.

CHAPTER 5

DUALITY

1. A Banach space X is *separable* if is contains a countable dense subset. Prove that if X^* is separable, then X is separable (but the converse is false). Hint: let $\{x_n^*\}$ be a sequence of unit vectors dense in the unit sphere of X^*. Pick unit vectors x_n in X such that $|x_n^* x_n| > 1/2$. Use Corollary 5.5 to show that span$\{x_n\}$ is dense in X; the same is true when the scalars are complex numbers with *rational* real and imaginary parts.

Solution.

If $X = \{0\}$, there is nothing to prove. So assume $X \neq \{0\}$, so that also $X^* \neq \{0\}$, by Corollary 5.7. Suppose X^* is separable; let then $y_n^* \neq 0$, $n = 1, 2, \cdots$, be dense in X^*. Denote $x_n^* = y_n^*/(||y_n^*||)$, and let S^* be the unit sphere of X^*. If $x^* \in S^*$, there exist $y_{n_k}^*$ converging to x^* in X^*. By continuity of the norm, $||y_{n_k}^*|| \to 1$, and therefore $x_{n_k}^*$ converge to x^*. Thus the unit vectors x_n^* are dense in S^*.
Since $\sup_{||x||=1} |x_n^* x| := ||x_n^*|| = 1$, there exist unit vectors x_n such that $|x_n^* x_n| > 1/2$. Suppose span $\{x_n\}$ is *not* dense in X. By Corollary 5.5, there exists $x^* \in S^*$ such that $x^* x_n = 0$ for all n. But then

$$||x^* - x_n^*|| \geq |(x^* - x_n^*)x_n| = |x_n^* x_n| > 1/2,$$

and therefore no subsequence of $\{x_n^*\}$ converges to x^*, contradicting the density of $\{x_n^*\}$ in S^*. Thus span $\{x_n\}$ is dense in X. Since $\mathbb{Q} + i\mathbb{Q}$ (where $\mathbb{Q}$ denotes the field of rational numbers) is dense in $\mathbb{C}$, it follows that the span Z of $\{x_n\}$ over the field $\mathbb{Q} + i\mathbb{Q}$ is dense in X, and Z is clearly countable. Hence X is separable.
(Given $\epsilon > 0$ and $x \in X$, there exist $\alpha_1, \cdots, \alpha_p \in \mathbb{C}$ and $x_{n_1}, \cdots, x_{n_p} \in \{x_n\}$ such that $||x - \sum_{j=1}^{p} \alpha_j x_{n_j}|| < \epsilon/2$, and there exist $\beta_1, \cdots, \beta_p \in \mathbb{Q} + i\mathbb{Q}$ such that $|\alpha_j - \beta_j| < \epsilon/(2p)$ for $j = 1, \cdots, p$. Then

$$||x - \sum_{j=1}^{p} \beta_j x_{n_j}|| \leq ||x - \sum_{j=1}^{p} \alpha_j x_{n_j}|| + \sum_{j=1}^{p} |\alpha_j - \beta_j| < \epsilon.)$$

The following example shows that the converse of the statement in the exercise is false in general. Take $X = l^1$. By the solution of Exercise 7, Chapter 4, each $x \in l^1$ has the representation $x = \sum x(m)e_m$, which converges in l^1. Thus $\text{span}_{\mathbb{C}}\{e_m\}$ is dense in l^1. As observed above, it follows that the countable set $\mathrm{span}_{\mathbb{Q}+i\mathbb{Q}}\{e_m\}$ is dense in l^1, that is, l^1 is separable. However $(l^1)^*$ (which is isometrically isomorphic to l^∞, cf. Exercise 7) is not separable. Indeed, assume that there exists a sequence of elements $x_n \in l^\infty$ which is dense in l^∞. Define $y \in l^\infty$ by $y(m) = x_m(m) + 1$. Then for all $m \in \mathbb{N}$,

$$||y - x_m||_\infty \geq |y(m) - x_m(m)| = 1,$$

and therefore no subsequence of $\{x_n\}$ converges to y, contradiction.

