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CHAPTER 6

BOUNDED OPERATORS

1. Let X be a Banach space, Y a normed space, and $T \in B(X,Y)$. Prove that if $TX \neq Y$, then TX is of Baire's first category in Y.

Solution.

Suppose TX of Baire's second category in Y. Then by the open mapping theorem (Theorem 6.9), T is an open mapping. Hence TX is an open neighbourhood of zero in Y, and contains therefore an open ball $B(0,r)$ in Y. For any $0 \neq y \in Y$, $\alpha y \in B(0,r) \subset TX$ for any scalar $0 < \alpha < r/(||y||)$, hence $y \in \alpha^{-1}TX = TX$, i.e., $TX = Y$, contradiction.

2. Let X, Y be normed spaces, and $T : X \to Y$ be linear. Prove that

$$||T|| = \sup\{|y^*Tx|;\, x \in X,\ y^* \in Y^*,\ ||x|| = ||y^*|| = 1\}.$$

Solution.

By the definition of operator norm and Corollary 5.8,

$$||T|| = \sup_{||x||=1} ||Tx|| = \sup_{||x||=1} \sup_{||y^*||=1} |y^*Tx|.$$

3. Let $(S, \mathcal{A}, \mu)$ be a positive measure space, and let $p, q \in [1, \infty]$ be conjugate exponents. Let $T : L^p(\mu) \to L^p(\mu)$ be linear. Prove that

$$||T|| = \sup\{|\int_S (Tf)g\, d\mu|;\, f \in L^p(\mu),\ g \in L^q(\mu),\ ||f||_p = ||g||_q = 1\}. \qquad (1)$$

(In case $p = 1$ or $p = \infty$, assume that the measure space is σ-finite.)

Solution.

Write $L^p := L^p(\mu)$. In case $1 \leq p < \infty$, there exists an isometric isomorphism $V : x^* \to g$ of $(L^p)^*$ onto L^q such that

$$x^*f = \int_S fg\, d\mu \qquad (f \in L^p). \qquad (2)$$

Let T be a linear map of L^p into itself. By Exercise 2,

$$||T|| = \sup_{||f||_p=1} \sup_{x^*\in(L^p)^*;||x^*||=1} |x^*Tf|,$$

and (1) follows from (2).

The case $p = \infty$ $(q = 1)$. For any given $f \in L^\infty$, the function $Tf \in L^\infty$ determines the continuous linear functional $\phi_f \in (L^1)^*$ by

$$\phi_f(g) = \int_S (Tf)g\, d\mu \qquad (g \in L^1), \tag{3}$$

and $||\phi_f|| = ||Tf||_\infty$ (cf. Theorem 4.6). Using the definition of the norms $||T||$ and ϕ, we obtain

$$\begin{aligned} ||T|| &= \sup_{||f||_\infty=1} ||Tf||_\infty = \sup_{||f||_\infty=1} ||\phi_f||, \\ &= \sup_{||f||_\infty=1} \sup_{||g||_1=1} |\phi_f(g)|, \end{aligned}$$

and (1) follows from (3).

4. Let X be a Banach space, $\{Y_\alpha; \alpha \in I\}$ a family of normed spaces, and $T_\alpha \in B(X, Y_\alpha)$, $(\alpha \in I)$. Define

$$Z = \{x \in X;\ \sup_{\alpha\in I} ||T_\alpha x|| = \infty\}.$$

Prove that Z is either empty or a dense G_δ in X.

Solution.

Consider the sets

$$V_n = \{x \in X;\ \sup_{\alpha\in I} ||T_\alpha x|| > n\} \qquad (n \in \mathbb{N}).$$

If $x \in V_n$, there exists $\alpha_0 \in I$ such that $||T_{\alpha_0} x|| > n$. Since T_{α_0} is continuous, there exists $r > 0$ such that $||T_{\alpha_0} y|| > n$ for all $y \in B(x,r)$. Hence $\sup_\alpha ||T_\alpha y|| > n$ for all $y \in B(x,r)$. This shows that V_n is open, and therefore Z $(= \bigcap_n V_n)$ is a G_δ in X. If V_n are dense in X for all n, it follows from Baire's theorem (Theorem 6.1) that Z is a dense G_δ in X. *Otherwise*, there exists $N \in \mathbb{N}$ such that V_N is *not dense* in X. Hence there exists a ball $B(a,r) \subset V_N^c$, i.e., $||T_\alpha y|| \le N$ for all $y \in B(a,r)$ $(= a + rB(0,1))$ and all $\alpha \in I$. Therefore, for all α and $x \in B(0,1)$,

$$||T_\alpha x|| = (1/r)||T_\alpha(a + rx) - T_\alpha a|| \le 2N/r,$$

i.e., $||T_\alpha|| \leq 2N/r$ for all α. Hence for all $x \in X$, $||T_\alpha x|| \leq (2N/r)||x||$ for all $\alpha \in I$, that is $\sup_\alpha ||T_\alpha x|| \leq (2N/r)||x|| < \infty$, and so $Z = \emptyset$.

5. Let X, Y be Banach spaces, and let $T : D(T) \subset X \to Y$ be a closed operator with range $R(T)$ of the second category in Y. Prove:

(a) $R(T) = Y$.

Solution.

Let $G(T) := \{[x, Tx];\, x \in D(T)\}$ be the graph of T. Since T is a closed operator, $G(T)$ is a Banach space (with the $X \times Y$-norm). Let $S := P_Y|_{G(T)}$ be the restriction to $G(T)$ of the projection P_Y of $X \times Y$ onto Y. Then S is a continuous linear map of $G(T)$ onto $R(T)$, and $R(T)$ is of Baire's second category in Y. By the open mapping theorem (Theorem 6.9), S is an open mapping. Therefore its range $R(T) = T(X)$ is an *open subspace* of Y, hence coincides with Y (since $0 \in R(T)$, some ball $rB_Y(0,1)$ is contained in $R(T)$; therefore, for every $0 \neq y \in Y$, $y = (2||y||/r)\, r[y/(2||y||)] \in (2||y||/r)\, R(T) = R(T)$, i.e., $R(T) = Y$).

(b) There exists a constant $c > 0$ such that, for each $y \in Y$, there exists $x \in D(T)$ such that $y = Tx$ and $||x|| \leq c||y||$.

Solution.

By Part (a), $S : G(T) \to Y$ is *onto*. Let $N = \ker S$. Then N is a closed subspace of $G(T)$, and therefore $G(T)/N$ is a Banach space (with the quotient norm; cf. Theorem 6.16). By the homomorphism theorem, S induces a (norm-decreasing) one-to-one linear map $\tilde{S}$ of $G(T)/N$ onto Y by

$$\tilde{S}([x, Tx] + N) = S[x, Tx] \ (= Tx).$$

By Corollary 6.11, $(\tilde{S})^{-1}$ is bounded; let $c/2$ be its norm. Given $y \in Y$, let $[x_0, Tx_0] + N = (\tilde{S})^{-1} y$. Then

$$\inf_{[x', Tx'] \in N} ||[x_0 + x', T(x_0 + x')]|| \leq (c/2)||y||.$$

Hence there exists $[x", Tx"] \in N$ such that

$$||[x_0 + x", T(x_0 + x")]|| \leq c\,||y||. \tag{4}$$

Set $x = x_0 + x"$. Then $x \in D(T)$, $Tx = y$, and by (4), $||x|| \leq ||[x, Tx]|| \leq c\,||y||$.

(c) If T is one-to-one, then T^{-1} is bounded.
(*This is a trivial consequence of Part (b).*)

6. Let m denote Lebesgue measure on the interval $[0,1]$, and let $1 \le p < r \le \infty$. Prove that the identity map of $L^r(m)$ into $L^p(m)$ is norm decreasing with range of Baire's first category in $L^p(m)$.

