.

CHAPTER 7

BANACH ALGEBRAS

1. A general Banach algebra $\mathcal{A}$ is not required to possess an identity. Consider then the cartesian product Banach space $\mathcal{A}_e := \mathcal{A} \times \mathbb{C}$ with the norm $|||[x, \lambda]||| = ||x|| + |\lambda|$ and the multiplication

$$[x, \lambda]\,[y, \mu] = [xy + \lambda y + \mu x, \lambda\mu].$$

Prove that $\mathcal{A}_e$ is a unital Banach algebra with the identity $e := [0, 1]$, commutative if $\mathcal{A}$ is commutative, and the map $x \in \mathcal{A} \to [x, 0] \in \mathcal{A}_e$ is an isometric isomorphism of $\mathcal{A}$ onto a maximal two-sided ideal (identified with $\mathcal{A}$) in $\mathcal{A}_e$. (With this identification, we have $\mathcal{A}_e = \mathcal{A} + \mathbb{C}e$.)

If ϕ is a homomorphism of the commutative Banach algebra $\mathcal{A}$ into $\mathbb{C}$, it extends uniquely to a homomorphism (also denoted by ϕ) of $\mathcal{A}_e$ into $\mathbb{C}$ by the identity $\phi([x, \lambda]) = \phi(x) + \lambda$. Conclude that $||\phi|| \leq 1$.

(*The solution consists of routine verifications. The last statement follows from from the observation at the top of page 179, and the trivial fact that the norm of the extended functional is* $\geq ||\phi||$.)

2. The requirement $||xy|| \leq ||x||\,||y||$ in the definition of a Banach algebra implies the joint continuity of multiplication. Prove:

(a) If $\mathcal{A}$ is a Banach space and also an algebra for which multiplication is *separately* continuous, then multiplication is jointly continuous. Hint: consider the bounded operators $L_x : y \to xy$ and $R_y : x \to xy$ on $\mathcal{A}$ and use the Uniform Boundedness theorem.

Solution. For each $x \in \mathcal{A}$, $L_x \in B(\mathcal{A})$ by hypothesis. For each $y \in \mathcal{A}$,

$$\sup_{||x||=1} ||L_x y|| = \sup_{||x||=1} ||R_y x|| := ||R_y|| < \infty,$$

since $R_y \in B(\mathcal{A})$ by hypothesis. By the Uniform Boundedness theorem (cf. Corollary 6.5),

$$\sup_{||x||=1} ||L_x|| := M < \infty.$$

For any $0 \neq x \in \mathcal{A}$, we then have for all $y \in \mathcal{A}$

$$||xy|| = ||x||\,||L_{x/||x||}y|| \leq ||x||\,||L_{x/||x||}||\,||y|| \leq M\,||x||\,||y||. \qquad (*)$$

Hence $||xy|| \leq M\,||x||\,||y||$ for all $x, y \in \mathcal{A}$, which implies joint continuity of the multiplication (cf. observation preceding Definition 7.2).

(b) (Notation as in Part (a); $\mathcal{A}$ has an identity e.) The norm $|x| := ||L_x||$ (where the norm on the right is the $B(\mathcal{A})$-norm) is equivalent to the given norm on $\mathcal{A}$, and satisfies the requirements $|xy| \leq |x|\,|y|$ and $|e| = 1$.

Solution.

By (*) we have for all $x \in \mathcal{A}$

$$|x| = ||L_x|| = \sup_{||y||=1} ||xy|| \leq M\,||x||.$$

On the other hand, $||x|| = ||xe|| = ||L_x e|| \leq ||L_x||\,||e|| = N\,|x|$, where $N := ||e||$. This proves the equivalence of the norms. The norm $|\cdot|$ satisfies $|e| = ||L_e|| = \sup_{||y||=1} ||ey|| = 1$ and

$$|xy| = ||L_{xy}|| = ||L_x L_y|| \leq ||L_x||\,||L_y|| = |x|\,|y|$$

for all $x, y \in \mathcal{A}$.

3. Let $\mathcal{A}$ be a unital complex Banach algebra. If $F \subset \mathcal{A}$ consists of commuting elements, denote by $\mathcal{C}_F$ the maximal commutative Banach subalgebra of $\mathcal{A}$ containing F. Prove:

(a) $\sigma_{\mathcal{C}_F}(a) = \sigma(a)$ for all $a \in \mathcal{C}_F$.

Solution.

The maximality assumption on $\mathcal{C}_F$ implies that if $b \in \mathcal{A}$ commutes with every element of $\mathcal{C}_F$, then $b \in \mathcal{C}_F$. In particular, $e \in \mathcal{C}_F$. Since $\mathcal{C}_F$ is a (unital) Banach *subalgebra* of $\mathcal{A}$, the inclusion $\rho_{\mathcal{C}_F}(a) \subset \rho(a)$ is trivial for any $a \in \mathcal{C}_F$. On the other hand, if $\lambda \in \rho(a)$, since $\lambda e - a$ commutes with every element of $\mathcal{C}_F$, the same is true of its inverse in $\mathcal{A}$, and therefore this inverse belongs to $\mathcal{C}_F$, i.e., $\lambda \in \rho_{\mathcal{C}_F}(a)$.

(b) If $a, b \in \mathcal{A}$ commute, then

$$\sigma(a+b) \subset \sigma(a) + \sigma(b) \quad \text{and} \quad \sigma(ab) \subset \sigma(a)\,\sigma(b).$$

Conclude that

$$r(a+b) \le r(a) + r(b) \quad \text{and} \quad r(ab) \le r(a)\, r(b).$$

Solution.

Let $F = \{a, b\}$ and $\mathcal{B} := \mathcal{C}_F$. Let Φ be the set of all homomorphisms of $\mathcal{B}$ onto $\mathbb{C}$. By Part (a) and Section 7.2 (1),

$$\sigma(a+b) = \sigma_{\mathcal{B}}(a+b) = \{\phi(a+b); \phi \in \Phi\}$$

$$= \{\phi(a) + \phi(b); \phi \in \Phi\} \subset \{\phi(a); \phi \in \Phi\} + \{\phi(b); \phi \in \Phi\}$$

$$= \sigma_{\mathcal{B}}(a) + \sigma_{\mathcal{B}}(b) = \sigma(a) + \sigma(b).$$

The corresponding relation for ab is proved in the same manner.

We then have

$$r(a+b) = \sup_{\lambda \in \sigma(a+b)} |\lambda| \le \sup_{\lambda \in \sigma(a)+\sigma(b)} |\lambda|$$

$$= \sup_{\alpha \in \sigma(a);\, \beta \in \sigma(b)} |\alpha + \beta| \le \sup_{\alpha \in \sigma(a)} |\alpha| + \sup_{\beta \in \sigma(b)} |\beta| = r(a) + r(b).$$

The corresponding inequality for the product is proved in the same manner.

(c) For all $a \in \mathcal{A}$ and $\lambda \in \rho(a)$,

$$r(R(\lambda; a)) = \frac{1}{d(\lambda, \sigma(a))}.$$

Solution.

Given $a \in \mathcal{A}$, let $\mathcal{B} = \mathcal{C}_{\{a\}}$ and let Φ be the set of homomorphisms of $\mathcal{B}$ onto $\mathbb{C}$. As observed in Part (a), $R(\lambda; a) \in \mathcal{B}$ for any $\lambda \in \rho(a)$, and clearly $\phi(R(\lambda; a)) = [\lambda - \phi(a)]^{-1}$. Thus (by Part (a))

$$\sigma(R(\lambda; a)) = \sigma_{\mathcal{B}}(R(\lambda; a)) = \{\phi(R(\lambda; a));\ \phi \in \Phi\}$$

$$= \{[\lambda - \phi(a)]^{-1};\ \phi \in \Phi\},$$

and $\sigma(a) = \sigma_{\mathcal{B}}(a) = \{\phi(a);\ \phi \in \Phi\}$. Hence

$$r(R(\lambda; a)) = \sup_{\phi \in \Phi} |\lambda - \phi(a)|^{-1} = \left[\inf_{\phi \in \Phi} |\lambda - \phi(a)| \right]^{-1}$$

$$= \left[\inf_{\mu \in \sigma(a)} |\lambda - \mu| \right]^{-1} = \frac{1}{d(\lambda, \sigma(a))}.$$

4. Let $\mathcal{A}$ be a commutative (unital) Banach algebra (over $\mathbb{C}$). A set $E \subset \mathcal{A}$ *generates* $\mathcal{A}$ if the minimal closed subalgebra of $\mathcal{A}$ containing E and the identity e coincides with $\mathcal{A}$. In that case, prove that the maximal ideal space of $\mathcal{A}$ is homeomorphic to a closed subset of the cartesian product $\prod_{a \in E} \sigma(a)$.

Solution.

Let $\mathcal{A}_0$ denote the set of all elements of $\mathcal{A}$ of the form $p(a_1, \cdots, a_n)$, where p is a polynomial in n variables ($n \geq 1$) and $a_k \in E$. Since E generates $\mathcal{A}$, $\mathcal{A}_0$ is a dense subalgebra of $\mathcal{A}$. Let Φ be the set of all homomorphisms of $\mathcal{A}$ onto $\mathbb{C}$, with the Gelfand topology. If ϕ_1, $\phi_2 \in \Phi$ coincide on E, then for each polynomial p in n variables and $a_k \in E$, $k = 1, \cdots, n$,

$$\phi_1(p(a_1, \cdots, a_n)) = p(\phi_1(a_1), \cdots, \phi_1(a_n)) = p(\phi_2(a_1), \cdots, \phi_2(a_n))$$

$$= \phi_2(p(a_1, \cdots, a_n)),$$

that is, $\phi_1 = \phi_2$ on $\mathcal{A}_0$, hence $\phi_1 = \phi_2$ by continuity of ϕ_1 and ϕ_2, and the density of $\mathcal{A}_0$ in $\mathcal{A}$. This shows that the map $\tau : \phi \in \Phi \to \Phi|_E$ is a one-to-one map of Φ into

$$\prod_{a \in E} \{\phi(a);\ \phi \in \Phi\} = \prod_{a \in E} \sigma(a)$$

(Cf. Section 7.2(1)). By definition of the Gelfand topology on Φ and the cartesian product topology, the map τ is a *continuous* map of the compact space Φ into a Hausdorff space; its range is necessarily closed, and since τ is one to one, it is a homeomorphism of Φ onto its range (=a closed subset of $\prod_{a \in E} \sigma(a)$).

5. Let $\mathcal{A}$ be a unital (complex) Banach algebra, and let G be the group of regular elements of $\mathcal{A}$. Suppose $\{a_n\} \subset G$ has the following properties:

(i) $a_n \to a$ and $a_n a = a a_n$ for all n;

(ii) the sequence $\{r(a_n^{-1})\}$ is bounded.