2. Consider the normed space

$$C_c^n(\mathbb{R}) := \{f \in C_c(\mathbb{R});\ f^{(k)} \in C_c(\mathbb{R}),\ k = 1, ..., n\},$$

with the norm

$$||f|| = \sum_{k=0}^{n} ||f^{(k)}||_u.$$

Given $\phi \in [C_c^n(\mathbb{R})]^*$, prove that there exist complex Borel measures μ_k $(k = 0, ..., n)$ such that

$$\phi(f) = \sum_k \int_{\mathbb{R}} f^{(k)} d\mu_k$$

for all $f \in C_c^n(\mathbb{R})$. Hint: consider the subspace

$$Z = \{[f, f', ..., f^{(n)}];\ f \in C_c^n(\mathbb{R})\}$$

of $C_c \times ... \times C_c$ $(n+1$ times). Define ψ on Z by $\psi([f, f', ..., f^{(n)}]) = \phi(f)$. (Cf. Exercise 3, Chapter 4.)

Solution.

Define Z and ψ as in the hint. Then ψ is linear, and

$$|\psi([f, f', \cdots, f^{(n)}]) = |\phi(f)| \leq ||\phi||\,||f|| = ||\phi||\,||[f, f', \cdots, f^{(n)}]||,$$

where the last norm is the norm in $C_c \times \cdots \times C_c$ $(n+1$ times). Thus $\psi \in Z^*$ (when Z is considered as a subspace of the latter product space). By the Hahn-Banach theorem (Theorem 5.2), there exists a functional $\tilde{\psi} \in (C_c \times \cdots \times C_c)^*$ such that $\tilde{\psi}|_Z = \psi$ and $||\tilde{\psi}|| = ||\psi|| \leq ||\phi||$. By Exercise 3 in Chapter 4, there exist unique functionals $x_k^* \in C_c^*$ $(k = 0, \cdots, n)$ such that

$$\tilde{\psi}([f_0, \cdots, f_n]) = \sum_{k=0}^{n} x_k^* f_k \tag{1}$$

for all $[f_0, \cdots, f_n] \in C_c \times \cdots \times C_c$, and $\max_k ||x_k^*|| = ||\tilde{\psi}||$ $(\leq ||\phi||)$.
By the Riesz Representation theorem (Theorem 4.9), there exist unique measures

$\mu_k \in M_r(\mathbb{R})$ such that $x_k^* f = \int_{\mathbb{R}} f\, d\mu_k$ for all $f \in C_c$, and $||\mu_k|| = ||x_k^*||$. Therefore by (1)

$$\tilde{\psi}([f_0, \cdots, f_n]) = \sum_{k=0}^{n} \int_{\mathbb{R}} f_k d\mu_k$$

for all $[f_0, \cdots, f_n] \in C_c \times \cdots \times C_c$. In particular, for all $f \in C_c^n(\mathbb{R})$,

$$\phi(f) = \psi([f, f', \cdots, f^{(n)}]) = \sum_{k=0}^{n} \int_{\mathbb{R}} f^{(k)} d\mu_k,$$

with $\mu_k \in M_r(\mathbb{R})$ and $||\mu_k|| \leq ||\phi||$ for all k.

3. Let X be a Banach space, and let $\Gamma \subset X^*$ be (norm) bounded and *weak**-closed. Prove:

(a) Γ is *weak**-compact.

Solution.

By Alaoglu's theorem (Theorem 5.24), the strongly closed unit ball $B^* = \{x^* \in X^*; ||x^*|| \leq 1\}$ is *weak**-compact. Since Γ is norm bounded (that is, there exists a positive number K such that $||x^*|| \leq K$ for all $x^* \in \Gamma$), we have $\Gamma \subset KB^*$. The set KB^* is *weak**-compact (because multiplication by a non-zero scalar is a homeomorphism of any t.v.s. onto itself), and therefore its *weak**-closed subset Γ is *weak**-compact.

(b) If Γ is also convex, then it is the *weak**-closed convex hull of its extremal points.

Solution.