Solution.

Case $r = \infty$. If $f \in L^\infty(m)$, $|f| \le ||f||_\infty$ a.e., hence $||f||_p \le ||f||_\infty ||1||_p = ||f||_\infty < \infty$, which shows that $f \in L^p(m)$ and the identity map id of $L^\infty(m)$ into $L^p(m)$ is norm decreasing.

Case $r < \infty$. We use Holder's inequality with the conjugate exponents r/p and $s = r/(r-p)$. For all $f \in L^r(m)$,

$$||f||_p^p = \int_0^1 |f|^p \cdot 1\, dm \le \Big(\int_0^1 (|f|^p)^{r/p}\Big)^{p/r} \Big(\int_0^1 1^s dm\Big)^{1/s} = ||f||_r^p < \infty,$$

hence $id : L^r(m) \to L^p(m)$ is norm-decreasing.

For $1 \le p < r \le \infty$, $id(L^r(m)) \ne L^p(m)$, and therefore $id(L^r(m))$ is of Baire's first category in $L^p(m)$, by Exercise 1.
(Since $1/r < 1/p$, we may pick any s such that $1/r < s < 1/p$; then $sp < 1$, and consequently $\int_0^1 t^{-sp} dm < \infty$; however $sr > 1$, and therefore $\int_0^1 t^{-sr} dm = \infty$ in case $r < \infty$, and $||t^{-s}||_\infty = \infty$. This shows that $t^{-s} \in L^p(m) - L^r(m)$.)

7. Let X be a Banach space, and let $T : D(T) \subset X \to X$ be a linear operator. Suppose there exists $\alpha \in \mathbb{C}$ such that $(\alpha I - T)^{-1} \in B(X)$. Let $p(\lambda) = \sum c_k \lambda^k$ be any polynomial (over $\mathbb{C}$) of degree $n \ge 1$. Prove that the operator $p(T) := \sum c_k T^k$ (with domain $D(T^n)$) is closed. (Hint: induction on n. Write $p(\lambda) = (\lambda - \alpha) q(\lambda) + r$, where the constant r may be assumed to be zero, without loss of generality, and q is a polynomial of degree $n-1$.)

Solution.

Consider the isometric isomorphism $J : [x,y] \to [y,x]$ of $X \times X$ onto itself. If T is one-to-one and T^{-1} denotes its inverse (with domain $R(T)$), then

$$G(T^{-1}) = \{[x, T^{-1}x];\ x \in R(T)\} = \{[Ty, y];\ y \in D(T)\} = JG(T).$$

It follows that T is closed iff T^{-1} is closed.
We proceed to prove that $p(T)$ is closed, using induction on the degree n of the polynomial p (we may assume that its highest coefficient is 1, without loss of generality).

If $n = 1$, say $p(\lambda) = \lambda - \beta$, then $p(T) = T - \beta I = (\alpha - \beta)I - (\alpha I - T)$ is closed, because $R := (\alpha I - T)^{-1}$ is in $B(X)$ by hypothesis, hence closed, and therefore its inverse $\alpha I - T$ is closed.
Suppose $q(T)$ is closed for any polynomial q of degree $n-1$ (where $n > 1$). If p is a polynomial of degree n, we have $p(\lambda) = (\lambda - \alpha)q(\lambda) + \beta$, where q is a polynomial of degree $n-1$ and β is a complex constant. Then $p(T) = (T - \alpha I)q(T) + \beta I$. We may assume $\beta = 0$ without loss of generality. Then $Rp(T) = -q(T)$ on $D(p(T))$ $(\subset D(q(T)))$. Let $x_n \in D(p(T))$ be such that $x_n \to x$ and $p(T)x_n \to y$. Then $x_n \in D(q(T))$, and $-q(T)x_n = Rp(T)x_n \to Ry$. Since $q(T)$ is closed by the induction hypothesis, it follows that $x \in D(q(T))$ and $-q(T)x = Ry \in D(\alpha I - T)$. Hence $x \in D((T - \alpha I)q(T)) = D(p(T))$ and $p(T)x = (T - \alpha I)q(T)x = -(T - \alpha I)Ry = y$. This shows that $p(T)$ is closed, as desired.

8. Let X, Y be Banach spaces. The operator $T \in B(X,Y)$ is *compact* if the set TB_X is conditionally compact in Y (where B_X denotes here the closed unit ball of X). Let $K(X,Y)$ be the set of all compact operators in $B(X,Y)$. Prove:

(a) $K(X,Y)$ is a (norm-)closed subspace of $B(X,Y)$.

Solution.

Let $S, T \in K(X,Y)$ and $\alpha, \beta \in \mathbb{C}$. If $\{x_n\} \in B_X$, there exists a subsequence $\{x'_n\}$ of $\{x_n\}$ such that $Sx'_n \to y$ in Y, and a subsequence $\{x_n"\}$ of $\{x'_n\}$ such that $Tx_n" \to z$. Then $(\alpha S + \beta T)x_n" \to \alpha y + \beta z$. This shows that $\alpha S + \beta T$ is compact.

Let $\{T_m\} \in K(X,Y)$ converge to T in the operator norm, and let $\{x_n\} \subset B_X$. For each m, there exists a subsequence $\{x_{n,m}\}_n$ of $\{x_n\}$ such that $\{T_m x_{n,m}\}_n$ converges. Consider the ("diagonal") subsequence $\{x'_n\} := \{x_{n,n}\}$. *For each* m, it differs from a subsequence of $\{x_{n,m}\}_n$ by only finitely many terms, and therefore $\{T_m x'_n\}_n$ converges in Y.
Let $\epsilon > 0$ be given. *Fix* m such that $||T - T_m|| < \epsilon/3$. For this m, there exists n_0 such that $||T_m x'_n - T_m x'_k|| < \epsilon/3$ for all $n, k > n_0$. Therefore, if $n, k > n_0$,

$$||Tx'_n - Tx'_k|| \le ||(T - T_m)(x'_n - x'_k)|| + ||T_m x'_n - T_m x'_k|| \le 2||T - T_m|| + ||T_m x'_n - T_m x'_k|| < \epsilon.$$

Hence $\{Tx'_n\}$ is Cauchy, and T is compact.

(b) If Z is a Banach space, then

$$K(X,Y)B(Z,X) \subset K(Z,Y) \quad \text{and} \quad B(Y,Z)K(X,Y) \subset K(X,Z).$$

In particular, $K(X) := K(X,X)$ is a closed two-sided ideal in $B(X)$.

Solution.

Let $S \in B(Z,X)$ and $T \in K(X,Y)$. Then $SB_Z \subset ||S||\,B_X$, hence $(TS)B_Z$ $(= T(SB_Z) \subset ||S||TB_X)$ is conditionally compact in Y, since TB_X is conditionally compact. Thus $TS \in K(Z,Y)$.
For T as before and $S \in B(Y,Z)$, TB_X is conditionally compact in Y, and therefore its continuous image $S(TB_X)$ is conditionally compact in Z. Hence $ST \in K(X,Z)$.

(c) $T \in B(X,Y)$ is a *finite range operator* if its range TX is finite dimensional. Prove that every finite range operator is compact.

Solution.

Let $T \in B(X,Y)$ be a finite range operator. Then TB_X is a bounded (by $||T||$) subset of the finite dimensional space TX, and is therefore conditionally compact (by the Heine-Borel theorem).

Adjoints

9. Let X,Y be Banach spaces, and let $T : X \to Y$ be a linear operator with domain $D(T) \subset X$ and range $R(T)$. If T is one-to-one, the inverse map T^{-1} is a linear operator with domain $R(T)$ and range $D(T)$.