Prove that $a \in G$. Hint: observe that $r(e - a_n^{-1}a) \leq r(a_n^{-1}) r(a_n - a) \to 0$, hence $1 - \sigma(a_n^{-1}a) = \sigma(e - a_n^{-1}a) \subset B(0, 1/2)$ for n large enough, and therefore $0 \notin \sigma(a_n^{-1}a)$.

Solution.

Let M be such that $r(a_n^{-1}) \leq M$ for all n (cf. (ii)). Then by Exercise 3(b) (cf. also (9), after Definition 7.5)

$$r(e - a_n^{-1}a) = r(a_n^{-1}(a_n - a)) \leq r(a_n^{-1}) r(a_n - a) \leq M\, ||a_n - a|| \to 0$$

as $n \to \infty$. Fix then n such that $r(e - a_n^{-1}a) < 1/2$. We have

$$1 - \sigma(a_n^{-1}a) = \sigma(e - a_n^{-1}a) \subset B(0, 1/2),$$

that is

$$\sigma(a_n^{-1}a) \subset 1 + B(0, 1/2) = B(1, 1/2).$$

Hence $0 \in \rho(a_n^{-1}a)$, i.e., $a_n^{-1}a \in G$, and therefore $a = a_n(a_n^{-1}a) \in G$.

6. Let $\mathcal{A}$ be a unital (complex) Banach algebra, $a \in \mathcal{A}$, and $\lambda \in \rho(a)$. Prove that $\lambda \in \rho(b)$ for all $b \in \mathcal{A}$ for which the series $s(\lambda) := \sum_n [(b-a)R(\lambda; a)]^n$ converges in $\mathcal{A}$, and for such b, $R(\lambda; b) = R(\lambda; a)\, s(\lambda)$.

Solution.

Let $\lambda \in \rho(a)$ be such that the series $s(\lambda)$ converges in $\mathcal{A}$. By continuity of multiplication in $\mathcal{A}$, we then have

$$[(b-a)R(\lambda; a)]\, s(\lambda) = s(\lambda)\, [(b-a)R(\lambda; a)]$$

$$= \sum_{n=0}^{\infty} [(b-a)R(\lambda; a)]^{n+1} = s(\lambda) - e.$$

Equivalently,

$$[e - (b-a)R(\lambda; a)]s(\lambda) = s(\lambda)[e - (b-a)R(\lambda; a)] = e. \tag{1}$$

Since $aR(\lambda; a) = \lambda R(\lambda; a) - e$, the expression in square brackets in (1) is equal to $(\lambda e - b)R(\lambda; a)$. Hence by (1)

$$(\lambda e - b)R(\lambda; a)s(\lambda) = e, \tag{2}$$

and

$$s(\lambda)(\lambda e - b)R(\lambda; a) = e. \tag{3}$$

If we multiply (3) on the left by $R(\lambda; a)$ and on the right by $\lambda e - a$, we obtain

$$R(\lambda; a)s(\lambda)(\lambda e - b) = e. \tag{4}$$

By (2) and (4), $\lambda \in \rho(b)$ and

$$R(\lambda; b) = R(\lambda; a)s(\lambda). \tag{5}$$

7. Let $\mathcal{A}$ and a be as in Exercise 6, and let V be an open set in $\mathbb{C}$ such that $\sigma(a) \subset V$. Prove that there exists $\delta > 0$ such that $\sigma(b) \subset V$ for all b in the ball $B(a, \delta)$. Hint: if

M is a bound for $R(\cdot;a)$ on the complement of V in the Riemann sphere, take $\delta = 1/M$ and apply Exercise 6.

Solution.

Let V be as in the statement of the exercise. Then $R(\cdot;a)$ is continuous on the complement V^c of V in the Riemann sphere $\overline{\mathbb{C}}$. Since V^c is closed, hence compact in $\overline{\mathbb{C}}$, $||R(\cdot;a)||$ is bounded (say by $M > 0$) on V^c. Let $\delta = 1/M$. If $b \in B(a,\delta)$, we have $||(b-a)R(\lambda;a)|| < \delta M = 1$ for all $\lambda \in V^c$, and therefore the series $s(\lambda)$ converges absolutely, hence converges in the *Banach* algebra $\mathcal{A}$ (cf. Exercise 6) for all $\lambda \in V^c$. By Exercise 6, it follows that $V^c \subset \rho(b)$, that is $\sigma(b) \subset V$ (whenever $b \in B(a,\delta)$).

8. Let ϕ be a non-zero linear functional on the Banach algebra $\mathcal{A}$. Trivially, if ϕ is *multiplicative*, then $\phi(e) = 1$ and $\phi \neq 0$ on $G(\mathcal{A})$. The following steps provide a proof of the *converse*. Suppose $\phi(e) = 1$ and $\phi \neq 0$ on $G(\mathcal{A})$. Denote $N = \ker(\phi)$ (note that $N \cap G(A) = \emptyset$). Prove:

(a) $d(e,N) = 1$. Hint: if $||e - x|| < 1$, then $x \in G(A)$, hence $x \notin N$.

Solution.

Since $0 \in N$, $d(e,N) \leq d(e,0) = ||e|| = 1$. If we had $d(e,N) < 1$, there would be an element $x \in N$ for which $d(e,x) < 1$. By Remark 7.4, this implies that $x \in G(\mathcal{A})$, contradicting the fact that $N \cap G(\mathcal{A}) = \emptyset$.

(b) $\phi \in \mathcal{A}^*$ and has norm 1. Hint: if $a \notin N$, $a_1 := e - \phi(a)^{-1}a \in N$, hence $d(e,a_1) \geq 1$ by Part (a).

Solution. Since $\phi \neq 0$, $N \neq \mathcal{A}$. Fix then $a \notin N$, and let $a_1 = e - \phi(a)^{-1}a$ (it is well-defined, because $\phi(a) \neq 0$). By linearity of ϕ and the assumption $\phi(e) = 1$, we have $\phi(a_1) = 0$, that is, $a_1 \in N$, hence $d(e,a_1) \geq d(e,N) = 1$ by Part (a). Therefore

$$1 \leq ||e - a_1|| = ||\phi(a)^{-1}a|| = |\phi(a)|^{-1}||a||,$$

i.e., $|\phi(a)| \leq ||a||$ for all $a \notin N$ (and trivially for $a \in N$). This shows that $||\phi|| \leq 1$ (hence $\phi \in \mathcal{A}^*$). But since $||e|| = 1$, $||\phi|| \geq |\phi(e)| = 1$; therefore $||\phi|| = 1$.

(c) Fix $a \in N$ with norm 1, and let $f(\lambda) := \phi(\exp(\lambda a))$ (where the exponential is defined by means of the usual power series, converging absolutely in $\mathcal{A}$ for all $\lambda \in \mathbb{C}$). Then f is an entire function with no zeros such that $f(0) = 1$, $f'(0) = 0$, and $|f(\lambda)| \leq e^{|\lambda|}$.

Solution.

For any $a \in \mathcal{A}$, the series $\sum \lambda^n a^n/n!$ defining $e^{\lambda a}$ converges absolutely (it is majorized by the convergent series $\sum |\lambda|^n ||a||^n/n! = e^{|\lambda|\,||a||}$), hence converges in the Banach algebra $\mathcal{A}$, and $||e^{\lambda a}|| \leq e^{|\lambda|\,||a||}$. Fix a unit vector $a \in N$, and define

$$f(\lambda) := \phi(e^{\lambda a}) = \sum (\phi(a^n)/n!)\lambda^n \tag{6}$$

(where the fact $\phi \in \mathcal{A}^*$ was used; cf. Part (b)). The series for f converges absolutely for all $\lambda \in \mathbb{C}$; hence f is entire, $f(0) = \phi(e) = 1$ (by hypothesis), $f'(0) = \phi(a) = 0$ (since $a \in N$), and $|f(\lambda)| \leq ||\phi||\,||e^{\lambda a}|| \leq e^{|\lambda|}$ (by Part (b) and the hypothesis $||a|| = 1$). Since $e^{\lambda a} \in G(\mathcal{A})$ (its inverse is $e^{-\lambda a}$!), it follows from the hypothesis on ϕ that $e^{\lambda a} \notin N$, hence $f(\lambda) := \phi(e^{\lambda a}) \neq 0$ for all $\lambda \in \mathbb{C}$.

(d) (*This is a result about entire functions.*) If f has the properties listed in Part (c), then $f = 1$ identically. *Sketch of proof*: since f has no zeros, it can be represented as $f = e^g$ with g entire; necessarily $g(0) = g'(0) = 0$, so that $g(\lambda) = \lambda^2 h(\lambda)$ with h entire, and $\Re g(\lambda) \leq |\lambda|$. For any $r > 0$, verify that $|2r - g| \geq |g|$ in the disc $|\lambda| \leq r$ and $|2r - g| > 0$ in the disc $|\lambda| < 2r$. Therefore $F(\lambda) := \frac{r^2 h(\lambda)}{2r - g(\lambda)}$ is analytic in $|\lambda| < 2r$, and $|F| \leq 1$ on the circle $|\lambda| = r$, hence in the disc $|\lambda| \leq r$ by the maximum modulus principle. Thus

$$\frac{|h|}{|2 - g/r|} \leq 1/r \qquad (|\lambda| < r).$$

Given λ, let $r \to \infty$ to conclude that $h = 0$.

(*Added details*: for $|\lambda| \leq r$, $\Re g \leq |\lambda| \leq r$, hence $|2r - g|^2 = 4r^2 - 4r\Re g + |g|^2 \geq |g|^2$. If $g(\lambda_0) = 2r$, then $2r = \Re g(\lambda_0) \leq |\lambda_0|$, hence $g(\lambda) \neq 2r$ (i.e., $|2r - g| > 0$) for $|\lambda| < 2r$. Therefore F (defined above) is analytic in $|\lambda| < 2r$. For $|\lambda| = r$,

$$|F(\lambda)| = \frac{|g(\lambda)|}{|2r - g(\lambda)|} \leq 1,$$

hence $|F| \leq 1$ for $|\lambda| \leq r$, by the maximum modulus principle.)

(e) If $a \in N$, then $a^2 \in N$. Hint: apply Parts (c) and (d) and look at the coefficient of λ^2 in the series for f.

(*Added details*: let a be a unit vector in N, and define f as in Part (c). By Parts (c) and (d), f is identically equal to 1. By the power series representation (6) of f, this implies that the coefficient $\phi(a^2)/2$ of λ^2 vanishes,i.e., $a^2 \in N$. For $a \in N$ arbitrary ($\neq 0$, w.l.o.g.), $a/(||a||)$ is a unit vector in N, hence $a^2/(||a||^2) = [a/(||a||)]^2 \in N$(by the preceding special case), and therefore $a^2 \in N$.)