If Γ is also convex, it is a compact convex set in the locally convex t.v.s. X^* with the *weak**-topology. By Corollary 5.31, it is the *weak**-closed convex hull of its extremal points.

4. Let X, Y be normed spaces and $T \in B(X, Y)$. Prove that T is continuous with respect to the weak topologies on X and Y, and $T^* : Y^* \to X^*$ is continuous with respect to the *weak**-topologies on Y^* and X^*. (Recall that the Banach adjoint T^* of $T \in B(X)$ is defined by means of the identity $(T^*y^*)x = y^*(Tx)$, $x \in X,\ y^* \in Y^*$.)

Solution.

(Let $T \in B(X, Y)$. For each $y^* \in Y^*$, we have $y^* \circ T \in X^*$, and the map $T^* : y^* \to y^* \circ T$ is clearly linear. By Corollary 5.8,

$$||T|| := \sup_{||x||=1} ||Tx|| = \sup\{|y^*(Tx)|\, ; x \in X,\, y^* \in Y^*,\, ||x|| = ||y^*|| = 1\}$$

$$= \sup_{||y^*||=1} \sup_{||x||=1} |(T^*y^*)x| = \sup_{||y^*||=1} ||T^*y^*|| := ||T^*||.$$

Thus $T^* \in B(Y^*, X^*)$ and the map $T \to T^*$ is is a (linear) isometry of $B(X, Y)$ into $B(Y^*, X^*)$.)

Let $T \in B(X, Y)$. If $\{x_\alpha\}$ is a net in X converging weakly to x, then for all $y^* \in Y^*$

$$y^*Tx_\alpha = (T^*y^*)x_\alpha \to (T^*y^*)x = y^*Tx.$$

Thus $Tx_\alpha \to Tx$ weakly in Y. This proves that T is a continuous map from X to Y with their weak topologies.

Similarly, if $y^*_\alpha \to y^*$ *weak** in Y^*, then for all $x \in X$,

$$(T^*y^*_\alpha)x = y^*_\alpha(Tx) \to y^*(Tx) = (T^*y^*)x,$$

that is, $T^*y^*_\alpha \to T^*y^*$ *weak** in X^*. Hence T^* is a continuous map of Y^* into X^* with their *weak** topologies.

5. Let $p, q \in [1, \infty]$ be conjugate exponents, let $(X, \mathcal{A}, \mu)$ be a σ-finite positive measure space, and let g be a complex measurable function on X such that $||g||_q \le M$ for some constant M. Then $||fg||_1 \le M\,||f||_p$ for all $f \in L^p(\mu)$ (by Theorems 1.26 and 1.33). *Prove the converse!*

Solution.

Let g be a complex measurable function on X, such that $||fg||_1 \le M\,||f||_p$ for all $f \in L^p(\mu)$ (for some constant M). Then $||g||_q \le M$.

Case $p = \infty$. This case is trivial: take $f = 1$ ($\in L^\infty(\mu)$)!

Case $p < \infty$. Since $fg \in L^1(\mu)$ for all $f \in L^p(\mu)$, the map

$$\phi : f \in L^p(\mu) \to \int fg\, d\mu \in \mathbb{C}$$

is a well-defined linear functional on $L^p(\mu)$, and by hypothesis, $|\phi(f)| \le ||fg||_1 \le M\,||f||_p$ for all $f \in L^p(\mu)$. Hence $\phi \in (L^p(\mu))^*$ and $||\phi|| \le M$. By Theorem 4.6, there exists a unique $h \in L^q(\mu)$ such that $\phi(f) = \int fh\, d\mu$ for all $f \in L^p(\mu)$, and $||h||_q = ||\phi|| \le M$. Hence $\int f(g-h)\, d\mu = 0$ for all $f \in L^p(\mu)$. If $f \in L^p(\mu)$, also $I_E f \in L^p(\mu)$ for all $E \in \mathcal{A}$ (since $||I_E f||_p \le ||f||_p$). Therefore $u := f(g-h)$ is an $L^1(\mu)$-function (by the hypothesis and Theorems 1.26 and 1.33) such that $\int_E u\, d\mu = 0$ for all $E \in \mathcal{A}$. By Proposition 1.22, it follows that $u = 0$ a.e., for all $f \in L^p(\mu)$. Let

$X_k \in \mathcal{A}$ be such that $\mu(X_k) < \infty$ and $\bigcup_k X_k = X$. Taking $f = I_{X_k}$ $(\in L^p(\mu))$, the preceding conclusion implies that $g = h$ a.e. on X_k for each k, hence $g = h$ a.e. on X, and therefore $||g||_q = ||h||_q \leq M$.