If $D(T)$ is *dense* in X, the (Banach) adjoint T^* of T is defined as follows:

$$D(T^*) = \{y^* \in Y^*;\ y^* \circ T \text{ is continuous on}\, D(T)\}.$$

Since $D(T)$ is dense in X, it follows that for each $y^* \in D(T^*)$ there exists a *unique* extension $x^* \in X^*$ of $y^* \circ T$ (cf. Exercise 1, Chapter 4); we set $x^* = T^*y^*$. Thus T^* is uniquely defined on $D(T^*)$ by the relation

$$(T^*y^*)x = y^*(Tx) \qquad (x \in D(T)).$$

Prove:

(a) T^* is closed. In case T is closed, $D(T^*)$ is *weak**-dense in Y^*, and if Y is reflexive, $D(T^*)$ is strongly dense in Y^*.

Solution.

Let $y_n^* \in D(T^*)$, $y_n^* \to y^*$ in Y^*, and $T^*y_n^* \to x^*$ in X^*. Then for all $x \in D(T)$, $y_n^*(Tx) \to y^*(Tx)$, and also $y_n^*(Tx) = (T^*y_n^*)x \to x^*x$. Hence $y^* \circ T = x^*|_{D(T)}$ is continuous, that is, $y^* \in D(T^*)$, and the unique continuous extension T^*y^* of $y^* \circ T$ to all of X is equal to x^*. This proves that T^* is closed.

Let Y_0^* denote the *weak**-closure of $D(T^*)$ in Y^*. If $Y_0^* \neq Y^*$, pick $y_0^* \in Y^*$ not in Y_0^*. By Corollary 5.21, there exist a *weak**-continuous linear functional F on Y^* and a real number λ such that

$$\Re F(y_0^*) > \lambda > \sup_{y^* \in Y_0^*} \Re F(y^*).$$

Since the *weak**-topology on Y^* is the Γ-topology on Y^* where $\Gamma = \kappa Y$, it follows from Theorem 5.23 that there exists $y_0 \in Y$ such that $F(y^*) = y^* y_0$ for all $y^* \in Y^*$. Hence

$$\Re y_0^* y_0 > \lambda > \sup_{y^* \in D(T^*)} \Re y^* y_0.$$

Since $\{\Re y^* y_0;\ y^* \in D(T^*)\}$ is an additive subgroup of $\mathbb{R}$, its upper boundedness implies that it is the trivial subgroup $\{0\}$. Thus $\Re y^* y_0 = 0$ for all $y^* \in D(T^*)$. For these y^*, also $-iy^* \in D(T^*)$, and therefore $\Re(-iy^* y_0) = 0$, that is, $\Im y^* y_0 = 0$. *We conclude that* $y^* y_0 = 0$ *for all* $y^* \in D(T^*)$.

Assume now that T is closed, and let $\Gamma(T)$ be its graph.
Suppose $y_0 \neq 0$. Then $[0, y_0] \notin \Gamma(T)$. Since $\Gamma(T)$ is a closed subspace of $X \times Y$, it follows from Corollary 5.3 that there exists $\phi \in (X \times Y)^*$ such that $\phi(\Gamma(T)) = \{0\}$ and $\phi([0, y_0]) \neq 0$. By Exercise 3, Chapter 4, there exist $x^* \in X^*$ and $y^* \in Y^*$ such that $\phi([x, y]) = x^* x + y^* y$ for all $[x, y] \in X \times Y$. Thus

$$x^* x + y^* T x = 0 \qquad (x \in D(T)),$$

and $y^* y_0 \neq 0$. Since $y^* \circ T = -x^*$ on $D(T)$, $y^* \circ T$ is continuous on $D(T)$, i.e., $y^* \in D(T^*)$, but $y^* y_0 \neq 0$, contradicting our preceding conclusion. This shows that $y_0 = 0$. But then we get the contradiction $0 > \lambda > 0$! Hence $Y_0^* = Y^*$, as wanted.

Let Y_1^* be the strong closure of $D(T^*)$ in Y^*. If $Y_1^* \neq Y^*$, pick $y_1^* \in Y^*$ not in Y_1^*. By Corollary 5.3, there exists $y^{**} \in Y^{**}$ such that $y^{**} Y_1^* = \{0\}$ and $y^{**} y_1^* \neq 0$. If Y is reflexive, $y^{**} = \hat{y}$ for some $y \in Y$. Thus $y^* y = 0$ for all $y^* \in D(T^*)$, and $y_1^* y \neq 0$. The preceding argument with y replacing y_0 shows that the first relation implies necessarily that $y = 0$, which contradicts the second relation. Hence $Y_1^* = Y^*$, as desired.

(b) If $T \in B(X, Y)$, then $T^* \in B(Y^*, X^*)$, and $||T^*|| = ||T||$. If $S, T \in B(X, Y)$, then $(\alpha S + \beta T)^* = \alpha S^* + \beta T^*$ for all $\alpha, \beta \in \mathbb{C}$. If $T \in B(X, Y)$ and $S \in B(Y, Z)$, then $(ST)^* = T^* S^*$.

Solution.

For each $y^* \in Y^*$, $y^* \circ T$ is continuous (i.e., $\in X^*$), hence $D(T^*) = Y^*$, and the identity

$$T^* y^* = y^* \circ T \qquad (y^* \in Y^*) \tag{5}$$

defines T^* as a (clearly linear) map of Y^* into X^*. By the definition of operator norm and Exercise 2,

$$||T^*|| = \sup_{||y^*||=1} ||T^*y^*|| = \sup_{||y^*||=||x||=1} |(T^*y^*)x|$$

$$= \sup_{||y^*||=||x||=1} |y^*Tx| = ||T||.$$

In particular, $T^* \in B(Y^*, X^*)$.
The linearity of the map $T \to T^*$ (from $B(X,Y)$ to $B(Y^*, X^*)$) is evident from the identity (5).
If $T \in B(X,Y)$ and $S \in B(Y,Z)$, then for all $z^* \in Z^*$, we have by (5)

$$(ST)^*z^* = z^* \circ (ST) = (z^* \circ S) \circ T = (S^*z^*) \circ T = T^*(S^*z^*) = (T^*S^*)z^*,$$

hence $(ST)^* = T^*S^*$.

(c) If $T \in B(X,Y)$, then $T^{**} := (T^*)^* \in B(X^{**}, Y^{**})$, $T^{**}|_X = T$, and $||T^{**}|| = ||T||$. In particular, if X is reflexive, then $T^{**} = T$ (note that κX is identified with X).

Solution.

By Part (b), $||T^{**}|| = ||T^*|| = ||T||$. Let κ_X and κ_Y be the canonical embeddings of X and Y in X^{**} and Y^{**} respectively. Then for all $x \in X$ and $y^* \in Y^*$,

$$[T^{**}(\kappa_X x)]y^* = (\kappa_X x)(T^*y^*) = (T^*y^*)x = y^*(Tx) = [\kappa_Y(Tx)]y^*,$$

that is,

$$T^{**}\kappa_X = \kappa_Y T. \tag{6}$$

With the identifications of $\kappa_X X$ and $\kappa_Y Y$ with X and Y respectively, this means that $T^{**}|_X = T$.

(d) If $T \in B(X,Y)$, then T^* is continuous with respect to the *weak**-topologies on Y^* and X^* (cf. Exercise 4, Chapter 5). *Conversely*, if $S \in B(Y^*, X^*)$ is continuous with respect to the *weak**-topologies on Y^* and X^*, then $S = T^*$ for some $T \in B(X,Y)$. Hint: given $x \in X$, consider the functional $\phi_x(y^*) = (Sy^*)x$ on Y^*.

Solution.

