ for all n, contradicting the fact that $||s_n||_1 \to \infty$ (cf. Exercise 6(b)).

Miscellaneous

8. Let $\{a_n\}$ and $\{b_n\}$ be two orthonormal sequences in the Hilbert space X such that

$$\sum_n ||b_n - a_n||^2 < 1. \tag{35}$$

Prove that $\{a_n\}$ is a Hilbert basis for X iff this is true for $\{b_n\}$.

Solution.

By the symmetry of Condition (35) with respect to the given orthonormal sequences, it suffices to prove that if $\{a_n\}$ is a Hilbert basis, the same is true of $\{b_n\}$. By Theorem 8.10(1) (and Terminology 8.11), it suffices to prove that $\{b_n\}^\perp = \{0\}$.
Suppose $0 \ne x \in \{b_n\}^\perp$. Since $\{a_n\}$ is a Hilbert basis, we have by Bessel's identity, Schwarz' inequality, and Condition (35) (cf. Theorem 8.10(7) and Terminology 8.11)

$$||x||^2 = \sum_n |(x, a_n)|^2 = \sum_n |(x, a_n - b_n)|^2$$

$$\le (\sum_n ||a_n - b_n||^2)||x||^2 < ||x||^2,$$

contradiction!

9. Let $\{a_k\}$ be a Hilbert basis for the Hilbert space X. Define $T \in B(X)$ by $Ta_k = a_{k+1}$, $k \in \mathbb{N}$. Prove:

(a) T is isometric.

(b) $T^n \to 0$ in the weak operator topology.

Solution.

(a) For any $x \in X$, we have $x = \sum (x, a_k) a_k$ (cf. Theorem 8.10(5) and Terminology 8.11), where the series converges (unconditionally) in X. Since $T \in B(X)$, it follows that

$$Tx = \sum (x, a_k) T a_k = \sum (x, a_k) a_{k+1}.$$

Hence, by Theorem 8.2(b) and Bessel's identity (cf.Theorem 8.10(7) and Terminology 8.11)

$$||Tx||^2 = \sum |(x, a_k)|^2 = ||x||^2.$$

(b) Since $T^n a_k = a_{k+n}$, the preceding argument shows that

$$T^n x = \sum_k (x, a_k) a_{k+n},$$

and therefore, by continuity of the inner product,

$$(T^n x, y) = \sum_k (x, a_k)(a_{k+n}, y) \qquad (x, y \in X).$$

By Schwarz' inequality in l^2 and Bessel's identity for the Hilbert basis $\{a_k\}$, for all $x, y \in X$,

$$|(T^n x, y)| \leq (\sum_k |(x, a_k)|^2)^{1/2} (\sum_k |(a_{k+n}, y)|^2)^{1/2}$$

$$= ||x|| \, (\sum_{j>n} |(y, a_j)|^2)^{1/2} \to 0$$

as $n \to \infty$, since $\sum_k |(y, a_j)|^2$ $(= ||y||^2)$ converges. This proves that $T^n \to 0$ in the weak operator topology. (However T^n does *not* converge to 0 in the s.o.t., because for any $x \neq 0$, $\lim_n ||T^n x|| = \lim_n ||x|| = ||x|| \neq 0$, by Part (a).)

10. If $\{a_n\}$ is an orthonormal (infinite) sequence in an inner product space, then $a_n \to 0$ weakly (however $\{a_n\}$ has no strongly convergent subsequence, because $||a_n - a_m||^2 = 2$; this shows in particular that the closed unit ball of an infinite dimensional Hilbert space is not stongly compact.)

(*Details.* By Bessel's inequality (Lemma 8.3), $\sum |(a_n, x)|^2 \leq ||x||^2 < \infty$ for all x in the i.p.s. X. In particular, $(a_n, x) \to 0$ for all x, that is, $a_n \to 0$ weakly. Since $a_n \perp (-a_m)$ for $n \neq m$, we have (cf. Lemma 8.1) $||a_n - a_m||^2 = ||a_n||^2 + ||a_m||^2 = 2$.)