(f) $\phi(x^2) = \phi(x)^2$ for all $x \in \mathcal{A}$. (Represent $x = x_1 + \phi(x)e$ with $x_1 \in N$ and apply Part (e).) In particular, $x \in N$ iff $x^2 \in N$.

(*Added details*: For any $x \in \mathcal{A}$, $x_1 := x - \phi(x)e \in N$ by linearity of ϕ and the hypothesis $\phi(e) = 1$. Hence $x_1^2 \in N$ by Part (e), and therefore (since $\phi(x_1^2) = 0 = \phi(x_1)^2$)

$$\phi(x^2) = \phi([x_1 + \phi(x)e]^2) = \phi(x_1^2 + 2\phi(x)x_1 + \phi(x)^2 e)$$

$$= \phi(x_1)^2 + 2\phi(x)\phi(x_1) + \phi(x)^2 = [\phi(x_1) + \phi(x)]^2$$

$$= [\phi(x_1 + \phi(x)e)]^2 = \phi(x)^2.)$$

(g) If either x or y belong to N, then (i) $xy + yx \in N$, (ii) $(xy)^2 + (yx)^2 \in N$, and (iii) $xy - yx \in N$. (For (i), apply Part (f) to $x + y$; for (ii), apply (i) to yxy instead of y, when $x \in N$; for (iii), write $(xy - yx)^2 = 2[(xy)^2 + (yx)^2] - (xy + yx)^2$ and use Part (f).) Conclude that N is a two-sided ideal in $\mathcal{A}$ and ϕ is multiplicative (use the representation $x = x_1 + \phi(x)e$ with $x_1 \in N$).

Solution.

By symmetry, we may assume $x \in N$.
(i) By Part (e), $\phi(x) = \phi(x^2) = 0$. Hence by Part (f)

$$\phi(xy + yx) = \phi(x^2 + xy + yx + y^2) - \phi(y^2) = \phi([x + y]^2) - \phi(y)^2$$

$$= [\phi(x + y)]^2 - \phi(y)^2 = \phi(y)^2 - \phi(y)^2 = 0.$$

(ii) Replacing y by yxy in (i), we get $(xy)^2 + (yx)^2 \in N$.
(iii) By Parts (g)(i) and (e), $(xy + yx)^2 \in N$, i.e., $(xy)^2 + (yx)^2 + [xy^2x + yx^2y] \in N$, and therefore by (ii) $xy^2x + yx^2y \in N$. Consequently $(xy - yx)^2 = (xy)^2 + (yx)^2 - [xy^2x + yx^2y] \in N$ by (ii), hence $xy - yx \in N$ by Part (f).
By (i) and (iii), if $x \in N$, then $xy = (1/2)[(xy + yx) + (xy - yx)] \in N$ and $yx = (1/2)[(xy + yx) - (xy - yx)] \in N$ for all $y \in \mathcal{A}$, i.e., N is a two-sided ideal in $\mathcal{A}$.
For any $x, y \in \mathcal{A}$, represent $x = x_1 + \phi(x)e$ with $x_1 \in N$ as in Part (f); then $x_1 y \in N$, and consequently

$$\phi(xy) = \phi(x_1 y + \phi(x)\, y) = \phi(x)\phi(y).$$

9. Let $\mathcal{A}$ be a (unital complex) Banach algebra. For $a, b \in \mathcal{A}$, denote $C(a, b) = L_a - R_b$ (cf. Section 7.1), and consider the series

$$b_L(\lambda) = \sum_{j=0}^{\infty} (-1)^j R(\lambda; a)^{j+1} [C(a, b)^j e];$$

$$b_R(\lambda) = \sum_{j=0}^{\infty} [C(b, a)^j e] R(\lambda; a)^{j+1}$$

for $\lambda \in \rho(a)$. Prove that if $b_L(\lambda)$ $(b_R(\lambda))$ converges in $\mathcal{A}$ for some $\lambda \in \rho(a)$, then its sum is a left inverse (right inverse, respectively) for $\lambda e - b$. In particular, if $\lambda \in \rho(a)$ is such that both series converge in $\mathcal{A}$, then $\lambda \in \rho(b)$ and $R(\lambda; b) = b_L(\lambda) = b_R(\lambda)$.

Solution.

Suppose the series $b_L(\lambda)$ converges in $\mathcal{A}$ for some $\lambda \in \rho(a)$. Since $aR(\lambda; a) = \lambda R(\lambda; a) - e$ and $C(a,b)$ commutes with $L_{R(\lambda;a)}$, we have

$$b_L(\lambda)b = R_b b_L(\lambda) = [L_a - C(a,b)]b_L(\lambda)$$

$$= \sum_{j=0}^{\infty}(-1)^j aR(\lambda;a)^{j+1}[C(a,b)^j e] - \sum_{j=0}^{\infty}(-1)^j R(\lambda;a)^{j+1}[C(a,b)^{j+1}e]$$

$$= \lambda b_L(\lambda) - \sum_{j=0}^{\infty}(-1)^j R(\lambda;a)^j[C(a,b)^j e] + \sum_{j=0}^{\infty}(-1)^{j+1}R(\lambda;a)^{j+1}[C(a,b)^{j+1}e]$$

$$= \lambda b_L(\lambda) - e.$$

Hence $b_L(\lambda)(\lambda e - b) = e$.
A similar calculation, starting with

$$b\, b_R(\lambda) = L_b b_R(\lambda) = [R_a + C(b,a)]b_R(\lambda),$$

shows that $b_R(\lambda)$ is a right inverse of $\lambda e - b$ whenever the series $b_R(\lambda)$ converges in $\mathcal{A}$. In particular, if λ is such that both series $b_L(\lambda)$ and $b_R(\lambda)$ converge in $\mathcal{A}$, the left and right inverses $b_L(\lambda)$ and $b_R(\lambda)$ of $\lambda e - b$ coincide, and are the inverse of $\lambda e - b$, that is, $\lambda \in \rho(b)$ and $R(\lambda; b) = b_L(\lambda) = b_R(\lambda)$.

10. (Notation as in Exercise 9.) Set

$$r(a,b) = \limsup_n ||C(a,b)^n e||^{1/n},$$

and consider the compact subsets of $\mathbb{C}$

$$\sigma_L(a,b) = \{\lambda \in \mathbb{C};\ d(\lambda, \sigma(a)) \le r(a,b)\};$$

$$\sigma_R(a,b) = \{\lambda \in \mathbb{C};\ d(\lambda, \sigma(a)) \le r(b,a)\};$$

$$\sigma(a,b) = \sigma_L(a,b) \cup \sigma_R(a,b).$$

Prove that the series $b_L(\lambda)$ $(b_R(\lambda))$ converge absolutely and uniformly on compact subsets of $\sigma_L(a,b)^c$ $(\sigma_R(a,b)^c$, respectively). In particular, $\sigma(b) \subset \sigma(a,b)$, and $R(\cdot; b) = b_L = b_R$ on $\sigma(a,b)^c$.

Solution.

Note that (cf. Theorem 7.9)

$$r(a,b) \leq \lim_j ||C(a,b)^j||_{B(\mathcal{A})}^{1/j} = r_{B(\mathcal{A})}(C(a,b)) \leq ||C(a,b)||_{B(\mathcal{A})} < \infty.$$

Note also that

$$\sigma_L(a,b)^c = \{\lambda;\ d(\lambda,\sigma(a)) > r(a,b)\} \subset \rho(a), \tag{7}$$

so that the series $b_L(\lambda)$ makes sense for $\lambda \in \sigma_L(a.b)^c$. Similarly,

$$\sigma_R(a,b)^c \subset \rho(a), \tag{8}$$

and therefore $b_R(\lambda)$ makes sense for $\lambda \in \sigma_R(a,b)^c$. By the Beurling-Gelfand spectral radius formula (Theorem 7.9) and Exercise 3(c), for all $\lambda \in \sigma_L(a,b)^c$,

$$\limsup_j ||R(\lambda;a)^{j+1}[C(a,b)^j e]||^{1/j} \leq \lim_j ||R(\lambda;a)^j||^{1/j} \limsup_j ||C(a,b)^j e||^{1/j}$$

$$= r(R(\lambda;a))r(a,b) = \frac{r(a,b)}{d(\lambda,\sigma(a))} < 1.$$

Therefore the series $b_L(\lambda)$ converges absolutely (hence converges in the Banach algebra $\mathcal{A}$).

Let $Q \subset \sigma_L(a,b)^c$ be compact, and denote

$$d = \inf_{\lambda \in Q} d(\lambda,\sigma(a)).$$

Since Q is compact and $d(\cdot,\sigma(a))$ is continuous on Q, there exists $\lambda_0 \in Q$ such that $d = d(\lambda_0,\sigma(a)) > r(a,b)$. Fix constants r, s such that $r(a,b) < r < s < d$. Let $Q_s := \{\zeta \in \mathbb{C}; d(\zeta;Q) \leq s\}$. Then Q_s is a compact subset of $\rho(a)$, and by continuity of the resolvent on $\rho(a)$, $M_s := \sup_{\zeta \in Q_s} ||R(\zeta;a)|| < \infty$.

We saw in the proof of Theorem 7.6 that $R(\lambda;a)' = -R(\lambda;a)^2$. A simple induction shows that

$$R(\lambda;a)^{j+1} = (-1)^j R(\lambda;a)^{(j)}/j! \qquad (\lambda \in \rho(a)).$$

By Cauchy's formula for the derivatives of analytic functions, we then have for all $\lambda \in Q$

$$R(\lambda;a)^{j+1} = (-1)^j/(2\pi i) \int_{|\zeta-\lambda|=s} \frac{R(\zeta;a)}{(\zeta-\lambda)^{j+1}} d\zeta.$$

The integrand has norm $\leq M_s/s^{j+1}$, and therefore

$$||R(\lambda;a)^{j+1}|| \leq M_s/s^j \qquad (j = 0,1,\cdots;\lambda \in Q).$$

Since $r(a,b) < r$, it follows from the definition of $r(a,b)$ that there exists $j_0 \in \mathbb{N}$ such that $||C(a,b)^j e|| < r^j$ for all integers $j > j_0$. Consequently, for all $j > j_0$ and $\lambda \in Q$,

$$||R(\lambda; a)^{j+1}[C(a,b)^j e]|| \leq ||R(\lambda; a)^{j+1}|| \, ||C(a,b)^j e|| \leq M_s (r/s)^j.$$

Since $r/s < 1$, it follows that the series $b_L(\lambda)$ converges absolutely and uniformly for $\lambda \in Q$.