Let $S \in B(Y^*, X^*)$ be continuous with respect to the *weak**-topologies on Y^* and X^*. Given $x \in X$, define

$$\phi_x(y^*) = (Sy^*)x \qquad (y^* \in Y^*). \tag{7}$$

The functional ϕ_x is trivially linear on Y^*. If $\{y^*_\alpha\}$ is a net in Y^* converging *weak** to y^*, then $Sy^*_\alpha \to Sy^*$ *weak** in X^* (by the continuity hypothesis on S), hence

$$\phi_x(y^*_\alpha) := (Sy^*_\alpha)x \to (Sy^*)x := \phi_x(y^*).$$

Thus ϕ_x is continuous with respect to the *weak**-topology on Y^* (for each $x \in X$). By Theorem 5.23, there exists an element of Y (which we denote by Tx, to indicate its dependence on x) such that $\phi_x(y^*) = y^*(Tx)$, that is

$$(Sy^*)x = y^*(Tx) \qquad (x \in X;\ y^* \in Y^*). \tag{8}$$

The map $T : X \to Y$ is well-defined (if $y, y' \in Y$ both satisfy the defining identity, then $y^*y = y^*y'$ for all y^*, and therefore $y = y'$ by Corollary 5.7); T is clearly linear, and by Corollary 5.8 and (8),

$$||T|| = \sup_{||x||=||y^*||=1} |y^*Tx| = \sup_{\cdots} |(Sy^*)x| = \sup_{||y^*||=1} ||Sy^*|| = ||S|| < \infty.$$

Thus $T \in B(X, Y)$, and by (8) and the definition of T^*, $T^*y^* = y^* \circ T = Sy^*$ for all $y^* \in Y^*$, i.e., $S = T^*$.

(e) $\overline{R(T)} = \bigcap\{\ker(y^*);\ y^* \in \ker(T^*)\}$. In particular, T^* is one-to-one iff $R(T)$ is dense in Y.

Solution.

If $y \in R(T)$ (say $y = Tx$ for some $x \in X$), then for each $y^* \in \ker T^*$, $y^*y = y^*Tx = (T^*y^*)x = 0$, that is, $R(T) \subset \bigcap_{y^* \in \ker T^*} \ker y^*$. Since y^* is continuous, its kernel is closed, hence the intersection of kernels is closed, and it follows that $\overline{R(T)}$ is contained in this intersection.
One the other hand, if $y_0 \notin \overline{R(T)}$, then by Corollary 5.3, there exists $y^* \in Y^*$ such that $y^*y_0 = 1$ and $y^*y = 0$ for all $y \in R(T)$. Hence $T^*y^* = y^* \circ T = 0$, that is $y^* \in \ker T^*$, but $y_0 \notin \ker y^*$. This shows that y_0 is not in the intersection, and the desired equality follows.

(f) Let $x^* \in X^*$ and $M > 0$ be given. Then there exists $y^* \in D(T^*)$ with $||y^*|| \leq M$ such that $x^* = T^*y^*$ if and only if

$$|x^*x| \leq M||Tx|| \qquad (x \in D(T)). \tag{9}$$

In particular, $x^* \in R(T^*)$ if and only if

$$\sup_{x \in D(T),\, Tx \neq 0} \frac{|x^*x|}{||Tx||} < \infty.$$

(Hint: Hahn-Banach.)

Solution.

Assume (9). The map $\phi : Tx \in R(T) \to x^*x$ is well-defined (if $Tx = Tx'$ for $x, x' \in D(T)$, then $|x^*x - x^*x'| = |x^*(x - x')| \leq M \,||T(x - x')|| = 0$). It is clearly linear, and $||\phi|| \leq M$. By the Hahn-Banach theorem (Theorem 5.2), there exists $y^* \in Y^*$ such that $||y^*|| = ||\phi|| \leq M$ and $y^*(Tx) = \phi(Tx) = x^*x$ for all $x \in D(T)$. In particular, $y^* \circ T = x^*|_{D(T)}$ is continuous, i.e., $y^* \in D(T^*)$, and by definition, T^*y^* is the unique continuous extension of $y^* \circ T$ to X, which is precisely x^*. Hence $x^* = T^*y^*$ (and $||y^*|| \leq M$).
The converse is trivial (if $x^* = T^*y^*$ for some $y^* \in D(T^*)$ with $||y^*|| \leq M$, then for all $x \in D(T)$

$$|x^*x| = |(T^*y^*)x| = |y^*(Tx)| \leq ||y^*||\,||Tx|| \leq M\,||Tx||.)$$

(g) Let $T \in B(X, Y)$ and let S^* be the (norm-)closed unit ball of Y^*. Then T^*S^* is *weak**-compact.

Solution.

By Alaoglu's theorem (Theorem 5.24), S^* is *weak**-compact. By Exercise 4, Chapter 5, T^* is continuous from Y^* to X^* with their *weak**-topologies. Therefore T^*S^* is *weak**-compact, as the continuous image of a compact set (in the appropriate *weak**-topologies).

(h) Let $T \in B(X, Y)$ have closed range TX. Suppose $x^* \in X^*$ vanishes on $\ker(T)$. Show that the map $\phi : TX \to \mathbb{C}$ defined by $\phi(Tx) = x^*x$ is a well-defined continuous linear functional, and therefore there exists $y^* \in Y^*$ such that $\phi = y^*|_{TX}$. (Hint: apply Corollary 6.10 to $T \in B(X, TX)$ to conclude that there exists $r > 0$ such that $\{y \in TX;\ ||y|| < r\} \subset TB_X(0, 1)$, and deduce that $||\phi|| \leq (1/r)||x^*||$.)

Solution.

The map ϕ is well-defined, because if $Tx = Tx'$, then $x - x' \in \ker(T) \subset \ker(x^*)$, hence $x^*x = x^*x'$. The linearity of ϕ is evident.
Since $T \in B(X, TX)$ is a continuous linear map of the Banach space X *onto the Banach space* TX (because TX is closed in Y), it follows that T is an open mapping (cf. Corollary 6.10). Hence $TB_X(0, 1)$ is open in TX (and contains 0). Let then $rB_{TX}(0, 1) \subset TB_X(0, 1)$. Thus

$$\{Tx;\ ||Tx|| < 1\} \subset \{Tx;\ ||x|| < 1/r\}.$$

Therefore

$$||\phi|| = \sup_{||Tx||<1} |\phi(Tx)| = \sup_{||Tx||<1} |x^*x|$$

$$\leq \sup_{||x||<1/r} |x^*x| = \sup_{||rx||<1} (1/r)|x^*(rx)| = (1/r)||x^*||.$$

Hence $\phi \in (TX)^*$, and therefore there exists $y^* \in Y^*$ such that $\phi = y^*|_{TX}$, by the Hahn-Banach theorem. (Thus x^* is in the range of T^*.)

(k) With T as in Part (h), prove that

$$T^*Y^* = \{x^* \in X^*; \ker(T) \subset \ker(x^*)\}. \tag{10}$$

In particular, T^* has (norm-)closed range in X^*.

Solution.

As observed at the end of Part (h), the set on the right hand side of (10) is contained in T^*Y^*. On the other hand, if $x^* = T^*y^*$ for some $y^* \in Y^*$, and $x \in \ker(T)$, then $x^*x = (T^*y^*)x = y^*(Tx) = 0$, i.e., $x \in \ker(x^*)$. This proves the inclusion $\subset$ in (10), and the conclusion follows.

10. Let X be a Banach space, and let T be a one-to-one linear operator with domain and range dense in X. Prove that $(T^*)^{-1} = (T^{-1})^*$, and T^{-1} is bounded (on its domain) iff $(T^*)^{-1} \in B(X^*)$.

Solution.