11. Let $\{x_\alpha\}$ be a net in the inner product space X. Then $x_\alpha \to x \in X$ strongly iff $x_\alpha \to x$ weakly and $||x_\alpha|| \to ||x||$.

Solution.

The " only if" statement follows from the continuity of the inner product and the norm on X. The "if" statement is a consequence of the identity (7) following Definition 1.34:

$$||x_\alpha - x||^2 = ||x_\alpha||^2 - 2\Re(x_\alpha, x) + ||x||^2 \to ||x||^2 - 2\Re(x, x) + ||x||^2 = 0.$$

12. Let A be an orthonormal basis for the Hilbert space X. Prove:

(a) If $f : A \to X$ is any map such that $(f(a), a) = 0$ for all $a \in A$, then it does not necessarily follow that $f = 0$ (the zero map on A).

(b) If $T : X \to X$ is linear and $(Tx, x) = 0$ for all $x \in X$, then $T = 0$ (the zero operator).

(c) If $S, T : X \to X$ are linear and $(Sx, x) = (Tx, x)$ for all $x \in X$, then $S = T$.

Solution.

(a) Suppose there are at least two distinct elements $a_1, a_2 \in A$. Let f be defined by $f(a_1) = a_2$; $f(a_2) = a_1$; and $f(a) = 0$ for all $a \in A$, $a \neq a_k$, $k = 1, 2$. Then $(f(a_1), a_1) = (a_2, a_1) = 0$; $(f(a_2), a_2) = (a_1, a_2) = 0$; and $(f(a), a) = (0, a) = 0$ for all $a \neq a_k$, $k = 1, 2$. Thus $(f(a), a) = 0$ for all $a \in A$, but f is not the zero map (because $||f(a_1)|| = ||a_2|| = 1$).

(b) For all $x, y \in X$, we have

$$0 = (T(x+y), x+y) = (Tx, x) + (Tx, y) + (Ty, x) + (Ty, y) = (Tx, y) + (Ty, x). \quad (36)$$

Replacing y by iy in (36), we obtain $0 = -i(Tx, y) + i(Ty, x)$, that is $(Ty, x) = (Tx, y)$, and we conclude from (36) that $(Tx, y) = 0$ for all $x, y \in X$. Taking in particular $y = Tx$, we get $||Tx||^2 = 0$ for all x, that is, $T = 0$.

(c) follows by applying Part (b) to the map $S - T$.

13. Let X be a Hilbert space, and let $\mathcal{N}$ be the set of normal operators in $B(X)$. Prove that the adjoint operation $T \to T^*$ is continuous on $\mathcal{N}$ in the s.o.t.

Solution.

Note that $T \in \mathcal{N}$ iff $||T^*x|| = ||Tx||$ for all $x \in X$ (indeed, this identity is equivalent to the identity $(T^*x, T^*x) = (Tx, Tx)$, that is, to $(TT^*x, x) = (T^*Tx, x)$; cf. Exercise 12(c)).
If $S, T \in \mathcal{N}$, then for all $x \in X$,

$$||T^*x - S^*x||^2 = ||T^*x||^2 - 2\Re(T^*x, S^*x) + ||S^*x||^2 = ||Tx||^2 - 2\Re(S(T^*x), x) + ||Sx||^2.$$

When $S \to T$ in the s.o.t., the third summand converges to $||Tx||^2$ and $S(T^*x) \to T(T^*x) = T^*Tx$, so that the second summand above converges to $-2\Re(Tx, Tx) = -2||Tx||^2$, and consequently $||(T^* - S^*)x|| \to 0$ for all x, i.e., $S^* \to T^*$ in the s.o.t.

14. Let X be a Hilbert space, and $T \in B(X)$. Denote by $P(T)$ and $Q(T)$ the orthogonal projections onto the closed subspaces $\ker T$ and $\overline{TX}$, respectively. Prove:

(a) The complementary orthogonal projections of $P(T)$ and $Q(T)$ are $Q(T^*)$ and $P(T^*)$, respectively.