A similar calculation gives the corresponding conclusion for $b_R(\lambda)$ on $\sigma_R(a,b)^c$. If $\lambda \in \sigma(a,b)^c = \sigma_L(a,b)^c \cap \sigma_R(a,b)^c$, both series $b_L(\lambda)$ and $b_R(\lambda)$ converge (absolutely) in $\mathcal{A}$, and it follows from Exercise 9 that $\lambda \in \rho(b)$ and $R(\lambda; b) = b_L(\lambda) = b_R(\lambda)$ (in particular, $\sigma(b) \subset \sigma(a,b)$). Furthermore, both series converge (absolutely and) *uniformly* on compact subsets of $\sigma(a,b)^c$.

11. (Notation as in Exercise 10.) Set

$$d(a,b) = \max\{r(a,b), \; r(b,a)\}$$

(not to be confused with the usual distance function in $\mathcal{A}$), so that trivially

$$\sigma(a,b) = \{\lambda; \; d(\lambda, \sigma(a)) \leq d(a,b)\} \tag{9}$$

and $\sigma(a,b) = \sigma(a)$ iff $d(a,b) = 0$. In this case, it follows from Exercise 10 (and symmetry) that $\sigma(b) = \sigma(a)$ (for this reason, elements a, b such that $d(a,b) = 0$ are said to be *spectrally equivalent*).

(*Additional details*: by (7) and (8), $\sigma(a) \subset \sigma(a,b)$ is always valid. If $d(a,b) = 0$, $\sigma(a,b) \subset \sigma(a)$, since $\sigma(a)$ is a closed set; hence $\sigma(a,b) = \sigma(a)$. On the other hand, if $\lambda \in \sigma(a,b)$ but $\lambda \notin \sigma(a)$, then since $\sigma(a)$ is closed, $d(a,b) \geq d(\lambda, \sigma(a)) > 0$, that is, $\sigma(a,b) = \sigma(a)$ iff $d(a,b) = 0$. In this case, we have $\sigma(b) \subset \sigma(a)$ (by Exercise 10), hence $\sigma(b) = \sigma(a)$ by summetry.)

12. Let D be a *derivation* on $\mathcal{A}$, that is, a linear map $D : \mathcal{A} \to \mathcal{A}$ such that $D(ab) = (Da)b + a(Db)$ for all $a, b \in \mathcal{A}$. (Example: given $s \in \mathcal{A}$, the map $D_s := L_s - R_s$ is a derivation; it is called an *inner derivation.*) Prove:

(a) If D is a derivation on $\mathcal{A}$ and Dv commutes with v for some v, then $Df(v) = f'(v)Dv$ for all polynomials f.

Solution.

Taking $a = b = e$ in the derivation relation, we have $D(e) = D(ee) = D(e)e + eD(e) = 2D(e)$, hence $D(e) = 0$. We verify next that $Dv^k = kv^{k-1}Dv$ for all $k \geq 1$.

This is trivial for $k = 1$. Proceeding by induction, assume the relation for some k. Then

$$Dv^{k+1} = (Dv^k)v + v^k Dv = kv^{k-1}(Dv)v + v^k Dv = (k+1)v^k Dv,$$

since Dv commutes with v. The claim now follows from the linearity of D.

(b) Let $s \in \mathcal{A}$. The element $v \in \mathcal{A}$ is *s-Volterra* if $D_s v = v^2$. (Example: in $\mathcal{A} = B(L^p([0,1]))$, take $S : f(t) \to tf(t)$ and $V : f(t) \to \int_0^t f(u)du$, the so-called *classical Volterra operator.*) Prove: (i) $D_s v^n = nv^{n+1}$; (ii) $v^{n+1} = D_s^n v/n!$; (iii) $C(s+\alpha v, s+\beta v)^n e = (-1)^n n! \binom{\beta-\alpha}{n} v^n$, for all $n \in \mathbb{N}$ and $\alpha, \beta \in \mathbb{C}$.

Solution.

(i) By Part (a), $D_s v^n = nv^{n-1}Dv = nv^{n+1}$.
(ii) Use induction: for $n = 1$, (ii) is the hypothesis that v is s-Volterra. Assuming (ii) for $n-1$, we get by (i)

$$D_s^n v/n! = D_s[D_s^{n-1}v/(n-1)!]/n = D_s(v^n)/n = v^{n+1}.$$

(iii) Use induction: for $n = 1$, (iii) is trivial. Assuming (iii) for n, we obtain from (i)

$$C(s+\alpha v, s+\beta v)^{n+1} e = (-1)^n n! \binom{\beta-\alpha}{n} [(s+\alpha v)v^n - v^n(s+\beta v)]$$

$$= (-1)^{n+1} n! \binom{\beta-\alpha}{n} (\beta-\alpha-n) v^{n+1} = (-1)^{n+1}(n+1)! \binom{\beta-\alpha}{n+1} v^{n+1}.$$

(c) If $v \in \mathcal{A}$ is s-Volterra, then (i) $||v^n||^{1/n} = O(1/n)$. In particular, v is quasi-nilpotent. (ii) $r(s+\alpha v, s+\beta v) = 0$ if $\beta - \alpha \in \mathbb{N} \cup \{0\}$, and $= \limsup(n!||v^n||)^{1/n}$ otherwise. (iii) $d(s+\alpha v, s+\beta v) = \limsup(n!||v^n||)^{1/n})$ if $\alpha \neq \beta$. (iv) For $\alpha \neq \beta$, $s+\alpha v$ and $s+\beta v$ are spectrally equivalent iff $||v^n||^{1/n} = o(1/n)$. (v) $d(s+\alpha v,\, s+\beta v) \le \operatorname{diam}\sigma(s)$. (vi) $d(S+\alpha V,\, S+\beta V) = 1$ when $\alpha \neq \beta$ (cf. Part (b) for notation). In particular $S+\alpha V$ and $S+\beta V$ are spectrally equivalent iff $\alpha = \beta$ (however they all have the same spectrum, but do not try to prove this here!) Note that if $\beta - \alpha \in \mathbb{N}$, then $r(S+\alpha V,\, S+\beta V) = 0$ while $r(S+\beta V,\, S+\alpha V) = 1$.

Solution.

(i) We verify by induction that

$$||v^n|| \le ||v|| \frac{(2||s||)^{n-1}}{(n-1)!} \qquad (n \in \mathbb{N}). \tag{10}$$

This is trivial for $n = 1$. Assuming (10) for n, we get from Part (b),(i)

$$||v^{n+1}|| = (1/n)||sv^n - v^n s|| \leq (2/n)||s||\,||v^n||$$

$$\leq (2/n)||s||\,||v||\frac{(2||s||)^{n-1}}{(n-1)!} = ||v||\frac{(2||s||)^n}{n!}.$$

By (10),

$$||v^n||^{1/n} \leq ||v||^{1/n}(2||s||)^{1-1/n}n^{1/n}(n!)^{-1/n}. \tag{11}$$

By Stirling's formula (cf. page 323), $(n!)^{-1/n}$ is asymptotic to e/n, and therefore (11) implies (i), since the other factor on the right hand side of (11) has limit equal to $2\,||s||$.

(ii) By definition (cf. Exercise 10) and Exercise 12 (b) (iii),

$$r(s+\alpha v,\, s+\beta v) = \limsup_n \left|\binom{\beta-\alpha}{n}\right|^{1/n}(n!||v^n||)^{1/n}. \tag{12}$$

If $\beta - \alpha \in \mathbb{N}\cup\{0\}$, the n-th binomial coefficient for $\beta-\alpha$ is equal to $(\beta-\alpha)\cdots(\beta-\alpha-n+1)/n! = 0$ for all $n \geq \beta-\alpha+1$, and (12) implies that $r(s+\alpha v,\, s+\beta v) = 0$. In case $\beta-\alpha \notin \mathbb{N}\cup\{0\}$, the binomial series $\sum\binom{\beta-\alpha}{n}z^n$ has radius of convergence $R = 1$ (cf. ratio test!). Hence by the Cauchy-Hadamard formula, $1 = 1/R = \limsup\left|\binom{\beta-\alpha}{n}\right|^{1/n}$, and (ii) follows then from (12).

(iii) This follows trivially from the definition of $d(a,b)$ in Exercise 11 and (ii).

(iv) By Stirling's formula and (iii),

$$\limsup(||v^n||^{1/n})/(1/n) = \limsup e(n!||v^n||)^{1/n} = e\,d(s+\alpha v,\, s+\beta v),$$

which implies (iv).

(v) If $\alpha = \beta$, this is trivial (since $d = 0$ in that case). For $\alpha \neq \beta$, we have by (iii) and Part (b) (ii)

$$d(s+\alpha v,\, s+\beta v) = \limsup(n!||v^n||)^{1/n} = \limsup(n!||v^{n+1}||)^{1/(n+1)}$$

$$= \limsup ||D_s^n v||^{1/(n+1)} \leq \limsup ||D_s^n||_{B(\mathcal{A})}^{1/(n+1)} = r_{B(\mathcal{A})}(D_s), \tag{13}$$

where we used the fact that $||v||^{1/(n+1)} \to 1$, and the Beurling-Gelfand formula for the spectral radius of $D_s \in B(\mathcal{A})$.

Since $D_s := L_s - R_s$ and L_s commutes with R_s, it follows from Exercise 3 (b) that (for spectra in the Banach algebra $B(\mathcal{A})$)

$$\sigma(D_s) \subset \sigma(L_s) - \sigma(R_s). \tag{14}$$

If $\lambda \in \rho(s)$, $L_{\lambda e - s}$ and $L_{R(\lambda;s)}$ commute, and their product is $L_e = I$ (the identity operator in $B(\mathcal{A})$), i.e., λ is in the resolvent set of L_s (relative to $B(\mathcal{A})$). Thus $\sigma_{B(\mathcal{A})}(L_s) \subset \sigma(s)$. A similar inclusion is valid for $\sigma_{B(\mathcal{A})}(R_s)$. Hence by (14),

$$\sigma(D_s) \subset \sigma(s) - \sigma(s). \tag{15}$$

By (13) and (15) , for $\alpha \neq \beta$,

$$d(s + \alpha v,\, s + \beta v) \leq \sup\{|\lambda - \mu|;\ \lambda, \mu \in \sigma(s)\} = \operatorname{diam} \sigma(s).$$

(vi) We verify by induction (and Fubini's theorem) that

$$(V^n f)(t) = [1/(n-1)!] \int_0^t (t-u)^{n-1} f(u)\, du \qquad (f \in L^p([0,1])). \tag{16}$$

If $k_n(t) = t^{n-1}/(n-1)!$ on $[0,1]$ and we extend all functions to $\mathbb{R}$ as zero outside $[0,1]$, then (16) means that $V^n f = k_n * f$. Hence (cf. Exercise 5, Chapter 4)

$$||V^n f||_p \leq ||k_n||_1\, ||f||_p = (1/n!)\, ||f||_p,$$

that is $||V^n|| \leq 1/n!$. Since $(V^n 1)(t) = t^n/n!$, we have

$$||V^n|| \geq ||V^n 1||_p = (1/n!)\, [1/(np+1)^{1/p}].$$

Thus

$$1/(np+1)^{1/np} \leq (n! ||V^n||)^{1/n} \leq 1,$$

and we conclude from (iii) that $d(S + \alpha V,\, S + \beta V) = 1$ for $\alpha \neq \beta$. The last observation follows from (ii).