Uniform convexity

6. Let X be a normed space, and let B and S denote its closed unit ball and its unit sphere, respectively. We say that X is *uniformly convex* (u.c.) if for each $\epsilon > 0$ there exists $\delta = \delta(\epsilon) > 0$ such that $||x - y|| < \epsilon$ whenever $x, y \in B$ are such that $||(x+y)/2|| > 1 - \delta$. Prove:

(a) X is u.c. iff whenever $x_n, y_n \in B$ are such that $||x_n + y_n|| \to 2$, it follows that $||x_n - y_n|| \to 0$.

Solution.

Suppose the ϵ, δ property holds in B, let $\epsilon > 0$ be given, and let $\delta = \delta(\epsilon)$ be the associated δ. If x_n, $y_n \in B$ are such that $||x_n + y_n|| \to 2$, there exists $n_0 \in \mathbb{N}$ such that $||x_n + y_n|| > 2 - 2\delta$ for all $n > n_0$, i.e., $||(x_n + y_n)/2|| > 1 - \delta$, hence $||x_n - y_n|| < \epsilon$ for all $n > n_0$, that is $||x_n - y_n|| \to 0$.
Conversely, suppose the above "sequential property" is valid in B, and the ϵ, δ property fails. Thus there exists $\epsilon > 0$, such that for all $n \in \mathbb{N}$, there exist x_n, $y_n \in B$ such that $||(x_n+y_n)/2|| > 1-1/n$, but $||x_n - y_n|| \geq \epsilon$. We have $2-2/n < ||x_n+y_n|| \leq 2$, so that $||x_n + y_n|| \to 2$, but $\lim_n ||x_n - y_n|| \neq 0$ (since $||x_n - y_n|| \geq \epsilon$ for all n), contradiction.

(b) Every inner product space is u.c.

Solution.

Let X be an inner product space, and let x_n, $y_n \in B$ satisfy $||x_n + y_n|| \to 2$. By the parallelogram identity (cf. Section 1.6, (8)),

$$0 \leq ||x_n - y_n||^2 = 2||x_n||^2 + 2||y_n||^2 - ||x_n + y_n||^2 \leq 4 - ||x_n + y_n||^2 \to 0.$$

Hence $||x_n - y_n|| \to 0$, and therefore X is u.c., by Part (a).

(c) Let X be a u.c. normed space and $\{x_n\} \subset X$. Then $x_n \to x \in X$ strongly iff $x_n \to x$ weakly and $||x_n|| \to ||x||$. (Hint: suppose $x_n \to x$ weakly and $||x_n|| \to ||x||$. We may assume that $x_n, x \in S$. Pick $x_0^* \in X^*$ such that $x_0^* x = 1 = ||x_0^*||$, cf. Corollary 5.7.)

Solution.

If $x_n \to x$ strongly, then trivially $x_n \to x$ weakly and $||x_n|| \to ||x||$ (in any normed space!), by the continuity of each $x^* \in X^*$ and of the norm.
Suppose X is u.c., and $\{x_n\} \subset X$ converges weakly to x, and $||x_n|| \to ||x||$. If $x = 0$, there is nothing to prove (because $||x_n - 0|| = ||x_n|| \to 0$, by the norms property alone). Assume then that $x \neq 0$. Since $||x_n|| \to ||x|| > 0$, we may assume without loss of generality that $||x_n|| > 0$ for all n. Let $y_n = x_n/||x_n||$ and $y = x/||x||$. Then y_n, y are unit vectors, and $y_n \to y$ weakly (because for all $x^* \in X^*$, $x^*y_n = x^*x_n/||x_n|| \to x^*x/||x|| = x^*y$.)
By Corollary 5.7, there exists $x_0^* \in X^*$ such that

$$x_0^* y = 1 = ||x_0^*||.$$

Hence

$$2 \geq ||y_n + y|| \geq |x_0^*(y_n + y)| = |x_0^* y_n + 1| \to |x_0^* y + 1| = 2.$$