Solution.

By continuity of the inner product, $x \in (\overline{TX})^{\perp}$ iff $(x, Ty) = 0$ for all $y \in X$. This is in turn equivalent to the identity $(T^*x, y) = 0$ for all y, that is, to $T^*x = 0$. Hence

$$(\overline{TX})^{\perp} = \ker T^*. \tag{37}$$

By the Orthogonal Decomposition theorem (Theorem 1.36) and (37),

$$X = \overline{TX} \oplus \ker T^*. \tag{38}$$

By (38), the complementary orthogonal projection of $Q(T)$ is $P(T^*)$. Applying this to T^* instead of T, we conclude that the complementary orthogonal projection of $P(T)$ $(= P(T^{**})$ by Exercise 13(a), Chapter 6) is $Q(T^*)$.

(b) $P(T^*T) = P(T)$ and $Q(T^*T) = Q(T^*)$.

Solution.

If $x \in \ker(T^*T)$ then $||Tx||^2 = (Tx, Tx) = (T^*Tx, x) = 0$, i.e., $x \in \ker T$, and trivially $\ker T \subset \ker(T^*T)$. Hence $\ker(T^*T) = \ker T$, that is, $P(T^*T) = P(T)$. Then, by Part (a), $Q(T^*T)$ has the complementary orthogonal projection $P((T^*T)^*) = P(T^*T) = P(T)$ (cf. Exercise 13(a,b), Chapter 6), and the latter is also the complementary orthogonal projection of $Q(T^*)$. Hence $Q(T^*T) = Q(T^*)$.

15. For any (non-empty) set A, denote by $\mathbb{B}(A)$ the B^*- algebra of all bounded complex functions on A with pointwise operations, the involution $f \to \overline{f}$ (complex conjugation), and the supremum norm $||f||_u = \sup_A |f|$.

Let A be an orthonormal basis of the Hilbert space X. For each $f \in \mathbb{B}(A)$ and $x \in X$, let

$$T_f x = \sum_{a \in A} f(a)(x, a)a. \tag{39}$$

Prove:

(a) The map $f \to T_f$ is an isometric *-isomorphism of the B^*- algebra $\mathbb{B}(A)$ into $B(X)$. (In particular, T_f is a normal operator.)

(b) T_f is selfadjoint (positive, unitary) iff f is real-valued ($f \geq 0,\ |f| = 1$, respectively).

Solution.

(a) The series in (39) converges (unconditionally) in X for each $x \in X$ and $f \in \mathbb{B}(A)$, because (cf. Theorems 8.2(a) and 8.10 (7))

$$\sum_{a\in A} |f(a)(x,a)|^2 \leq ||f||_u^2 \sum_{a\in A} |(x,a)|^2 = ||f||_u^2 ||x||^2 < \infty. \tag{40}$$

Thus $T_f x$ is well-defined for each $x \in X$ and $f \in \mathbb{B}(A)$. The linearity of T_f on X is now trivial, and by (40) and Theorem 8.2(b), $||T_f|| \leq ||f||_u$. Hence $T_f \in B(X)$. Since $T_f a = f(a)a$ for each $a \in A$, we have $||T_f|| \geq ||T_f a|| = |f(a)|$ for all $a \in A$, i.e., $||T_f|| \geq ||f||_u$, and we conclude that $||T_f|| = ||f||_u$ for each $f \in \mathbb{B}(A)$. The map $f \to T_f$ is trivially linear; it is therefore a linear isometry of $\mathbb{B}(A)$ into $B(X)$. For each $a \in A$ and $f, g \in \mathbb{B}(A)$, $T_{fg}a = (fg)(a)a = f(a)g(a)a$ and $T_f T_g a = T_f g(a)a = g(a)T_f a = g(a)f(a)a$. Thus $T_{fg} = T_f T_g$ on A, and therefore, by linearity and continuity of the operators and the density of span$A$ in X (cf. Theorem 8.10(3) and Terminology 8.11), the last identity is valid on X.
Let $f \in \mathbb{B}(A)$ and fix $a \in A$. For all $b \in A$,