(d) If v is s-Volterra, then

$$R(\lambda; v) = \lambda^{-1} e + \lambda^{-2} \exp(s/\lambda) v \exp(-s/\lambda) \qquad (\lambda \neq 0).$$

Solution.

By Part (c) (i), v is quasi-nilpotent. Hence $R(\lambda; v)$ has the (absolutely) convergent Neumann expansion (for all $\lambda \neq 0$)

$$R(\lambda; v) = \sum_{n=0}^{\infty} v^n/\lambda^{n+1} = \lambda^{-1} e + \lambda^{-2} \sum_{n=0}^{\infty} v^{n+1}/\lambda^n.$$

By Part (b) (ii), we obtain (for all $\lambda \neq 0$)

$$R(\lambda; v) = \lambda^{-1}e + \lambda^{-2}\sum_{n=0}^{\infty}\lambda^{-n}(D_s^n/n!)v$$

$$= \lambda^{-1}e + \lambda^{-2}\exp(D_s/\lambda)\,v. \tag{17}$$

Since $D_s = L_s - R_s$ and L_s commutes with R_s in $B(\mathcal{A})$, we have

$$\exp(D_s/\lambda) = \exp(L_s/\lambda)\,\exp(-R_s/\lambda),$$

and clearly $\exp(L_s/\lambda)x = \exp(s/\lambda)\,x$ and $\exp(-R_s/\lambda)\,x = x\,\exp(-s/\lambda)$ for all $x \in \mathcal{A}$. The formula in Part (d) follows then from (17).

(e) If v is s-Volterra, then for all $\alpha, \lambda \in \mathbb{C}$

$$\exp[\lambda(s + \alpha v)] = \exp(\lambda s)(e + \lambda v)^{\alpha} = (e - \lambda v)^{-\alpha}\exp(\lambda s), \tag{18}$$

where the binomials are defined by means of the usual series (note that v is quasi-nilpotent, so that the binomial series converge for all complex λ).

Solution.

For any $a, b \in \mathcal{A}$, since L_a commutes with $C(a,b)$ in $B(\mathcal{A})$, we have for all $\lambda \in \mathbb{C}$

$$\exp(\lambda b) = \exp(\lambda R_b)\,e = \exp[\lambda L_a - \lambda C(a,b)]\,e = \exp(\lambda L_a)\,\exp[-\lambda C(a,b)]\,e. \tag{19}$$

Similarly, since R_a commutes with $C(b,a)$,

$$\exp(\lambda b) = \exp(\lambda L_b)\,e = \exp[\lambda R_a + \lambda C(b,a)]\,e = \exp(\lambda R_a)\,\exp[\lambda C(b,a)]\,e. \tag{20}$$

If v is s-Volterra ($s, v \in \mathcal{A}$) and $\alpha \in \mathbb{C}$, take $a = s$ and $b = s + \alpha v$ in (19). By Part (b) (iii), for all $n = 0, 1, \ldots$, $C(s,\, s + \alpha v)^n\,e = (-1)^n n!\binom{\alpha}{n}v^n$. Hence by (19)

$$\exp[\lambda(s + \alpha v)] = \exp(\lambda s)\sum_{n=0}^{\infty}\binom{\alpha}{n}\lambda^n v^n = \exp(\lambda s)\,(e + \lambda v)^{\alpha}. \tag{21}$$

The second formula in (18) is obtained in a similar fashion from (20) .

(f) If v is s-Volterra and $\rho(s)$ is connected, then $\sigma(s + kv) \subset \sigma(s)$ for all $k \in \mathbb{Z}$. For all $\lambda \in \rho(s)$,

$$R(\lambda; s + kv) = \sum_{j=0}^{k}\binom{k}{j}j!R(\lambda; s)^{j+1}v^j \qquad (k \geq 0); \tag{22}$$

$$= \sum_{j=0}^{|k|} (-1)^j \binom{|k|}{j} j! v^j R(\lambda; s)^{j+1} \qquad (k < 0). \tag{23}$$

(Apply Exercise 9.) If $\rho(s)$ and $\rho(s + kv)$ are both connected for some integer k, then $\sigma(s + kv) = \sigma(s)$. In particular, if $\sigma(s) \subset \mathbb{R}$, then $\sigma(s + kv) = \sigma(s)$ for all $k \in \mathbb{Z}$.

Solution.

Denote $\delta = \limsup(n!||v^n||)^{1/n}$ ($= d(s + \alpha v,\ s + \beta v)$ whenever $\alpha \neq \beta$ by Part (c) (iii), and is $\leq \operatorname{diam} \sigma(s)$ by Part (c) (v)). With notation as in Exercises 9 and 10, with $a = s$ and $b = s + \alpha v$ for any $\alpha \neq 0$, the compact set

$$\Lambda := \{\lambda;\ d(\lambda, \sigma(s)) \leq \delta\} \tag{24}$$

coincides with $\sigma(s,\ s + \alpha v)$, contains $\sigma(s + \alpha v)$, and on its complement (which is a subset of $\rho(s + \alpha v)$ and of $\rho(s)$), one has $R(\lambda; s + \alpha v) = b_L(\lambda) = b_R(\lambda)$. (The dependence of b_L and b_R on α is not shown for simplicity of notation.) In the present case, by Part (b) (iii),

$$b_L(\lambda) = \sum_{j=0}^{\infty} R(\lambda; s)^{j+1} j! \binom{\alpha}{j} v^j, \tag{25}$$

and

$$b_R(\lambda) = \sum_{j=0}^{\infty} (-1)^j j! \binom{-\alpha}{j} v^j R(\lambda; s)^{j+1}. \tag{26}$$

Note that b_L and b_R reduce to the *finite* sums in (22) and (23) when $\alpha = k \in \mathbb{N} \cup \{0\}$ and $-\alpha = k \in \mathbb{N}$ respectively. As observed above, for all $\lambda \in \Lambda^c$ and $k \in \mathbb{N} \cup \{0\}$,

$$b_L(\lambda)\,[\lambda e - (s + kv)] = [\lambda e - (s + kv)]\,b_L(\lambda) = e.$$

Since b_L is a finite sum in the present case, the functions above are analytic throughout $\rho(s)$ and coincide on the non-empty open subset Λ^c. Since $\rho(s)$ is connected, they coincide throughout $\rho(s)$, that is, $\rho(s) \subset \rho(s + kv)$, and (22) is valid. The same argument (with b_R instead of b_L and with k a negative integer) yields the conclusion $\rho(s) \subset \rho(s + kv)$ and (23) in case $k < 0$.

Observe that if v is s-Volterra, and $s' := s + kv$, then v is s'-Volterra as well. Therefore, if $\rho(s')$ is assumed connected, our previous result shows that $\sigma(s) = \sigma(s' - kv) \subset \sigma(s')$. If $\rho(s)$ is also connected, we have $\sigma(s') \subset \sigma(s)$. Therefore $\sigma(s') = \sigma(s)$ when both resolvent sets are connected. In particular, if $\sigma(s) \subset \mathbb{R}$ (so that $\rho(s)$ is trivially connected), we have $\sigma(s') \subset \sigma(s) \subset \mathbb{R}$, hence $\rho(s')$ is also connected, and the conclusion $\sigma(s') = \sigma(s)$ follows (when $\sigma(s) \subset \mathbb{R}$).

13. Let $\mathcal{A}$ be a (unital, complex) Banach algebra, and let $a, b, c \in \mathcal{A}$ be such that $C(a,b)c = 0$ (i.e., $ac = cb$). Prove:

(a) $C(e^a, e^b)c = 0$ (i.e., $e^a c = ce^b$, where the exponential function e^a is defined by the usual absolutely convergent series; the base of the exponential should not be confused with the identity of $\mathcal{A}$!)

(A trivial induction shows that $a^n c = cb^n$ for all $n = 0, 1, 2, \cdots$. Therefore, by continuity of multiplication in $\mathcal{A}$, $e^a c = ce^b$.)

(b) If $\mathcal{A}$ is a B^*-algebra, then e^{x-x^*} is unitary, for any $x \in \mathcal{A}$. (In particular, $||e^{x-x^*}|| = 1$.)

(By continuity of the involution on a B^*-algebra, $[e^{x-x^*}]^* = e^{x^*-x} = [e^{x-x^*}]^{-1}$.)

(c) If $\mathcal{A}$ is a B^*-algebra and a, b are normal elements (such that $ac = cb$, as before!), then

$$e^{a^*} ce^{-b^*} = e^{a^*-a} ce^{b-b^*}; \tag{27}$$

$$||e^{a^*} ce^{-b^*}|| \leq ||c||. \tag{28}$$

(Formula (27) follows from Part (a) by multiplying on the left by e^{a^*-a} and on the right by e^{-b^*}. Indeed, for *commuting* elements $x, y \in \mathcal{A}$, the "exponential identity" $e^{x+y} = e^x e^y$ follows as usual from the power series representation. Since a, b are normal (that is, commute with their adjoints), $e^{a^*-a} = e^{a^*}(e^a)^{-1}$ and $e^b e^{-b^*} = e^{b-b^*}$. The inequality (28) follows then from (27) and Part (b).)

(d) For a, b, c as in Part (c), define

$$f(\lambda) = e^{\lambda a^*} ce^{-\lambda b^*} \qquad (\lambda \in \mathbb{C}). \tag{29}$$

Prove that $||f(\lambda)|| \leq ||c||$ for all $\lambda \in \mathbb{C}$, and conclude that $f(\lambda) = c$ for all λ (that is, $e^{\lambda a^*} c = ce^{\lambda b^*}$ for all $\lambda \in \mathbb{C}$).

Solution.

If a, b, c are as in Part (c) and $\lambda \in \mathbb{C}$, then $\overline{\lambda} a, \overline{\lambda} b, c$ are as in Part (c). Replacing a, b by $\overline{\lambda} a$, $\overline{\lambda} b$ in (28), we obtain $||f(\lambda)|| \leq ||c||$. For any continuous linear functional ϕ on $\mathcal{A}$, $\phi \circ f$ is a complex valued entire function, bounded by $||\phi||\,||c||$ for all $\lambda \in \mathbb{C}$. By Liouville's theorem, $\phi \circ f$ is constant, that is, $\phi(f(\lambda)) = \phi(f(0)) = \phi(c)$ for all ϕ. By Corollary 5.7, $f(\lambda) = c$ for all $\lambda \in \mathbb{C}$.

(e) If $\mathcal{A}$ is a B^*-algebra, and a, b are normal elements of $\mathcal{A}$ such that $ac = cb$ for some $c \in \mathcal{A}$, then $a^* c = cb^*$.