Hence $||y_n + y|| \to 2$, and therefore, by uniform convexity and Part (a) with the sequence $\{y_n\}$ and the constant sequence $\{y\}$, it follows that $||y_n - y|| \to 0$. Consequently $x_n = ||x_n|| y_n \to ||x|| y = x$ strongly in X, by the continuity of the map $[\alpha, x] \to \alpha x$ from $\mathbb{C} \times X$ to X.

(d) The "distance theorem" (Theorem 1.35) is valid for a u.c. Banach space X.

Solution.

Statement: *Let K be a non-empty closed subset of the u.c. Banach space X, such that $(x+y)/2 \in K$ whenever $x, y \in K$. Let $x \in X$ and $d := d(x, K)$. Then there exists a unique $k \in K$ such that $d = d(x, k)$.*

Proof. We may assume $d > 0$ (otherwise $x \in K$, and the claim is trivial).
There exist $y_n \in K$ such that

$$(d \leq\ )\ ||x - y_n|| \leq d + 1/n. \tag{2}$$

Let $\epsilon > 0$, and let $\delta = \delta(\epsilon)$ as in the definition of uniform convexity (since any $\delta' < \delta$ yields the same conclusion, we may assume that $\delta < 1/2$). Let $0 < r < d\delta/(1-\delta)$ $(< d)$, and define $x_n = (x - y_n)/(d + r)$. Consider the sequence $\{x_n\}$ for $n > 1/r$. We have $||x_n|| = ||x - y_n||/(d + r) \leq (d + 1/n)/(d + r) < 1$, i.e., $x_n \in B$ for $n > 1/r$. Also, for $n, m > 1/r$,

$$||(x_n + x_m)/2|| = (d + r)^{-1} ||x - (y_n + y_m)/2|| \geq d/(d + r) > 1 - \delta$$

by the choice of r. Hence $||x_n - x_m|| < \epsilon$ for all $n, m > 1/r$. Therefore, for all such n, m,

$$||y_n - y_m|| = ||(x - y_m) - (x - y_n)|| = (d + r)||x_m - x_n|| < 2d\epsilon.$$

Thus $\{y_n\}$ is Cauchy in K, which is complete as a closed subset of the Banach space X. Let then $k := \lim y_n \quad (\in K)$. Letting $n \to \infty$ in (2), it follows from the continuity of the norm that $||x - k|| = d$, as desired.
To prove uniqueness, suppose $k' \in K$ satisfies also $||x - k'|| = d\ (> 0)$. Consider the two constant sequences of unit vectors $x_n := (x-k)/d$ and $y_n := (x-k')/d$. Since $(k+k')/2 \in K$, we have

$$2 \geq ||x_n + y_n|| = (2/d)||x - (k+k')/2|| \geq (2/d)\, d = 2,$$

i.e., $||x_n + y_n|| = 2$ (for all n). Hence

$$||k' - k|| = d\, ||(x-k)/d - (x-k')/d|| = d\, ||x_n - y_n|| \to 0,$$

so that $k' = k$.

The following parts are steps in the proof of the result stated in Part (i) below.

(e) Let X be a u.c. Banach space, and let ϵ, δ be as in the definition above. Denote by S^* and S^{**} the unit spheres of X^* and X^{**} respectively. Given $x_0^{**} \in S^{**}$, there exists $x_0^* \in S^*$ such that $|x_0^{**}x_0^* - 1| < \delta$. Also there exists $x \in B$ such that $|x_0^*x - 1| < \delta$. Define

$$E_\delta = \{x \in B;\ |x_0^*x - 1| < \delta\} (\neq \emptyset!).$$

Show that $||x - y|| < \epsilon$ for all $x, y \in E_\delta$.