By hypothesis, both T^* and $(T^{-1})^*$ exist. Let $x^* \in D((T^{-1})^*)$. We have $[(T^{-1})^*x^*]x = x^*T^{-1}x$ for all $x \in R(T)$. Setting $x = Ty$ with $y \in D(T)$, the relation is $[(T^{-1})^*x^*] \circ T = x^*|_{D(T)}$, which shows that the left hand side is continuous on $D(T)$. Thus $(T^{-1})^*x^* \in D(T^*)$ and

$$T^*[(T^{-1})^*x^*] = x^* \quad \text{for all } x^* \in D((T^{-1})^*). \tag{11}$$

If $x^* \in D(T^*)$, $(T^*x^*)x = x^*Tx$ for all $x \in D(T)$. Write $Tx = y$ $(\in R(T))$. The relation takes the form $(T^*x^*) \circ T^{-1} = x^*|_{D(T^{-1})}$. The left hand side is thus continuous on its domain; therefore $T^*x^* \in D((T^{-1})^*$, and

$$(T^{-1})^*(T^*x^*) = x^* \quad \text{for all } x^* \in D(T^*). \tag{12}$$

By (12), T^* is one-to-one, hence $(T^*)^{-1}$ exists, and by (11), $(T^{-1})^* \subset (T^*)^{-1}$. Writing $T^*x^* = y^*$ $(\in R(T^*) = D((T^*)^{-1}))$ in (12), we have $(T^{-1})^*y^* = x^* = (T^*)^{-1}y^*$, that is $(T^*)^{-1} \subset (T^{-1})^*$. Together with the preceding inclusion, this proves the desired equality.

If T^{-1} is bounded on its domain, $y^* \circ T^{-1}$ is continuous on $D(T^{-1})$ for all $y^* \in X^*$, hence $D((T^{-1})^*) = X^*$. As an adjoint, $(T^{-1})^*$ is closed (cf. Part (a)). By the Closed Graph theorem (Corollary 6.13), $(T^{-1})^* \in B(X^*)$, as claimed. Conversely, if the latter relation holds, $y^* \circ T^{-1}$ is continuous on $D(T^{-1})$ for all $y^* \in X^*$. Consider the family of linear maps

$$\{\phi_x;\ x \in D(T^{-1}),\ ||x|| = 1\}, \tag{13}$$

where $\phi_x : X^* \to \mathbb{C}$ is defined by $\phi_x(y^*) = y^*T^{-1}x$ for each $x \in D(T^{-1})$. Since $||\phi_x|| = \sup_{||y^*||=1} |y^*(T^{-1}x)| = ||T^{-1}x|| < \infty$ (cf. Corollary 5.8), the maps in (13) are bounded. Also, for each $y^* \in X^*$,

$$\sup\{|\phi_x(y^*)|;\ x \in D(T^{-1}),\ ||x|| = 1\}$$

$$= \sup\{|(y^* \circ T^{-1})x|;\ x \in D(T^{-1}),\ ||x|| = 1\} < \infty,$$

by the continuity of $y^* \circ T^{-1}$ on its domain. It follows from the Uniform Boundedness theorem (Corollary 6.5) that

$$\sup\{||\phi_x||;\ x \in D(T^{-1}),\ ||x|| = 1\} < \infty,$$

That is,

$$\sup\{||T^{-1}x||;\ x \in D(T^{-1}),\ ||x|| = 1\} < \infty,$$

which means that T^{-1} is bounded on its domain.

11. Let $T : D(T) \subset X \to X$ have dense domain in the Banach space X. Prove:

(a) If the range $R(T^*)$ of T^* is *weak**-dense in X^*, then T is one-to-one.

Solution.

Let $x \in \ker(T)$ and $x^* \in R(T^*)$. Write $x^* = T^*y^*$ with $y^* \in D(T^*)$. Then $x^*x = (T^*y^*)x = y^*(Tx) = y^*0 = 0$. Since $R(T^*)$ is *weak**-dense in X^*, it follows that $x^*x = 0$ for all $x^* \in X^*$, and therefore $x = 0$ (cf. Corollary 5.7). This shows that $\ker(T) = \{0\}$, i.e., T is one-to-one.

(b) T^{-1} exists and is bounded (on its domain) iff $R(T^*) = X^*$.

Solution.

Suppose T^{-1} exists and is bounded (on its domain). Let $x^* \in X^*$, $x \in D(T)$, and denote $y = Tx$ $(\in R(T) = D(T^{-1}))$. By hypothesis, $x^* \circ T^{-1}$ is continuous on its domain. Hence $x^* \in D((T^{-1})^*)$ and $y^* := (T^{-1})^* x^* = x^* \circ T^{-1}$ on $D(T^{-1})$ $(= R(T))$. We have $y^* T x = y^* y = x^* T^{-1} T x = x^* x$, which shows that $y^* \in D(T^*)$ and $T^* y^* = x^*$. Hence $R(T^*) = X^*$.

Conversely, suppose $R(T^*) = X^*$. By Part (a), T is one-to-one, hence T^{-1} exists. Next, we apply Exercise 5(b) to T^* (the hypothesis in that exercise are satisfied by $T^* : D(T^*) \subset X^* \to X^*$: it is closed, by Exercise 9(a), and its range is all of X^* by hypothesis). Thus there exists a constant $c > 0$ such that, for each $x^* \in X^*$, there exists $y^* \in D(T^*)$ such that $x^* = T^* y^*$ and $||y^*|| \le c\,||x^*||$. Hence for all $x \in D(T^{-1})$

$$|x^* T^{-1} x| = |(T^* y^*) T^{-1} x| = |y^* x| \le ||y^*||\,||x|| \le c\,||x^*||\,||x||,$$

that is, $||T^{-1}||_{B(D(T^{-1}),X)} \le c$ (cf. Exercise 2).

12. Let X be a Banach space, and $T \in B(X)$. We say that T is *bounded below* if

$$\inf_{0 \ne x \in X} \frac{||Tx||}{||x||} > 0. \tag{14}$$

Prove:

(a) If T is bounded below, then it is one-to-one and has closed range.

Solution.

Denote the infimum in (14) by $1/c$. Then $||x|| \le c\,||Tx||$ for all $x \in X$. This trivially implies that $\ker(T) = \{0\}$. Next, if $y_n := Tx_n \in TX$ converge to y, then $||x_n - x_m|| \le c\,||T(x_n - x_m)|| \to 0$ as $n, m \to \infty$. If x denotes the limit of the Cauchy sequence $\{x_n\}$, then $Tx = \lim Tx_n = y$, which proves that TX is closed.

(b) T is non-singular (that is, invertible in $B(X)$) if and only if it is bounded below and T^* is one-to-one.

Solution.

Suppose T is non-singular. Then

$$\inf_{x \ne 0} \frac{||Tx||}{||x||} = \inf_{y \ne 0} \frac{||y||}{||T^{-1}y||} = \Big(\sup_{y \ne 0} \frac{||T^{-1}y||}{||y||}\Big)^{-1} = ||T^{-1}||^{-1} > 0. \tag{15}$$

Thus T is bounded below. If $x^* \in \ker(T^*)$, then $x^* \circ T = T^* x^* = 0$, and therefore $x^* = 0$ because T is onto. Hence T^* is one-to-one.

Conversely, suppose T is bounded below and T^* is one-to-one. By Part (a), TX is closed and T is one-to-one. If $x^* \in X^*$ satisfies $x^*(TX) = \{0\}$, this means that $T^*x^* = 0$, hence $x^* = 0$ by hypothesis. Therefore $TX = X$ by Corollary 5.4. Consequently T ($\in B(X)$) is one-to-one and onto, hence invertible in $B(X)$, by Corollary 6.11 ("continuity of the inverse").