$$(b, T_f^* a) = (T_f b, a) = f(b)(b,a) = f(b)\delta_{a,b} = f(a)\delta_{a,b}$$

$$= f(a)(b,a) = (b, \overline{f(a)}a) = (b, T_{\overline{f}}a),$$

that is, $(T_f^* - T_{\overline{f}})a \in A^\perp$, and therefore $(T_f^* - T_{\overline{f}})a = 0$ (cf. Theorem 8.10(1) and Terminology 8.11). By linearity and continuity of the operators, and the density of span$A$ in X (cf. Theorem 8.10(3)), we conclude that $T_f^* = T_{\overline{f}}$. Collecting, we proved that the map $f \to T_f$ is an isometric $*$-isomorphism of the B^*-algebra $\mathbb{B}(A)$ into $B(X)$.

(b) By Part (a), T_f is selfadjoint iff $T_f = T_{\overline{f}}$, and since the map $f \to T_f$ is one-to-one, this is equivalent to $f = \overline{f}$.
T_f is positive iff $(T_f x, x) \geq 0$ for all $x \in X$ (cf. Theorem 7.23(iii)). In particular, if T_f is positive, then for all $a \in A$, $f(a) = (f(a)a, a) = (T_f a, a) \geq 0$. Conversely, if $f \geq 0$, then for all $x \in X$,

$$(T_f x, x) = (\sum_a f(a)(x,a)a, x) = \sum_a f(a)(x,a)(a,x) = \sum_a f(a)|(x,a)|^2 \geq 0.$$

T_f is unitary iff $T_f T_{\overline{f}} = I$, that is, iff $T_{|f|^2} = I$. This is equivalent to $T_{|f|^2}a = a$ for all $a \in A$ (by linearity and continuity of the operators, and density of span$A$ in X), i.e., to $|f(a)| = 1$ for all $a \in A$.

16. Let $(S, \mathcal{A}, \mu)$ be a σ-finite positive measure space. Consider $L^\infty(\mu)$ as a B^*-algebra with pointwise multiplication and complex conjugation as involution. Let $p \in [1, \infty)$. For each $f \in L^\infty(\mu)$ define

$$T_f g = fg \qquad (g \in L^p(\mu)).$$

Prove:

(a) The map $f \to T_f$ is an isometric isomorphism of $L^\infty(\mu)$ into $B(L^p(\mu))$; in case $p = 2$, the map is an isometric *-isomorphism of $L^\infty(\mu)$ onto a commutative B^*-algebra of (normal) operators on $L^2(\mu)$.

Solution.

Let $p \in [1, \infty)$ and $f \in L^\infty := L^\infty(\mu)$. For all $g \in L^p := L^p(\mu)$, $|T_f g| = |f|\,|g| \leq ||f||_\infty |g|$ a.e., hence

$$||T_f g||_p \leq ||f||_\infty ||g||_p,$$

that is, $||T_f|| \leq ||f||_\infty$. Suppose $||T_f|| < ||f||_\infty$. Pick c such that $||T_f|| < c < ||f||_\infty$. Then $\mu([|f| > c]) > 0$. Since the measure space is σ-finite, there exist $X_k \in \mathcal{A}$ such that $\mu(X_k) < \infty$ and $X = \bigcup_k X_k$. Let $E_k = [|f| > c] \cap X_k$, $k = 1, 2, \cdots$. Since $\sum_k \mu(E_k) = \mu([|f| > c]) > 0$, there exists k (which we fix from now on) such that $\mu(E_k) > 0$ (and of course $\mu(E_k) < \infty$). Let $g = I_{E_k}$. Then $g \in L^p$ (with $||g||_p = \mu(E_k)^{1/p}$) and