In particular, if c commutes with a normal element a, it commutes also with its adjoint; this is *Fuglede's theorem.*

(By Part (d), $e^{\lambda a^*}c = ce^{\lambda b^*}$ for all $\lambda \in \mathbb{C}$. Writing the power series for both sides of this identity, it follows from the uniqueness of power series representation (cf. Corollary 5.7!) that the coefficient of λ on both sides coincides, that is, $a^*c = cb^*$.)

14. Consider $L^1(\mathbb{R})$ (with respect to Lebesgue measure) *with convolution as multiplication.* Prove that $L^1(\mathbb{R})$ is a commutative Banach algebra with no identity, and the Fourier transform F is a contractive (i.e., norm-decreasing) homomorphism of $L^1(\mathbb{R})$ into $C_0(\mathbb{R})$. (Cf. Exercise 7, Chapter 2.)

Solution.

By Theorem 1.29, $L^1(\mathbb{R})$ is a Banach space. By Exercise 7, Chapter 2, $f*g \in L^1(\mathbb{R})$ and $||f*g||_1 \le ||f||_1||g||_1$ for all $f, g \in L^1(\mathbb{R})$.
The associative law for convolution: by Exercise 7(d), Chapter 2, if $g, h \in L^1(\mathbb{R})$, then $g(y-z)h(z) \in L^1(\mathbb{R}^2)$; also $f(x-y)$ is measurable on $\mathbb{R}^2$ for $f \in L^1(\mathbb{R})$; hence $f(x-y)g(y-z)h(z)$ is measurable on $\mathbb{R}^2$. Since $|f|*(|g|*|h|) \in L^1(\mathbb{R})$, $f(x-y)g(y-z)h(z) \in L^1(\mathbb{R}^2)$ for almost all $x \in \mathbb{R}$. For such x, an application of Fubini's theorem and the invariance of the Lebesgue measure give

$$[f*(g*h)](x) = \int_{\mathbb{R}}\int_{\mathbb{R}} f(x-y)g(y-z)h(z)dzdy = \int \Big(\int f(x-y)g(y-z)dy\Big)h(z)dz$$

$$= \int \Big(\int f(x-z-u)g(u)du\Big)h(z)dz = [(f*g)*h](x).$$

The commutative law $f*g = g*f$ follows from the invariance of the Lebesgue measure; the distributive law is a trivial consequence of the linearity of the integral. We then conclude from these verifications that $L^1(\mathbb{R})$ is a commutative Banach algebra in the sense of Exercise 1 (no identity required). Actually, if there were an identity $e \in L^1(\mathbb{R})$, then (cf. Exercise 7, Chapter 2) $(Fe)^2 = F(e*e) = Fe$, so that Fe has the range $\{0,1\}$. Since Fe is continuous and vanishes at infinity (cf. Parts (f) and (k) of Exercise 7, Chapter 2), Fe is identically 0; but then $Ff = F(f*e) = Ff\,Fe = 0$ identically for all $f \in L^1(\mathbb{R})$, which is absurd (cf. I.3.12, page 303). (The remaining statements of the exercise are contained in Parts (f) and (k) of Exercise 7, Chapter 2.)

15. Let ϕ be a non-zero homomorphism of the Banach algebra $L^1 = L^1(\mathbb{R})$ into $\mathbb{C}$ (cf. Exercises 14 and 1). Prove:

(a) There exists a unique $h \in L^\infty = L^\infty(\mathbb{R})$ such that $\phi(f) = \int fh\,dx$ for all $f \in L^1$, and $||h||_\infty = 1$. (All $\int$ in this exercise mean $\int_{\mathbb{R}}$.) Moreover

$$\phi(f_y)\phi(g) = \phi(f)\phi(g_y) \qquad (f, g \in L^1;\ y \in \mathbb{R}), \tag{30}$$

where $f_y(x) = f(x-y)$.

Solution.

By Exercise 1, $\phi \in L^1(\mathbb{R})^*$ and $||\phi|| = 1$. By Theorem 4.6, there exists a unique $h \in L^\infty(\mathbb{R})$ such that $\phi(f) = \int fh\,dx$ for all $f \in L^1(\mathbb{R})$, and $||h||_\infty = ||\phi|| = 1$. Finally (30) follows from the relation $f_y * g = f * g_y$ and the multiplicativity of ϕ. (The above relation is verified as follows:

$$(f_y*g)(x) = \int f(x-y-t)g(t)dt = (f*g)_y(x) = (g*f)_y(x) = (g_y*f)(x) = (f*g_y)(x).)$$

(b) For any $f \in L^1$ such that $\phi(f) \neq 0$,

(i) $h(y) = \frac{\phi(f_y)}{\phi(f)}$ *a.e.*; (in particular, h may be chosen to be continuous).

Solution.

If $G \in L^1(\mathbb{R}^2)$, then $G(x,\cdot) \in L^1(\mathbb{R})$ for almost all x, $G(\cdot,y) \in L^1(\mathbb{R})$ for almost all y, and the almost everywhere defined function $\int G(\cdot,y)dy$ belongs to $L^1(\mathbb{R})$ (cf. Fubini's theorem, Theorem 2.17). By linearity and continuity of ϕ on $L^1(\mathbb{R})$, it follows that

$$\phi(\int G(\cdot,y)dy) = \int \phi[G(\cdot,y)]dy. \tag{31}$$

Let f be as in the hypothesis and $g \in L^1(\mathbb{R})$. By (31) with $G(x,y) = f(x-y)g(y)$ (cf. Exercise 7(d), Chapter 2)

$$\phi(f)\phi(g) = \phi(f*g) = \phi(\int f_y(x)g(y)dy) = \int \phi(f_y)\,g(y)\,dy.$$

Dividing by the non-zero constant $\phi(f)$, we obtain

$$\phi(g) = \int g(y)\frac{\phi(f_y)}{\phi(f)}\,dy \qquad (g \in L^1(\mathbb{R})). \tag{32}$$

By (32), the unique $h \in L^\infty(\mathbb{R})$ in the integral representation of ϕ coincides a.e. with $\phi(f_y)/\phi(f)$ (which is continuous and bounded, hence in $L^\infty(\mathbb{R})$; indeed, by the invariance of Lebesgue's measure and Exercise 1, Chapter 3,

$$|\phi(f_y) - \phi(f_t)| = |\phi(f_y - f_t)| \le ||\phi||\,||f_y - f_t||_1$$

$$= ||f_y - f_t||_1 = ||f_{y-t} - f||_1 \to 0 \quad \text{as } y \to t,$$

and $|\phi(f_y)| \leq ||\phi|| \, ||f_y||_1 = ||f||_1$.) We may then choose h as the *continuous* representative $\phi(f_y)/\phi(f)$ of its equivalence class (*this choice is independent on the particular* $f \in L^1(\mathbb{R})$ *such that* $\phi(f) \neq 0$, *as follows from (30).*)

(ii) $\phi(f_y) \neq 0$ for all $y \in \mathbb{R}$.

(If $\phi(f_{y_0}) = 0$ for some y_0, then by (30), we would have $\phi(g_{y_0}) = 0$ for all $g \in L^1(\mathbb{R})$. Take in particular $g = f_{-y_0}$. Then $0 = \phi(g_{y_0}) = \phi(f)$, contradicting the hypothesis on f.)

(iii) $|h(y)| = 1$ for all $y \in \mathbb{R}$.

Solution.

By (ii), we may take f_{-y} instead of f in the definition of h, for any $y \in \mathbb{R}$. Then for any y,

$$h(y) = \frac{\phi[(f_{-y})_y]}{\phi(f_{-y})} = \frac{\phi(f)}{\phi(f_{-y})} = h(-y)^{-1}. \tag{33}$$

Since $|h| \leq 1$, it follows from (33) that $|h| \geq 1$, hence $|h| = 1$ identically.

(iv) $h(x+y) = h(x)h(y)$ for all $x, y \in \mathbb{R}$ and $h(0) = 1$.

Conclude that $h(y) = e^{-ity}$ for some $t \in \mathbb{R}$ (for all y) and that $\phi(f) = (Ff)(t)$, where F is the Fourier transform.

Conversely, each $t \in \mathbb{R}$ determines the homomorphism $\phi_t(f) = (Ff)(t)$. Conclude that the map $t \to \phi_t$ is a homeomorphism of $\mathbb{R}$ onto the *Gelfand space* Φ of L^1 (that is, the space of non-zero complex homomorphisms of L^1 with the Gelfand topology). (Hint: the Gelfand topology is Hausdorff and is weaker than the metric topology on $\mathbb{R}$.)

Solution.

We have $h(0) = \phi(f_0)/\phi(f) = 1$, since $f_0 = f$. Fix $x, y \in \mathbb{R}$. By (ii), $\phi(f_x) \neq 0$; by the observation at the end of (i),

$$h(y) = \frac{\phi[(f_x)_y]}{\phi(f_x)} = \frac{\phi(f_{x+y})}{\phi(f)} \frac{\phi(f)}{\phi(f_x)} = \frac{h(x+y)}{h(x)},$$

i.e., $h(x+y) = h(x)h(y)$.
Let $g(x) = e^{-x^2}$. We have $\phi(g) = \phi([g(x/n)]^{n^2}) = [\phi(g(x/n))]^{n^2}$. Suppose $\phi(g) = 0$. Then $\phi(g(x/n)) = 0$ for all $n \in \mathbb{N}$. We have $f_n(x) := f(x)g(x/n) \to f(x)$ pointwise and $|f_n| \leq |f| \in L^1(\mathbb{R})$. Therefore $f_n \to f$ in $L^1(\mathbb{R})$ by the Lebesgue Dominated