Hilbert adjoint

13. Let X be a Hilbert space, and $T : D(T) \subset X \to X$ a linear operator with dense domain. The *Hilbert adjoint* T^* of T is defined in a way analogous to that of Exercise 9, through the Riesz representation:

$$D(T^*) := \{y \in X; \phi_y : x \to (Tx, y) \text{ is continuous on } D(T)\}.$$

Since $D(T)$ is dense, given $y \in D(T^*)$, there exists a unique vector in X, which we denote by T^*y, such that

$$(Tx, y) = (x, T^*y) \qquad (x \in D(T)). \tag{16}$$

Prove:

(a) If $T \in B(X)$, then $T^* \in B(X)$, $||T^*|| = ||T||$, $T^{**} = T$, and $(\alpha T)^* = \overline{\alpha} T^*$ for all $\alpha \in \mathbb{C}$. Also $I^* = I$.

Solution.

For each $y \in D(T^*)$, the continuous linear functional ϕ_y on the normed space $D(T)$ has a unique extension as a continuous linear functional on X (cf. Exercise 1, Chapter 4), and there exists therefore a unique vector, denoted T^*y, such that $\phi_y(x) = (x, T^*y)$ for all $x \in D(T)$ (cf. Theorem 1.37), which is precisely the identity (16).

If $T \in B(X)$, ϕ_y is continuous on X for all y; thus T^* is everywhere defined and clearly linear.
For any $0 \neq z \in X$, we get from Schwarz' inequality

$$||z|| = ((1/||z||)z, z) \leq \sup_{||x||=1} |(x, z)| \leq ||z||,$$

that is,

$$||z|| = \sup_{||x||=1} |(x, z)| \tag{17}$$

for all $z \in X$ (trivially for $z = 0$). By the operator norm's definition, (17) and (16),

$$||T^*|| = \sup_{||y||=1} ||T^*y|| = \sup_{||x||=||y||=1} |(x, T^*y)|$$

$$= \sup_{\cdots} |(Tx, y)| = \sup_{||x||=1} ||Tx|| = ||T||.$$

In particular $T^* \in B(X)$.
Many of the routine verifications to follow depend on the observation that if $A, B \in B(X)$ satisfy $(Ax, y) = (Bx, y)$ (or $(x, Ay) = (x, By)$) for all $x, y \in X$, then $A = B$ (indeed, taking $y = (A-B)x$ in the identity $((A-B)x, y) = 0$, one gets $||(A-B)x||^2 = 0$ for all x).
By (16), $(Tx, y) = (x, T^*y) = (T^{**}x, y)$ for all $x, y \in X$, hence $T^{**} = T$. If $\alpha \in \mathbb{C}$, $(x, \overline{\alpha}T^*y) = \alpha(x, T^*y) = \alpha(Tx, y) = ((\alpha T)x, y) = (x, (\alpha T)^*y)$ for all x, y, hence $(\alpha T)^* = \overline{\alpha}T^*$. The fact $I^* = I$ is trivial.

(b) If $S, T \in B(X)$, then $(S+T)^* = S^* + T^*$ and $(ST)^* = T^*S^*$.

Solution. For all $x, y \in X$, we have by (16)

$$(x, (S+T)^*y) = ((S+T)x, y) = (Sx, y)+(Tx, y) = (x, S^*y)+(x, T^*y) = (x, (S^*+T^*)y),$$

and therefore $(S+T)^* = S^* + T^*$ by the observation in Part (a). Also

$$(x, (ST)^*y) = (STx, y) = (Tx, S^*y) = (x, T^*S^*y),$$

hence $(ST)^* = T^*S^*$.

(c) $T \in B(X)$ is called a *normal* operator if $T^*T = TT^*$. Prove that T is normal iff

$$(T^*x, T^*y) = (Tx, Ty) \qquad (x, y \in X). \tag{18}$$

Solution.

By the observation in Part (a), T is normal iff $(TT^*x, y) = (T^*Tx, y)$ for all $x, y \in X$; by (16), this is equivalent to (18).

(d) If $T \in B(X)$ is normal, then $||T^*x|| = ||Tx||$ and $||T^*Tx|| = ||T^2x||$ for all $x \in X$. Conclude that $||T^*T|| = ||T^2||$ and $||T^2|| = ||T||^2$.

Solution.

Taking $y = x$ in (18), we get $||T^*x|| = ||Tx||$ for all $x \in X$. Replacing x by Tx in this last identity, we obtain $||T^*Tx|| = ||T^2x||$. Hence $||T^*T|| = \sup_{||x||=1} ||T^*Tx|| = \sup_{||x||=1} ||T^2x|| = ||T^2||$. Finally, using Schwarz' inequality, we get

$$||T||^2 = \sup_{||x||=1} ||Tx||^2 = \sup(Tx, Tx) = \sup(T^*Tx, x)$$

$$\leq \sup ||T^*Tx|| = ||T^*T|| = ||T^2|| \leq ||T||^2.$$

Hence $||T||^2 = ||T^2||$.

14. Let X be a Hilbert space, and $T : D(T) \subset X \to X$ be a linear operator. T is *symmetric* if $(Tx, y) = (x, Ty)$ for all $x, y \in D(T)$. Prove that if T is symmetric and everywhere defined, then $T \in B(X)$ and $T = T^*$. (Hint: Corollary 6.13.)

Solution.

We show that a symmetric everywhere defined operator T is *closed*, hence $\in B(X)$ by the Closed Graph theorem (Corollary 6.13). Let $x_n \in X$ converge to x and Tx_n converge to y. Then for all $z \in X$,

$$(y, z) = \lim_n (Tx_n, z) = \lim_n (x_n, Tz) = (x, Tz) = (Tx, z),$$

hence $y = Tx$, and T is indeed closed. By (16), and the symmetry condition, $(x, T^*y) = (Tx, y) = (x, Ty)$ for all $x, y \in X$, hence $T^* = T$ (cf. observation in Exercise 13(a)).

15. Let X be a Hilbert space, and $B : X \times X \to \mathbb{C}$ be a sesquilinear form such that

$$|B(x, y)| \leq M||x||\,||y|| \quad \text{and} \quad B(x, x) \geq m||x||^2 \tag{19}$$

for all $x, y \in X$, for some constants $M < \infty$ and $m > 0$. Prove that there exists a unique non-singular $T \in B(X)$ such that $B(x, y) = (x, Ty)$ for all $x, y \in X$. Moreover,

$$||T|| \leq M \quad \text{and} \quad ||T^{-1}|| \leq 1/m. \tag{20}$$

(This is the *Lax-Milgram theorem.*) Hint: apply Theorem 1.37 to get T; show that $R(T)$ is closed and dense (cf. Theorem 1.36), and apply Corollary 6.11.

Solution.

By (19), $B(\cdot, y)$ is a continuous linear functional on X, for each given $y \in X$. By the "little" Riesz representation theorem (Theorem 1.37), there exists a unique vector in X (which we denote by Ty) such that $B(x, y) = (x, Ty)$ for all x (and y). The linearity of T follows from the uniqueness of the representation and the conjugate linearity of $B(x, \cdot)$. We have (cf. (17))

$$||T|| = \sup_{||x||=||y||=1} |(x, Ty)| = \sup_{\cdots} |B(x, y)| \leq M.$$

Thus $T \in B(X)$.

For all $x \neq 0$, we have by (19) and Schwarz' inequality

$$m\,||x||^2 \leq (x, Tx) \leq ||Tx||\,||x||,$$

hence

$$\inf_{x\neq 0} \frac{||Tx||}{||x||} \geq m.$$

If $T^*x = 0$ for some x, then $m\,||x||^2 \leq (x, Tx) = (T^*x, x) = (0, x) = 0$, hence $x = 0$. Thus T is *bounded below* and T^* is one-to-one. By Exercise 12(b) in its Hilbert space version, $\exists T^{-1} \in B(X)$ and $||T^{-1}||$ is equal to the reciprocal of the above infimum, hence is $\leq 1/m$.