$$||T_f|| \mu(E_k)^{1/p} = ||T_f||\,||g||_p \geq ||T_f g||_p$$

$$= \Big(\int_{E_k} |f|^p d\mu \Big)^{1/p} \geq c\mu(E_k)^{1/p}.$$

Since $\mu(E_k) > 0$, we get the contradiction $||T_f|| \geq c$. Hence $||T_f|| = ||f||_\infty$.
The map $f \in L^\infty \to T_f \in B(L^p)$ is clearly linear and multiplicative. Since it is norm-preserving, we conclude that it is an isometric isomorphism of L^∞ into $B(L^p)$.
In case $p = 2$, we have for all $g, h \in L^2$

$$(T_f g, h) = (fg, h) = \int (fg)\overline{h}\, d\mu = \int g\overline{\overline{f}h} d\mu$$

$$= (g, \overline{f}h) = (g, T_{\overline{f}} h).$$

Hence $(T_f)^* = T_{\overline{f}}$. We conclude that the map $f \to T_f$ is an isometric *-isomorphism of the B^*-algebra L^∞ into $B(L^2)$; its range is necessarily a commutative B^*-subalgebra of $B(L^2)$ (and consists necessarily of normal operators).

(b) (Case $p = 2$.) T_f is selfadjoint (positive, unitary) iff f is real-valued ($f \geq 0,\ |f| = 1$, respectively) almost everywhere.

Solution.

Since the map $f \to T_f$ is a *-isomorphism of L^∞ (with complex conjugation as its involution) into $B(L^2)$, T_f is selfadjoint iff $f = \overline{f}$ (as elements of L^∞, that is, iff f is real almost everywhere. Similarly, the normal operator T_f is unitary (i.e., $T_f T_f^* = I$) iff $|f|^2 = 1$ a.e.
By Theorem 7.23(iii), T_f is positive iff $(T_f g, g) \geq 0$ for all $g \in L^2$, that is, iff $\int f|g|^2 d\mu \geq 0$ for all $g \in L^2$. By the averages lemma (Lemma 1.38), this is equivalent to $f|g|^2 \geq 0$ a.e. for all $g \in L^2$. With notation as in Part (a), take $g = I_{X_k}$ $(k = 1, 2, \cdots)$ to conclude that the last inequality is equivalent to $f \geq 0$ a.e.

17. Let X be a Hilbert space. Show that multiplication in $B(X)$ is *not* (jointly) continuous in the w.o.t., even on the norm-closed unit ball $B(X)_1$ of $B(X)$ in the relative w.o.t.; however it is continuous on $B(X)_1$ in the relative s.o.t.

Solution.

Let X, $\{a_k\}$, and T be as in Exercise 9. Then, for any $x \in X$, $T^n x = \sum_k (x, a_k) a_{k+n}$. By Exercise 9(b), $T^n \to 0$ in the w.o.t. Set $a_k = 0$ for $k \leq 0$, and define $S \in B(X)$ by $S a_k = a_{k-1}$ for $k \in \mathbb{N}$. Then for all $x \in X$ and $n \in \mathbb{N}$,

$$S^n x = \sum_{k>n} (x, a_k) a_{k-n}. \tag{41}$$

Hence for all x (cf. Theorem 8.2(b))

$$||S^n x||^2 = \sum_{k>n} |(x, a_k)|^2 \to 0 \tag{42}$$

as $n \to \infty$, since $\sum_k |(x, a_k)|^2$ converges (to $||x||^2$, by Bessel's identity). Thus $S^n \to 0$ in the s.o.t. (hence in the w.o.t.). However $S^n T^n a_k = a_k$ for all $k \in \mathbb{N}$, hence $S^n T^n = I$ do not converge to 0 in the w.o.t. This example shows that multiplication in $B(X)$ is *not* continuous in general with respect to the weak operator topology. This is so even on $B(X)_1$, in the relative w.o.t. Indeed, since T is an isometry by Exercise 9(a), $||T^n|| = 1$ for all $n \in \mathbb{N}$. Also by (42) and Bessel's inequality, $||S^n x|| \leq ||x||$ for all x, hence $||S^n|| \leq 1$. Thus the sequences $\{S^n\}$ and $\{T^n\}$ are in $B(X)_1$, both converge to 0 in the w.o.t., but $\{S^n T^n\}$ does not.
However, with respect to the relative s.o.t., multiplication is continuous on $B(X)_1$ for any Banach space X. Indeed, if $\{T_\alpha\}$ and $\{S_\alpha\}$ are nets in $B(X)_1$ converging to T and S respectively in the s.o.t., then for all $x \in X$

$$||S_\alpha T_\alpha - ST|| \leq ||S_\alpha||\, ||(T_\alpha - T)x|| + ||(S_\alpha - S)(Tx)|| \to 0,$$

since $||S_\alpha|| \le 1$.