Convergence theorem (Theorem 1.20). Hence $0 = \phi(g(x/n)) = \phi(g(x/n))\phi(f) = \phi(f_n) \to \phi(f)$, i.e., $\phi(f) = 0$, contradicting our hypothesis on f. This argument shows that $\phi(g) \neq 0$, and therefore, by the observation at the end of (i), $h(y) = \phi(g_y)/\phi(g)$. For $0 < |y| < 1$,

$$|y^{-1}[g_y(x) - g(x)]| \leq 2(|x| + 1)\, e^{-(|x|-1)^2} \in L^1(\mathbb{R}),$$

and $y^{-1}[g_y(x) - g(x)] \to 2x\, e^{-x^2}$ pointwise as $y \to 0$. By dominated convergence, $y^{-1}(g_y - g) \to 2x\, e^{-x^2}$ in $L^1(\mathbb{R})$, and therefore $y^{-1}[\phi(g_y) - \phi(g)] \to \phi(2x\, e^{-x^2})$ (by linearity and continuity of ϕ). In particular, the limit $\lim_{y\to 0} y^{-1}[h(y) - h(0)]$ exists (i.e., h is differentiable at 0). It follows now from the identity $h(x+y) = h(x)h(y)$ that h is differentiable at every point x and $h'(x) = h'(0)h(x)$:

$$y^{-1}[h(x+y) - h(x)] = h(x)\, y^{-1}[h(y) - h(0)] \to h(x)h'(0).$$

Denote $h'(0) = a$. The unique solution of the "initial value problem"

$$h' = ah, \qquad h(0) = 1$$

is $h(x) = e^{ax}$. Since $|h(x)| = 1$ identically, a is a pure imaginary constant, say $a = -it$ (thus $h(x) = e^{-itx}$ and $\phi(f) = \int f(x)h(x)dx = (Ff)(t)$).
Conversely, for each $t \in \mathbb{R}$, the map $\phi_t : f \to (Ff)(t)$ is a non-zero homomorphism of $L^1(\mathbb{R})$ into $\mathbb{C}$ (cf. Exercise 7(f), Chapter 2; note that for each t there is an $f \in L^1$ such that $(Ff)(t) \neq 0$; actually, there are f such that $Ff \neq 0$ everywhere, cf. I.3.12; thus ϕ_t is *not* the zero homomorphism). The map $\tau : t \to \phi_t$ is thus a map of $\mathbb{R}$ onto Φ. It is one-to-one: indeed, if $\phi_t = \phi_s$ for some $t, s \in \mathbb{R}$, consider the L^1 function f defined by $f(x) = e^{-x}$ for $x > 0$ and $f(x) = 0$ for $x \leq 0$. Then $(Ff)(u) = (1 + iu)^{-1}$ (cf. I.3.15). Hence $(1 + it)^{-1} = \phi_t(f) = \phi_s(f) = (1 + is)^{-1}$, and therefore $t = s$. The map τ is continuous: indeed, if $t_n \to t$ in $\mathbb{R}$, then for all $f \in L^1(\mathbb{R})$, $\phi_{t_n}(f) = (Ff)(t_n) \to (Ff)(t) = \phi_t(f)$ (cf. Exercise 7(f), Chapter 2). This means that $\phi_{t_n} \to \phi_t$ in Φ (with the Gelfand topology, which is the relative *weak**-topology of $(L^1)^*$ on Φ). Thus τ is continuous. Extend τ to the one-point compactification $\overline{\mathbb{R}} = \mathbb{R} \cup \{\infty\}$ by defining $\tau(\infty) = 0$ (the zero homomorphism). The extended τ is one-to-one and onto $\Phi_0 := \Phi \cup \{0\}$. It is also continuous (when Φ_0 is endowed with the relative *weak**-topology of $(L^1)^*$), because if $t_n \to \infty$, then $\phi_{t_n}(f) = (Ff)(t_n) \to 0 = \phi_\infty(f)$ for all $f \in L^1$, that is, $\tau(t_n) \to \tau(\infty)$ (cf. Exercise 7(k), Chapter 2). If $E \subset \overline{\mathbb{R}}$ is closed (hence compact!), $\tau(E)$ is a compact subset of the *Hausdorff* space Φ_0, and is therefore closed. This shows that τ is a homeomorphism of $\overline{\mathbb{R}}$ onto Φ_0. Its restriction to $\mathbb{R}$ is then a homeomorphism of $\mathbb{R}$ onto Φ.

16. Let $\mathcal{A}$, $\mathcal{B}$ be commutative Banach algebras, $\mathcal{B}$ semi-simple. Let $\tau : \mathcal{A} \to \mathcal{B}$ be an algebra homomorphism. Prove that τ is continuous. Hint: for each $\phi \in \Phi(\mathcal{B})$ (the Gelfand space of $\mathcal{B}$), $\phi \circ \tau \in \Phi_0(\mathcal{A}) : \Phi(\mathcal{A}) \cup \{0\}$. Use the Closed Graph theorem.

Solution.

For each $\phi \in \Phi(\mathcal{B})$, $\phi \circ \tau \in \Phi_0(\mathcal{A})$ (it is a homomorphism of $\mathcal{A}$ into $\mathbb{C}$, as the composition of two homomorphisms). By Section 7.2, Subsection 3, $\phi \circ \tau$ is continuous. Let $a_n \in \mathcal{A}$ be such that $a_n \to a$ in $\mathcal{A}$ and $\tau(a_n) \to b$ in $\mathcal{B}$. By the continuity of ϕ on $\mathcal{B}$ and of $\phi \circ \tau$ on $\mathcal{A}$, we have (for all $\phi \in \Phi(\mathcal{B})$)

$$\phi(b) = \lim_n \phi(\tau(a_n)) = \lim(\phi \circ \tau)(a_n) = (\phi \circ \tau)(a) = \phi(\tau(a)). \tag{34}$$

Since $\mathcal{B}$ is semi-simple, $\Phi(\mathcal{B})$ separates the points of $\mathcal{B}$ (cf. Section 7.2, end of Subsection 6). It then follows from (34) that $b = \tau(a)$. This shows that $\tau : \mathcal{A} \to \mathcal{B}$ is a closed (everywhere defined) operator between the Banach spaces $\mathcal{A}$ and $\mathcal{B}$. By the Closed Graph theorem (Corollary 6.13), τ is continuous.

17. Let X be a compact Hausdorff space, and let $\mathcal{A} = C(X)$. Prove that the Gelfand space Φ of $\mathcal{A}$ (terminology as in Exercise 15) is homeomorphic to X. Hint: consider the map $t \in X \to \phi_t \in \Phi$, where $\phi_t(f) = f(t)$, $(f \in C(X))$ (this is the so-called "evaluation at t" homomorphism). If $\exists \phi \in \Phi$ such that $\phi \neq \phi_t$ for all $t \in X$ and $M = \ker\phi$, then for each $t \in X$ there exists $f_t \in M$ such that $f_t(t) \neq 0$. Use continuity of the functions and compactness of X to get a finite set $\{f_{t_j}\} \subset M$ such that $h := \sum |f_{t_j}|^2 > 0$ on X, hence $h \in G(\mathcal{A})$; however $h \in M$, contradiction. Thus $t \to \phi_t$ is onto Φ, and one-to-one (by Urysohn's lemma). Identifying Φ with X through this map, observe that the Gelfand topology is weaker than the given topology on X and is Hausdorff.

Solution.

Following the hint (and its notations), we consider the map $\tau : t \to \phi_t$ of X into Φ (for any $t \in X$, ϕ_t is a homomorphism of $\mathcal{A}$, since the operations in $\mathcal{A}$ are defined pointwise, and it is not the zero homomorphism, because $\phi_t(e) = 1$, where $e(t) = 1$ identically).
If $t, s \in X$, $t \neq s$, there exists $f \in \mathcal{A}$ such that $f(t) = 1$ and $f(s) = 0$ (cf. Urysohn's lemma, Theorem 3.1), i.e., $\phi_t(f) \neq \phi_s(f)$, hence $\tau(t) \neq \tau(s)$, showing that τ is one-to-one.
Suppose τ is *not* onto, and let then $\phi \in \Phi$ be such that $\phi_t \neq \phi$ for all $t \in X$. The corresponding maximal ideals $M_t := \ker(\phi_t)$ and $M := \ker(\phi)$ are then distinct (cf. Section 7.2, Subsection 2). Hence, for each $t \in X$, there exists a function $f_t \in M$ such that $f_t \notin M_t$. Thus $f_t(t) \neq 0$, and since f_t is continuous, there exists a neighbourhood V_t of t on which $f_t \neq 0$. By compactness of X, there exist $t_1, \cdots, t_n \in X$ such that $X = \bigcup_{j=1}^n V_{t_j}$. Consider the function $h = \sum_{j=1}^n |f_{t_j}|^2 = \sum_j f_{t_j} \overline{f}_{t_j} \in M$. For each $x \in X$, there exists j such that $x \in V_{t_j}$, and therefore $h(x) \geq |f_{t_j}(x)|^2 > 0$. Hence $1/h \in C(X) = \mathcal{A}$, which means that $h \in G(\mathcal{A})$. But $h \in M$, contradicting the fact

that $M \cap G(\mathcal{A}) = \emptyset$ for any (maximal) ideal M (cf. Section 7.2, Subsection 3). We then conclude that τ is *onto*.
If $\{t_\alpha\}$ is a net in X converging to $t \in X$, then for each $f \in \mathcal{A}$, we have $\phi_{t_\alpha}(f) \to \phi_t(f)$ by continuity of f (this is just another way of writing that $f(t_\alpha) \to f(t)$). Hence $\phi_{t_\alpha} \to \phi_t$ (by definition of the Gelfand topology on Φ), i.e., $\tau(t_\alpha) \to \tau(t)$. Collecting, we established that τ is a bijective continuous map of the compact space X onto the Hausdorff space Φ; it is therefore a homeomorphism of X and Φ.

18. Let U be the open unit disk in $\mathbb{C}$. For $n \in \mathbb{N}$, let $\mathcal{A} = \mathcal{A}(U^n)$ denote the Banach algebra of all complex functions analytic in $U^n := U \times \ldots \times U$ and continuous on the closure $\overline{U^n}$ of U^n in $\mathbb{C}^n$, with pointwise operations and supremum norm $||f||_u := \sup\{|f(z)|; z \in \overline{U^n}\}$. Let Φ be the Gelfand space of $\mathcal{A}$ (terminology as in Exercise 15). Given $f \in \mathcal{A}$ and $0 < r < 1$, denote $f_r(z) = f(rz)$ and $Z(f) = \{z \in \overline{U^n}; f(z) = 0\}$. Prove:

(a) f_r is the sum of an absolutely and uniformly convergent power series in $\overline{U^n}$. Conclude that the polynomials (in n variables) are dense in $\mathcal{A}$.

Solution.

Fix $0 < r < 1$. For all $z \in \overline{U^n}$, $rz \in \{z = (z_1, \cdots, z_n) \in \mathbb{C}^n; |z_j| \leq r\}$, and this set is a compact subset of U^n. Therefore, for each f analytic in U^n, the function $f_r \in \mathcal{A}$ (defined on $\overline{U^n}$ by $f_r(z) := f(rz)$) has an absolutely and uniformly convergent power series representation on $\overline{U^n}$. In particular, the partial sums of the series (which belong to $\mathcal{P}$, the algebra of all complex polynomials in n variables) converge uniformly to f_r on $\overline{U^n}$. If μ is a regular complex Borel measure on $\overline{U^n}$ such that $\int p\, d\mu = 0$ for all $p \in \mathcal{P}$ (integration in the sequel is over $\overline{U^n}$), then $\int f_r\, d\mu = 0$ for all $f \in \mathcal{A}$ and $0 < r < 1$. By continuity of such f on $\overline{U^n}$, $f_r \to f$ pointwise on $\overline{U^n}$ as $r \to 1-$, and $|f_r| \leq ||f||_u \in L^1(|\mu|)$. By the Dominated Convergence theorem (Theorem 1.20), $0 = \int f_r\, d\mu \to \int f\, d\mu$. Thus $\int f\, d\mu = 0$ for all $f \in \mathcal{A}$. By Theorem 5.2, Corollary 5.6, and Theorem 4.9, it follows that $\mathcal{P}$ is dense in $\mathcal{A}$.