16. Let X, Y be normed spaces, and $T : X \to Y$ be linear. Prove that T is an open map iff $T\overline{B}_X(0,1)$ contains $\overline{B}_Y(0,r)$ for some $r > 0$. When this is the case, T is *onto*.

Solution.

If T is an open (linear) map, $TB_X(0,1)$ is an open neighbourhood of 0 in Y, hence contains some ball $B_Y(0,s)$. Then for any $0 < r < s$,

$$\overline{B}_Y(0,r) \subset B_Y(0,s) \subset TB_X(0,1) \subset T\overline{B}_X(0,1).$$

Conversely, suppose the "balls condition" is satisfied by T, let $V \subset X$ be open, and let $y \in TV$. Then $y = Tx$ for some $x \in V$, and since V is open, there exists $s > 0$ such that $B_X(x,s) \subset V$. For any $0 < c < s$,

$$x + c\overline{B}_X(0,1) = \overline{B}_X(x,c) \subset B_X(x,s) \subset V,$$

hence by linearity of T,

$$Tx + cT\overline{B}_X(0,1) \subset TV, \tag{21}$$

and therefore, by the "balls condition" and (21),

$$\overline{B}_Y(Tx, cr) = Tx + cr\overline{B}_Y(0,1) \subset Tx + cT\overline{B}_X(0,1) \subset TV.$$

This proves that TV is open in Y, and we conclude that T is an open map. In particular, TX is an open subspace of Y, hence coincides with Y (cf. solution of Exercise 5(a)).

17. Let X be a Banach space, Y a normed space, and $T \in B(X,Y)$. Suppose the closure of $T\overline{B}_X(0,1)$ contains some ball $\overline{B}_Y(0,r)$. Prove that T is open. (Hint: adapt the *proof* of Lemma 2 in the proof of Theorem 6.9, and use Exercise 16.)

Solution.

Let r be as in the hypothesis. By homogenity of T, for all $n = 0, 1, 2, \cdots$,

$$\overline{B}_Y(0, r/2^{n+1}) \subset \text{closure}\Big(T\overline{B}_X(0, 1/2^{n+1})\Big). \tag{22}$$

Claim:

$$\overline{B}_Y(0, r/2) \subset T\overline{B}_X(0, 1). \tag{23}$$

Let y be in the set on the left hand side of (23). By (22) for $n = 0$, there exists $x_0 \in \overline{B}_X(0, 1/2)$ such that

$$||y - Tx_0|| \le r/4. \tag{24}$$

Suppose we have found vectors $x_k \in \overline{B}_X(0, 1/2^{k+1})$ ($k = 0, \cdots, n-1$, for some $n \ge 1$) such that

$$||y - T(x_0 + \cdots + x_{n-1})|| \le r/2^{n+1}. \tag{25}$$

(By (24), this is true for $n = 1$.) By (25) and (22), there exists $x_n \in \overline{B}_X(0, 1/2^{n+1})$ such that

$$||[y - T(x_0 + \cdots + x_{n-1})] - Tx_n|| \le r/2^{n+2}.$$

Thus we obtain inductively a sequence $\{x_n\}_{n=0}^{\infty}$ such that $||x_n|| \le 1/2^{n+1}$ for $n = 0, 1, \cdots$, which satisfies (25) for all $n \ge 1$. The series $\sum_{k \ge 0} x_k$ converges in X, since it converges absolutely ($\sum_{k\ge 0} ||x_k|| \le \sum 1/2^{k+1} = 1$) and X is complete (cf. Theorem 6.15). Let x denote the sum of the series. Then $||x|| \le \sum_k ||x_k|| \le 1$ and by (25) and the continuity of T and of the norm, $||y - Tx|| = 0$. Hence $y \in T\overline{B}_X(0, 1)$, and the claim is proved.
By Exercise 16, it follows that T is open.

18. Let X be a Banach space, and let $P \in B(X)$ be such that $P^2 = P$. Such an operator is called a *projection*. Verify:

(a) $I - P$ is a projection (called *the complementary projection*).

(Trivial: $(I - P)^2 = I - 2P + P^2 = I - P$.)

(b) The ranges PX and $(I-P)X$ are *closed* subspaces such that $X = PX \oplus (I-P)X$. Moreover $PX = \ker(I - P) = \{x; Px = x\}$ and $(I - P)X = \ker(P)$.

Solution.

If $y \in PX$, then $y = Px$ for some $x \in X$, hence $Py = P^2x = Px = y$ (conversely, if $Py = y$, then $y \in PX$ trivially). Thus

$$PX = \{y \in X;\ Py = y\} = \{y \in X;\ (I - P)y = 0\} = \ker(I - P). \tag{26}$$

In particular, PX is closed, as the kernel of the continuous operators $I-P$. Applying (26) to the projection $I-P$, we get $(I-P)X = \ker(I-(I-P)) = \ker(P)$. Finally, $PX \cap [(I-P)X] = [\ker(I-P)] \cap [\ker(P)] = \{0\}$.

(c) Conversely, if Y, Z are closed subspaces of X such that $X = Y \oplus Z$ ("complementary subspaces"), and $P : X \to Y$ is defined by $P(y+z) = y$ for all $y \in Y$, $z \in Z$, then P is a projection with $PX = Y$.

Solution.

Since the representation $x = y + z$ of each $x \in X$ (with $y \in Y$ and $z \in Z$) is unique (because $X = Y \oplus Z$) the operator P is well-defined and linear. We have $P^2(y+z) = Py = y = P(y+z)$ for all $y \in Y$ and $z \in Z$. Thus $P^2 = P$ and P is onto Y. Suppose $y_n + z_n \to x$ and $P(y_n + z_n) \to v$. Then $y_n \to v$, and therefore $v \in Y$ (since Y is closed). Also $z_n = (y_n + z_n) - y_n \to x - v$, hence $x - v \in Z$ (since Z is closed), and therefore $x = v + (x - v)$ is the unique representation of x according to the direct sum decompsition $X = Y \oplus Z$. It follows that $Px = v$, which proves that $P : X \to X$ is a closed operator (defined on the Banach space X). By the Closed Graph theorem (Corollary 6.13), $P \in B(X)$, and we conclude that P is a projection, and $PX = Y$.

(d) If Y, Z are closed subspaces of X such that $Y \cap Z = \{0\}$, then $Y + Z$ is closed iff there exists a positive constant c such that

$$||y|| \leq c||y+z|| \quad \text{for all } y \in Y,\ z \in Z. \tag{27}$$

Solution.

If $V := Y \oplus Z$ is closed, it is a Banach space, and Y, Z are complementary subspaces of V. Let P be the projection of V onto Y as in Part (c). Since P is bounded, we have $||y|| = ||P(y+z)|| \leq ||P||\,||y+z||$ for all $y \in Y$ and $z \in Z$, hence (27) is valid with $c = ||P|| > 0$ if $P \neq 0$ (in case $P = 0$, $Y = \{0\}$ and (27) is trivial, with $c > 0$ arbitrary).

Conversely, suppose (27) is valid. Let $v_n \in V$ converge to $x \in X$. Represent $v_n = y_n + z_n$ with $y_n \in Y$ and $z_n \in Z$. By (27), $||y_n - y_m|| \leq c||v_n - v_m|| \to 0$ as $n, m \to \infty$. Since Y is a *closed* subspace of the *Banach* space X, there exists $y \in Y$ such that $y_n \to y$. Hence $z_n = v_n - y_n \to x - y$. Since Z is closed, it follows that $x - y \in Z$, and therefore $x = y + (x - y) \in Y \oplus Z = V$. This proves that V is closed.

19. Let X, Y be Banach spaces, and let $\{T_n\}_{n \in \mathbb{N}} \subset B(X,Y)$ be Cauchy in the s.o.t. (that is, $\{T_n x\}$ is Cauchy for each $x \in X$). Prove that $\{T_n\}$ is convergent in $B(X,Y)$ in the strong operator topology.