Solution.

Let $x,\ y \in E_\delta$. Then, by the triangle inequality,

$$||(x+y)/2|| \geq |x_0^*(x+y)/2| = (1/2)|(x_0^*x - 1) + (x_0^*y - 1) + 2|$$
$$\geq (1/2)[2 - |x_0^*x - 1| - |x_0^*y - 1|] > 1 - \delta.$$

Hence $||x - y|| < \epsilon$.

(f) In any normed space X, the set

$$U := \{x^{**} \in X^{**};\ |x^{**}x_0^* - 1| < \delta\}$$

is a *weak**-neighbourhood of x_0^{**}.

Solution.

By definition, $x_0^{**} \in U$. If $x^{**} \in U$, pick any real number r such that $0 < r < \delta - |x^{**}x_0^* - 1|$, and let $N(x^{**}; x_0^*; r)$ be the basic neighbourhood of the *weak**-topology with the indicated parameters (cf. paragraph preceding Theorem 5.24). For all $z^{**} \in N(x^{**}; x_0^*; r)$, we have

$$|z^{**}x_0^* - 1| \leq |z^{**}x_0^* - x^{**}x_0^*| + |x^{**}x_0^* - 1| < r + |x^{**}x_0^* - 1| < \delta.$$

Thus $N(x^{**}; x_0^*; r) \subset U$, and we conclude that U is an *open* neighbourhood of x_0^{**} in the *weak**-topology.

(g) For any *weak**-neighbourhood V of x_0^{**}, the *weak**-neighbourhood $W := V \cap U$ of x_0^{**} meets κB. (κ denotes the canonical embedding of X in X^{**}.) Thus $V \cap \kappa(E_\delta) \neq \emptyset$, and therefore x_0^{**} belongs to the *weak**-closure of $\kappa(E_\delta)$. (Cf. Goldstine's theorem.)

Solution.

By Part (f), U is an (open) *weak**-neighbourhood of x_0^{**}; the same is then true for W. Since κB is *weak**-dense in the norm-closed unit ball B^{**} of X^{**} (cf. Goldstine's theorem, 5.25), it follows that $(\kappa B) \cap W \neq \emptyset$. Pick $\kappa z \in (\kappa B) \cap W$ (where $z \in B$). Then $\kappa z \in U$, so that $|(\kappa z)x_0^* - 1| < \delta$, i.e., $|x_0^* z - 1| < \delta$, which means that $z \in E_\delta$. Hence $V \cap (\kappa E_\delta) \neq \emptyset$ (the intersection contains κz). This proves that x_0^{**} belongs to the *weak**-closure of κE_δ.

(h) Fix $x \in E_\delta$. Then $x_0^{**} \in \kappa x + \epsilon B^{**}$. (Hint: apply Parts (e) and (g), and the fact that B^{**} is *weak**-compact, hence *weak**-closed.)

Solution.

By Part (g), there exists a net $\{x_\alpha\} \subset E_\delta$ such that $\kappa x_\alpha \to x_0^{**}$ *weak**. By Part (e), $||x - x_\alpha|| < \epsilon$. Since κ is a linear isometry, it follows that

$$\kappa x_\alpha = \kappa(x_\alpha - x) + \kappa x \in \kappa x + \epsilon B^{**}. \tag{3}$$

By Alaoglu's theorem (Theorem 5.24), B^{**} is *weak**-compact; therefore $\kappa x + \epsilon B^{**}$ is *weak**-compact, hence *weak**-closed. Consequently, by (3), $x_0^{**} \in \kappa x + \epsilon B^{**}$.

(i) Conclude from Part (h) that $d(x_0^{**}, \kappa B) = 0$, and therefore $x_0^{**} \in \kappa B$ since κB is norm-closed in X^{**} (cf. paragraph preceding Theorem 5.9). This proves the following theorem: *uniformly convex Banach spaces are reflexive.*

Solution.