18. (Notation as in Exercise 17.) Prove that $B(X)_1$ is compact in the w.o.t. but *not* in the s.o.t.

Solution.

The compactness of $B(X)_1$ in the w.o.t. is proved in the same manner as Alaoglu's theorem (Theorem 5.24). For each $[x,y] \in X \times X$, let

$$\Delta[x,y] = \{\lambda \in \mathbb{C};\ |\lambda| \le ||x||\,||y||\},$$

and

$$\Delta = \prod_{[x,y]\in X\times X} \Delta[x,y]$$

with the Cartesian product topology. By Tychonoff's theorem, Δ is compact.
If $T \in B(X)_1$, then for all $x, y \in X$, $|(Tx,y)| \le ||x||\,||y||$, i.e., $(Tx,y) \in \Delta(x,y)$. Thus the map

$$\beta(T) : [x,y] \in X \times X \to (Tx,y)$$

belongs to Δ. This shows that $\beta(B(X)_1) \subset \Delta$. Let $\{T_\alpha\}$ be a net in $B(X)_1$ such that $\beta(T_\alpha)$ converge in the Δ-topology to some $F \in \Delta$. This means that $(T_\alpha x, y) \to F(x,y)$ for all $[x,y] \in X \times X$. One verifies in a routine way that F is linear in the first variable, conjugate- linear in the second, and $|F(x,y)| \le ||x||\,||y||$ for all $x, y \in X$. Therefore, for each fixed $x \in X$, $F(x,\cdot)$ is a continuous conjugate- linear functional on X. By the "little" Riesz representation theorem (Theorem 1.37 in a conjugate-linear version), there exists a unique vector $Tx \in X$ such that $F(x,y) = (Tx,y)$ for all $x, y \in X$. The linearity of $F(\cdot,y)$ for all y implies the linearity of T, and $||T|| = \sup_{||x||=||y||=1} |F(x,y)| \le 1$, that is, $T \in B(X)_1$, and by definition, $\beta(T) = F$. This shows that $\beta(B(X)_1)$ is a *closed* subset of Δ, and is therefore compact in the Δ-topology. Equivalently, $B(X)_1$ is compact in the w.o.t.

The following example shows that in general $B(X)_1$ is not compact in the s.o.t. Let T be as in Exercise 9. By Exercise 9(a), T is isometric, and therefore $||T^n|| = 1$ for all n, i.e., $\{T^n\} \subset B(X)_1$. Since $T^n a_1 = a_{1+n}$, we have for all $n > m$

$$||T^n a_1 - T^m a_1||^2 = ||a_{1+n} - a_{1+m}||^2 = ||a_{1+n}||^2 + ||a_{1+m}||^2 = 2,$$

and therefore $\{T^n\}$ has no subsequence converging in the s.o.t.

19. Let X, Y be Hilbert spaces, and $T \in B(X,Y)$. Imitate the definition of the Hilbert adjoint of an operator in $B(X)$ to define the (Hilbert) adjoint $T^* \in B(Y,X)$. Observe that T^*T is a positive operator in $B(X)$. Let $\{a_n\}_{n\in\mathbb{N}}$ be an orthonormal basis for X

and let $Vx = \{(x, a_n)\}$. We know that V is a Hilbert space isomorphism of X onto l^2. What are V^*, V^{-1}, and V^*V?

Solution.