(b) Each $\phi \in \Phi$ is an "evaluation homomorphism" ϕ_w for some $w \in \overline{U^n}$, where $\phi_w(f) = f(w)$. Hint: consider the polynomials $p_j(z) = z_j$ (where $z = (z_1, \ldots, z_n)$). Then $w := (\phi(p_1), \ldots, \phi(p_n)) \in \overline{U^n}$ and $\phi(p_j) = p_j(w)$. Hence $\phi(p) = p(w)$ for all polynomials p. Apply Part (a) to conclude that $\phi = \phi_w$. The map $w \to \phi_w$ is the wanted homeomorphism of $\overline{U^n}$ onto Φ.

Solution.

Let $\phi \in \Phi$ be given, and use the notation of the "hint". Note that $w \in \overline{U^n}$, because $|w_j| = |\phi(p_j)| \leq ||\phi||\, ||p_j||_u = 1$. We have $\phi_w(p_j) = p_j(w) = \phi(p_j)$ for $j = 1, \cdots, n$, hence $\phi_w(p) = \phi(p)$ for all $p \in \mathcal{P}$, because $p_1, \cdots, p_n$ generate the algebra $\mathcal{P}$, and ϕ_w, ϕ are algebra homomorphisms which coincide on p_j, $j = 1, \cdots, n$. Since $\mathcal{P}$ is dense in $\mathcal{A}$ (by Part (a)) and the homomorphisms ϕ_w, ϕ are necessarily continuous (and coincide on $\mathcal{P}$), we conclude that $\phi = \phi_w$. This shows that the map $\tau : w \in \overline{U^n} \to \phi_w \in \Phi$ is *onto*. Suppose $v, w \in \overline{U^n}$, $v \neq w$. There exists then $j \in \{1, \cdots, n\}$ such that $v_j \neq w_j$, that is, $p_j(v) \neq p_j(w)$. Thus $\phi_v(p_j) \neq \phi_w(p_j)$, hence $\phi_v \neq \phi_w$, and τ is one-to-one. If $w_n \to w$ in $\overline{U^n}$, then $f(w_n) \to f(w)$ for all $f \in C(\overline{U^n})$, that is, $\phi_{w_n} \to \phi_w$ in Φ, by definition of the Gelfand topology on Φ. Collecting, we see that τ is a bijective continuous map of the *compact space* $\overline{U^n}$ onto the *Hausdorff space* Φ; it follows that τ is a homeomorphism.

(c) Given $f_1, ..., f_m \in \mathcal{A}$ such that $\bigcap_{k=1}^m Z(f_k) = \emptyset$, there exist $g_1, ..., g_m \in \mathcal{A}$ such that $\sum_k f_k g_k = 1$ (on $\overline{U^n}$). Hint: otherwise, the ideal J generated by $f_1, ..., f_m$ is proper, and therefore there exists $\phi \in \Phi$ vanishing on J. Apply Part (b) to reach a contradiction.

Solution.

Let J be as in the hint. If $J \neq \mathcal{A}$, it is contained in a maximal ideal M (cf. Section 7.2, Subsection 3). Let $\phi = \phi_M$ (notation as in Section 7.2, Subsection 2). Since $\ker(\phi_M) = M$, ϕ vanishes on $J \subset M$. By Part (b), there exists $w \in \overline{U^n}$ such that $\phi = \phi_w$. Hence $f(w) = 0$ for all $f \in J$, and in particular, $f_k(w) = 0$ for $k = 1, \cdots, m$. This means that $w \in \bigcap_{k=1}^m Z(f_k) = \emptyset$, contradiction. Hence $J = \mathcal{A}$. Equivalently, $1 \in J$, which means that there exist $g_1, \cdots, g_m \in \mathcal{A}$ such that $1 = \sum_k f_k g_k$.

19. Let $\mathcal{A}$ be a (unital, complex) Banach algebra such that

$$K := \sup_{0 \neq a \in \mathcal{A}} \frac{||a||^2}{||a^2||} < \infty. \tag{35}$$

Prove:

(a) $||a|| \leq K\, r(a)$ for all $a \in \mathcal{A}$.

Solution.

By (35),

$$||a|| \leq K^{1/2} ||a^2||^{1/2} \qquad (a \in \mathcal{A}). \tag{36}$$

We verify by induction that for all $a \in \mathcal{A}$ and $n = 1, 2, \cdots$,

$$||a|| \leq K^{(1/2)+\cdots+(1/2^n)} ||a^{2^n}||^{1/2^n}. \tag{37}$$

(For $n = 1$, this is (36); assuming (37) for n, we get (37) for $n+1$ by using (36) with a^{2^n} replacing a.)
Letting $n \to \infty$, we conclude from (37) and the Beurling-Gelfand formula (Theorem 7.9) that $||a|| \le K\, r(a)$ for all $a \in \mathcal{A}$.

(b) $||p(a)|| \le K\, ||p||_{C(\sigma(a))}$ for all $p \in \mathcal{P}$, where $\mathcal{P}$ denotes the algebra of all polynomials of one complex variable over $\mathbb{C}$.

Solution.

By the spectral mapping theorem (Theorem 7.8), in any (unital, complex) Banach algebra $\mathcal{A}$, if $a \in \mathcal{A}$ and $p \in \mathcal{P}$,

$$r(p(a)) := \sup_{\lambda \in \sigma(p(a))} |\lambda| = \sup_{\lambda \in p(\sigma(a))} |\lambda| = \sup_{\mu \in \sigma(a)} |p(\mu)| := ||p||_{C(\sigma(a))}. \tag{38}$$

Now (b) follows from Part (a) with $p(a)$ replacing a and (38).

(c) If $a \in \mathcal{A}$ has the property that $\mathcal{P}$ is dense in $C(\sigma(a))$, then there exists a continuous algebra homomorphism (with norm $\le K$) $\tau : C(\sigma(a)) \to \mathcal{A}$ such that $\tau(p) = p(a)$ for all $p \in \mathcal{P}$.

Solution.

Define τ on $\mathcal{P}$ by $\tau(p) = p(a)$. Then τ is an algebra homomorphism of $\mathcal{P}$ into $\mathcal{A}$; it is continuous with norm $\le K$ by Part (b), when $\mathcal{P}$ is considered as a subspace of $C(\sigma(a))$. By Exercise 1, Chapter 4, τ extends uniquely as an element of $B(C(\sigma(a)), \mathcal{A})$, with norm $\le K$, by the assumed density of $\mathcal{P}$ in $C(\sigma(a))$. The multiplicativity of τ on $C(\sigma(a))$ follows trivially from the continuity of τ, the density of $\mathcal{P}$, and the multiplicativity of τ on $\mathcal{P}$.

20. Let $K \subset \mathbb{C}$ be compact $\neq \emptyset$, and let $C(K)$ be the corresponding Banach algebra of continuous functions with the supremum norm $||f||_K := \sup_K |f|$. Denote $\mathcal{P}_1 := \{p \in \mathcal{P};\ ||p||_K \le 1\}$ (cf. Exercise 19 (b)).

Let X be a Banach space, and $T \in B(X)$. For $x \in X$, denote

$$||x||_T := \sup_{p \in \mathcal{P}_1} ||p(T)x||; \tag{39}$$

$$Z_T := \{x \in X;\ ||x||_T < \infty\}. \tag{40}$$

Prove:

(a) Z_T is a Banach space for the norm $||\cdot||_T$ (which is greater than the given norm on X).

Solution.

The constant polynomial $e(\lambda) = 1$ (identically) belongs to $\mathcal{P}_1$ and $e(T) = I$ (the identity operator on X); hence $||x||_T \geq ||e(T)x|| = ||x||$ for all $x \in X$. Therefore $||\cdot||_T$ is positive definite; it is trivially homogeneous and satisfies the triangle identity. Thus Z_T is a subspace of X and $||\cdot||_T$ is a norm on Z_T. Suppose $\{x_n\}$ is a Cauchy sequence with respect to the norm $||\cdot||_T$. Since this norm is greater than the given norm, the sequence is Cauchy in the Banach space X, hence converges in X to some x.
Given $\epsilon > 0$, there exists $n_0 \in \mathbb{N}$ such that $||x_n - x_m||_T < \epsilon$ for all $n, m > n_0$. Hence

$$||p(T)(x_n - x_m)|| < \epsilon \qquad (n, m > n_0;\ p \in \mathcal{P}_1). \tag{41}$$

Letting $m \to \infty$, we obtain (by the continuity of $p(T)$ and of the norm)

$$||p(T)(x_n - x)|| \leq \epsilon \qquad (n > n_0;\ p \in \mathcal{P}_1),$$

that is, $||x_n - x||_T \leq \epsilon$ for all $n > n_0$; thus $x_n \to x$ in the $||\cdot||_T$-norm. This shows that $(Z_T, ||\cdot||_T)$ is a Banach space.

(b) $||p(T)||_{B(Z_T)} \leq 1$ (for all $p \in \mathcal{P}_1$).

Solution.

Note that if we let $\tau_x(p) = p(T)x$, then $x \in Z_T$ iff $\tau_x \in B(\mathcal{P}, X)$ (where $\mathcal{P}$ is considered as a subspace of $C(K)$), and (by definition of the norms!)

$$||x||_T = ||\tau_x||_{B(\mathcal{P},X)}. \tag{42}$$

In particular

$$||p(T)x|| \leq ||p||_K ||x||_T \qquad (x \in Z_T,\, p \in \mathcal{P}). \tag{43}$$

Fix $x \in Z_T$ and $p \in \mathcal{P}$. For any $q \in \mathcal{P}_1$, we have by (43)

$$||q(T)[p(T)x]|| = ||(qp)(T)x|| \leq ||qp||_K ||x||_T \leq ||p||_K ||x||_T,$$

hence

$$||p(T)x||_T \leq ||p||_K ||x||_T \qquad (x \in Z_T,\, p \in \mathcal{P}). \tag{44}$$

This means that $||p(T)||_{B(Z_T)} \leq ||p||_K$ for all $p \in \mathcal{P}$ (which is eqivalent to (b)).

(c) If the compact set K is such that $\mathcal{P}$ is dense in $C(K)$, there exists a contractive algebra homomorphism $\tau : C(K) \to B(Z_T)$ such that $\tau(p) = p(T)$ for all $p \in \mathcal{P}$.

(This follows from Part (b), in the same manner as Part (c) follows from Part (b) in Exercise 19.)