Solution.

For each $x \in X$, $\{T_n x\}$ is a Cauchy sequence in the Banach space Y; denote its limit in Y by Tx. The map $x \to Tx$ is linear, by the properties of limits. For each x in the Banach space X, $\sup_n ||T_n x|| < \infty$. By the Uniform Boundedness theorem (Corollary 6.5), $\sup_n ||T_n|| := M < \infty$. Thus $||T_n x|| \le ||T_n|| \, ||x|| \le M \, ||x||$ for all $x \in X$. Letting $n \to \infty$, we get $||Tx|| \le M \, ||x||$ for all x, that is, $T \in B(X)$, and by definition, $T_n x \to Tx$ for all x, i.e., $T_n \to T$ in the s.o.t.

20. Let X, Y be Banach spaces and $T \in B(X,Y)$. Prove that T is one-to-one with closed range iff there exists a positive constant c such that $||Tx|| \ge c||x||$ for all $x \in X$. In that case, $T^{-1} \in B(TX, X)$.

Solution.

The condition on the norms means that T is bounded below, and it was shown in Exercise 12(a) (for the case $Y = X$, but the same proof applies in general) that bounded below operators are one-to-one with closed range. Considering then T as a one-to-one and onto operator in $B(X, TX)$ (with X and TX Banach spaces), it follows from Corollary 6.11 that $T^{-1} \in B(TX, X)$.
Conversely, if T is one-to-one with closed range, we just observed that $T^{-1} \in B(TX, X)$; denote its (non-zero!) norm (in $B(TX, X)$) by $1/c$. Then for all $x \in X$,

$$c \, ||x|| = c \, ||T^{-1}(Tx)|| \le c \, ||T^{-1}|| \, ||Tx|| = ||Tx||.$$

21. Let X be a Banach space, and let $C \in B(X)$ be a *contraction*, that is, $||C|| \le 1$. Prove:

(a) $e^{t(C-I)}$ (defined by means of the usual series) is a contraction for all $t \ge 0$.

Solution.

For any $T \in B(X)$, the series $\sum_{n \ge 0} T^n/n!$ converges absolutely (because $||T^n/n!|| \le ||T||^n/n!$ and $\sum ||T||^n/n! = e^{||T||} < \infty$), and therefore it converges in $B(X)$ (since $B(X)$ is a Banach space; cf. Theorem 6.15). If $S, T \in B(X)$ commute, one verifies as in the scalar case that

$$e^{S+T} = e^S e^T. \tag{28}$$

With $S = tC$ and $T = -tI$, (28) gives

$$e^{t(C-I)} = e^{-tI} e^{tC} = e^{-t} e^{tC} \tag{29}$$

hence $||e^{t(C-I)}|| = e^{-t} ||e^{tC}|| \le e^{-t} e^{||tC||} \le 1$ for all $t \ge 0$.

(b) $||C^m x - x|| \le m\,||Cx - x||$ for all $m \in \mathbb{N}$.

Solution.

For $m = 1$, (b) is an identity. If $m \ge 2$,

$$C^m - I = (C^{m-1} + \cdots + I)(C - I),$$

hence

$$||C^m x - x|| = ||(C^{m-1} + \cdots + I)(Cx - x)|| \le (||C||^{m-1} + \cdots + 1)||Cx - x|| \le m\,||Cx - x||.$$

(c) Let $Q_n := e^{n(C-I)} - C^n$ $(n \in \mathbb{N})$. Then

$$||Q_n x|| \le e^{-n} \sum_{k=0}^{\infty} (n^k/k!)||C^{|k-n|}x - x|| \tag{30}$$

for all $n \in \mathbb{N}$. Hint: note that $C^n x = e^{-n} \sum_k (n^k/k!) C^n x$; break the ensuing series for $Q_n x$ into series over $k \le n$ and over $k > n$.

Solution.

By (29) with $t = n$,

$$Q_n = e^{-n} \sum_{k=0}^{\infty} (n^k/n!)(C^k - C^n). \tag{31}$$

For $k \le n$, we have for all $x \in X$

$$||(C^k - C^n)x|| = ||C^k(x - C^{n-k}x)|| \le ||C||^k ||C^{n-k}x - x|| \le ||C^{n-k}x - x||.$$

For $k > n$, we have

$$||(C^k - C^n)x|| = ||C^n(C^{k-n}x - x)|| \le ||C||^n ||C^{k-n}x - x|| \le ||C^{k-n}x - x||.$$

Hence

$$||(C^k - C^n)x|| \le ||C^{|k-n|}x - x||$$

for all k, and (30) follows then from (31).

(d) $||Q_n x|| \le \sum_{k \ge 0} e^{-n}(n^k/k!)|k - n|\,||Cx - x||$.

Solution.

This follows trivially from (30) and Part (b).

(e) $||Q_n x|| \leq \sqrt{n}||(C-I)x||$ for all $n \in \mathbb{N}$. Hint: consider the *Poisson probability measure* μ (with "parameter" n) on $\mathbb{P}(\mathbb{N} \cup \{0\})$, defined by $\mu(\{k\}) = e^{-n}n^k/k!$; apply Schwarz' inequality in $L^2(\mu)$ and Part (d) to get the inequality

$$||Q_n x|| \leq ||k-n||_{L^2(\mu)}||Cx - x|| = \sqrt{n}||Cx - x||. \tag{32}$$

Solution. Since μ is a probability measure on $\mathbb{N} \cup \{0\}$, it follows from Schwarz' inequality that $||\cdot||_1 \leq ||\cdot||_2$ on the measure space specified in the hint. By Part (d),

$$||Q_n x|| \leq ||\{k-n\}_k||_1 \, ||Cx - x|| \leq ||\{k-n\}_k||_2 ||Cx - x||.$$

The Poisson distribution with parameter n has the "expectation" n; its "standard deviation" is then $||\{k-n\}_k||_2$, and is known to be equal to $\sqrt{n}$ (cf. Example I.3.9, page 302).

(f) Let $F : [0,\infty) \to B(X)$ be contraction-valued. For $t > 0$ fixed, set $A_n := (n/t)[F(t/n) - I]$, $n \in \mathbb{N}$. Suppose $\sup_n ||A_n x|| < \infty$ for all x in a dense subspace D of X. Then

$$\lim_{n\to\infty} ||e^{tA_n}x - F(t/n)^n x|| = 0 \tag{33}$$

for all $t > 0$. Hint: by Part (a), $||e^{tA_n}|| \leq 1$, and therefore $||e^{tA_n} - F(t/n)^n|| \leq 2$. By Part (e) with $C = F(t/n)$, the limit in (33) is 0 for all $x \in D$.

Solution.

Fix $t > 0$. For $C = F(t/n)$, we have $Q_n = e^{tA_n} - F(t/n)^n$. As observed in the hint,

$$||Q_n|| \leq 2 \quad \text{for all } n \in \mathbb{N}. \tag{34}$$

By Part (e),

$$||Q_n x|| \leq \sqrt{n}||[F(t/n) - I]x|| = \frac{t}{\sqrt{n}}||A_n x||.$$

Hence $Q_n x \to 0$ for all $x \in D$. By (34) and the density of D in X, it follows that $Q_n x \to 0$ for all $x \in X$ (given $\epsilon > 0$ and $x \in X$, pick $y \in D$ such that $||x - y|| < \epsilon/3$, and let n_0 be such that $||Q_n y|| < \epsilon/3$ for all $n > n_0$. Then for all $n > n_0$,

$$||Q_n x|| \leq ||Q_n(x-y)|| + ||Q_n y|| < ||Q_n|| \, ||x-y|| + \epsilon/3 < \epsilon).$$