Given $y \in Y$, the map $x \in X \to (Tx, y)$ is a continuous linear functional on X. By the "little" Riesz representation theorem (Theorem 1.37), there exists a unique vector in X, which we denote by T^*y, such that $(Tx, y) = (x, T^*y)$ for all $x \in X$ (and $y \in Y$). The conjugate linearity of the left hand side of this identity with respect to $y \in Y$ and the uniqueness of the Riesz representation imply that T^* is linear on Y, and

$$||T^*|| = \sup_{||x||=||y||=1} |(x, T^*y)| = \sup_{\cdots} |(Tx, y)| = ||T||.$$

Thus $T^* \in B(Y, X)$. It follows that $T^*T \in B(X)$ and $(x, T^*Tx) = (Tx, Tx) \geq 0$ for all x, that is, T^*T is a positive operator (cf. Theorem 7.23).

Let $\{a_n\}$ be a Hilbert basis for the Hilbert space X. Consider the Hilbert space isomorphism $V \in B(X, l^2)$ defined by $Vx = \{(x, a_n)\}$ (cf. Theorem 8.10(7)). If Λ_n denotes the sequence whose n-th component equals 1 and all other components vanish, then clearly $Va_n = \Lambda_n$.
For all $x \in X$ and $\Lambda := \{\lambda_n\} \in l^2$,

$$(x, V^*\Lambda) = (Vx, \Lambda)_{l^2} = (\{(x, a_n)\}, \{\lambda_n\})_{l^2}$$

$$= \sum (x, a_n)\overline{\lambda_n} = (x, \sum \lambda_n a_n),$$

where the last sum converges unconditionally in X (cf. Theorem 8.2(b)). Hence

$$V^*\Lambda = \sum \lambda_n a_n \qquad (\Lambda \in l^2). \tag{43}$$

By (43) and Theorem 8.10(5), we have for all $x \in X$

$$V^*Vx = V^*\{(x, a_n)\} = \sum (x, a_n)a_n = x. \tag{44}$$

Also for all $\Lambda \in l^2$,

$$VV^*\Lambda = V(\sum \lambda_n a_n) = \sum \lambda_n V a_n = \sum \lambda_n \Lambda_n = \Lambda. \tag{45}$$

Thus

$$V^*V = I_X \quad \text{and } V^*V = I_{l^2}, \tag{46}$$

the respective identity operators on X and l^2. Thus $V^{-1} = V^*$.

20. Let $\{a_n\}_{n\in\mathbb{N}}$ be an orthonormal basis for the Hilbert space X, and let $Q \in B(X)$ be invertible in $B(X)$. Let $b_n = Qa_n$, $n \in \mathbb{N}$. Prove that there exist positive constants A, B such that

$$A\sum|\lambda_k|^2 \le ||\sum\lambda_k b_k||^2 \le B\sum|\lambda_k|^2 \tag{47}$$

for all finite sets of scalars λ_k.

Solution.

We have

$$||\sum\lambda_k b_k||^2 = ||Q(\sum\lambda_k a_k)||^2 \le ||Q||^2\sum|\lambda_k|^2$$

and

$$\sum|\lambda_k|^2 = ||\sum\lambda_k a_k||^2 = ||\sum\lambda_k Q^{-1}b_k||^2 = ||Q^{-1}\sum\lambda_k b_k||^2 \le ||Q^{-1}||^2||\sum\lambda_k b_k||^2.$$

Hence (47) is valid with $B = ||Q||^2$ and $A = ||Q^{-1}||^{-2}$.

21. Let X be a Hilbert space. A sequence $\{a_n\} \subset X$ is *upper (lower) Bessel* if there exists a positive constant B (A) such that

$$\sum|(x, a_n)|^2 \le B\,||x||^2 \qquad (\ge A\,||x||^2, \text{ respectively}) \tag{48}$$

for all $x \in X$. The sequence is *two-sided Bessel* if it is both upper and lower Bessel (for example, an orthonormal sequence is upper Bessel with $B = 1$, and is two-sided Bessel with $A = B = 1$ iff it is an orthonormal basis for X).