By Part (h), for any $x \in E_\delta$ ($\subset B$), $||x_0^{**} - \kappa x|| \leq \epsilon$. Hence $d(x_0^{**}, \kappa B) \leq \epsilon$, and the conclusions follow from the arbitrariness of ϵ.

7. Let $\{\beta_n\}_{n=0}^{\infty} \in l^{\infty}$ be such that there exists a positive constant K for which

$$|\sum_{n=0}^{N} \alpha_n \beta_n| \leq K \max_{t\in[0,1]} |\sum_{n=0}^{N} \alpha_n t^n| \tag{4}$$

for all $\alpha_0, ..., \alpha_N \in \mathbb{C}$ and $N = 0, 1, 2, ...$ Prove that there exists a unique complex Borel measure μ on $[0,1]$ such that $\beta_n = \int_0^1 t^n d\mu$ for all $n = 0, 1, 2, ...$. Moreover $||\mu|| \leq K$. Formulate and prove the converse.

Solution.

Consider the Banach space $C([0,1])$, and its dense subspace $\mathcal{P}$ of all polynomials with complex coefficients (restricted to the interval $[0,1]$). Define $\phi : \mathcal{P} \to \mathbb{C}$ by

$$\phi(\sum_{n=0}^{N} \alpha_n t^n) = \sum_{n=0}^{N} \alpha_n \beta_n. \tag{5}$$

The map ϕ is trivially linear, and (4) means that $||\phi|| \leq K$. Let $x^* = \tilde{\phi}$ be the unique extension of ϕ as a continuous linear functional on $C([0,1])$ (cf. Exercise 1, Chapter 4). We have $||x^*|| = ||\phi|| \leq K$. By the Riesz Representation theorem (Theorem 4.4), there exists a unique regular complex Borel measure μ on $[0,1]$ such that $x^* f = \int_0^1 f\, d\mu$ for all $f \in C([0,1])$, and we also have $||\mu|| = ||x^*|| \leq K$. Taking in particular $f = t^n \in \mathcal{P}$, we then have $\int_0^1 t^n d\mu = \phi(t^n) = \beta_n$. Conversely, if there exists a complex Borel measure μ on $[0,1]$ with $||\mu|| \leq K$ such that $\beta_n = \int_0^1 t^n d\mu$ for $n = 0, 1, \cdots$, then (4) is valid. Indeed, for any $\alpha_j \in \mathbb{C}$ and $N = 0, 1, \cdots$,

$$\Big|\sum_{n=0}^{N} \alpha_n \beta_n\Big| = \Big|\int_0^1 (\sum_n \alpha_n t^n)\, d\mu\Big|$$

$$\leq ||\sum_n \alpha_n t^n||_u ||\mu|| \leq K \max_{t\in[0,1]} |\sum_n \alpha_n t^n|.$$

8. Prove the converse of Theorem 4.4

Solution.

Statement: let X, Y be normed spaces such that $B(X,Y)$ is complete. Then Y is complete.

Let $\{y_n\}$ be Cauchy in Y. Fix a unit vector $x_0 \in X$, and then choose $x_0^* \in X^*$ such that $x_0^* x_0 = 1 = ||x_0^*||$ (cf. Corollary 5.7). Define the linear operators $T_n : X \to Y$ by

$$T_n x = (x_0^* x) y_n \qquad n = 1, 2, \cdots . \tag{6}$$

Since $||T_n x|| \leq ||y_n||\,||x||$, the operators T_n are bounded (with $||T_n|| \leq ||y_n||$). Moreover, for all $x \in X$,

$$||(T_n - T_m)x|| = ||(x_0^* x)(y_n - y_m)|| \leq ||x_0^*||\,||y_n - y_m||\,||x|| = ||y_n - y_m||\,||x||,$$

hence $||T_n - T_m|| \leq ||y_n - y_m|| \to 0$ as $n, m \to \infty$. Since $B(X,Y)$ is complete, there exists $T \in B(X,Y)$ such that $||T_n - T|| \to 0$. In particular, it follows that $T_n x_0 \to T x_0$ in Y, i.e., $y_n \to y$ with $y := T x_0$. This proves the completeness of Y.