(a) Let $\{a_n\}$ be an upper Bessel sequence in X, and define V as in Exercise 19. Then $V \in B(X, l^2)$ and $||V|| \le B^{1/2}$. On the other hand, for any $\{\lambda_n\} \in l^2$, the series $\sum\lambda_n a_n$ converges in X and its sum equals $V^*\{\lambda_n\}$. The operator $S := V^*V \in B(X)$ is a positive operator with norm $\le B$.

Solution.

For all $x \in X$, we have by (48)

$$||Vx||_{l^2}^2 = \sum|(x, a_n)|^2 \le B\,||x||^2,$$

hence $||V|| \le B^{1/2}$.
Given $\Lambda := \{\lambda_n\} \in l^2$, we have for all $x \in X$ and all integers $p \ge m \ge 1$,

$$|(\sum_{n=m}^{p}\lambda_n a_n, x)| = |\sum_{n=m}^{p}\lambda_n(a_n, x)| \le (\sum_{n=m}^{p}|\lambda_n|^2)^{1/2}(\sum_{n=m}^{p}|(x, a_n)|^2)^{1/2}$$

$$\leq (\sum_{n=m}^{p} |\lambda_n|^2)^{1/2} B^{1/2} ||x||, \tag{49}$$

where we used Schwarz' inequality for finite dimensional Hilbert space and (48). Taking the supremum over all x with $||x|| = 1$, we get

$$|| \sum_{n=m}^{p} \lambda_n a_n || \leq B^{1/2} (\sum_{n=m}^{p} |\lambda_n|^2)^{1/2}. \tag{50}$$

Since $\Lambda \in l^2$, it follows from (50) that the series $\sum \lambda_n a_n$ converges in X. Denote its sum by $T\Lambda$. This defines an obviously linear map $T : l^2 \to X$. Taking $m = 1$ and letting $p \to \infty$ in (50), we get

$$||T\Lambda|| \leq B^{1/2} ||\Lambda||_{l^2}. \tag{51}$$

Hence $T \in B(l^2, X)$ and $||T|| \leq B^{1/2}$. Furthermore, for all $x \in X$ and $\Lambda \in l^2$,

$$(x, T\Lambda) = (x, \sum \lambda_n a_n) = \sum (x, a_n) \overline{\lambda_n} = (Vx, \Lambda)_{l^2} = (x, V^*\Lambda),$$

by definition of the adjoint V^* (cf. Exercise 19). Hence $V^* = T$. The operator $S := V^*V \in B(X)$ (with norm $\leq ||V^*|| \, ||V|| \leq B$, by the preceding estimates) is positive, since for all $x \in X$, $(Sx, x) = (Vx, Vx)_{l^2} \geq 0$ (cf. Theorem 7.23(iii)).

(b) If $\{a_n\}$ is two-sided Bessel, $S - AI$ is positive, and therefore $\sigma(S) \subset [A, B]$. In particular, S is *onto*. Conclude that every $x \in X$ can be represented as $x = \sum (x, S^{-1} a_n) a_n$ (convergent in X).

Solution.

Dy definition,

$$Sx = \sum (x, a_n) a_n \qquad (x \in X), \tag{52}$$

where the series converges in X. Therefore, if $\{a_n\}$ is two-sided Bessel, then for all $x \in X$

$$((S - AI)x, x) = \sum |(x, a_n)|^2 - A \, ||x||^2 \geq 0,$$

that is, $S - AI$ is positive. Equivalently, $\sigma(S) - A \subset [0, \infty)$, i.e., $\sigma(S) \subset [A, \infty)$. But $\sigma(S) \subset [-B, B]$ since $||S|| \leq B$. Therefore $\sigma(S) \subset [A, B]$. In particular $0 \in \rho(S)$. Hence, by (52), we have the following series representation (converging in X)

$$x = S(S^{-1}x) = \sum (S^{-1}x, a_n) a_n = \sum (x, S^{-1} a_n) a_n, \qquad (x \in X)$$

since $S := V^*V$ (and therefore S^{-1}) is selfadjoint.