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CHAPTER 4

CONTINUOUS LINEAR FUNCTIONALS

1. Let X, Y be Banach spaces, Z a dense subspace of X, and $T \in B(Z,Y)$. Then there exists a unique $\tilde{T} \in B(X,Y)$ such that $\tilde{T}|_Z = T$. Moreover, the map $T \to \tilde{T}$ is an isometric isomorphism of $B(Z,Y)$ onto $B(X,Y)$.

Solution.

Let $x \in X$. By the density of Z in X, there exists a sequence $\{x_n\} \subset Z$ such that $x_n \to x$ in the X-norm (all norms are denoted $||\cdot||$, for simplicity of notation, unless distinction is required by the context). The sequence $\{Tx_n\}$ is Cauchy, because

$$||Tx_n - Tx_m|| = ||T(x_n - x_m)|| \leq ||T||\,||x_n - x_m|| \to 0$$

as $n, m \to \infty$. Since Y is complete, there exists a unique vector $y \in Y$ such that $Tx_n \to y$ in Y. The vector y does not depend on the particular sequence $\{x_n\}$ converging to x, because if $x'_n \to x$ as well, and $y' = \lim Tx'_n$, then $||T(x'_n - x_n)|| \leq ||T||\,||x'_n - x_n|| \to 0$, hence $||y' - y|| = 0$, by the continuity of the norm. Set $\tilde{T}x = y$. By the last observation, $\tilde{T}$ is well defined on X and clearly linear (if $x, y \in X$ and $\alpha, \beta \in \mathbb{C}$, then $\alpha x_n + \beta y_n \to \alpha x + \beta y$ for any sequences $\{x_n\}$ and $\{y_n\}$ in Y converging to x and y respectively. Therefore

$$\tilde{T}(\alpha x + \beta y) = \lim_n T(\alpha x_n + \beta y_n) = \lim_n [\alpha T x_n + \beta T y_n] = \alpha \tilde{T} x + \beta \tilde{T} y).$$

If $||T||_Z$ denote the $B(Z,Y)$-norm of T, then for x and x_n as above, since $||Tx_n|| \leq ||T||_Z ||x_n||$, it follows that $||\tilde{T}x|| \leq ||T||_Z ||x||$ by continuity of norms. Consequently $\tilde{T} \in B(X,Y)$ and $||\tilde{T}||_{B(X,Y)} \leq ||T||_Z$.
If $x \in Z$, the constant sequence $\{x_n\}$ with $x_n = x$ for all n belongs to Z and converges to x; hence $\tilde{T}x = \lim_n Tx_n = Tx$, that is $\tilde{T}|_Z = T$. In particular, it follows that $||T||_Z \leq ||\tilde{T}||_{B(X,Y)}$, and therefore these norms are equal (by the preceding observation).
The uniqueness assertion in the exercise is trivial (any two extensions S, $S' \in B(X,Y)$ of T coincide, by the continuity of S, S' and the density of Z in X). In particular, $S = \widetilde{S|_Z}$ for any $S \in B(X.Y)$; this shows that the map $T \to \tilde{T}$ is *onto* $B(X,Y)$. It is trivially linear (by the properties of limits) and isometric (since it is norm-preserving and linear).

2. Let X be a locally compact Hausdorff space. Prove that $C_0(X)^*$ is isometrically isomorphic to $M_r(X)$, the space of all *regular* complex Borel measures on X. (Hint: Theorems 3.24 and 4.9, and Exercise 1.)

Solution.

By Theorem 3.24, the normed space $C_c(X)$ is dense in the Banach space $C_0(X)$. By Exercise 1, there exists a unique isometric isomorphism $\phi \to \tilde{\phi}$ of $C_c(X)^* := B(C_c(X), \mathbb{C})$ onto $B(C_0(X), \mathbb{C}) = C_0(X)^*$. The map $\mu \in M_r(X) \to \phi \in C_c(X)^*$ of Theorem 4.9(1) is an isometric isomormphism of $M_r(X)$ and $C_c(X)^*$. The composition map $\mu \to \tilde{\phi}$ is then an isometric isomorphism of $M_r(X)$ onto $C_0(X)^*$.
(It follows from the definitions that

$$\tilde{\phi}(f) = \int_X f\, d\mu \qquad (f \in C_0(X))$$

when ϕ and μ are associated as in Theorem 4.9.)

3. Let X_k $k = 1, ..., n$ be normed spaces, and consider $\prod_k X_k$ as a normed space with the norm $|||[x_1, ..., x_n]||| = \sum_k ||x_k||$. Prove that there exists an isometric isomorphism of $(\prod_k X_k)^*$ and $\prod_k X_k^*$ with the norm $|||[x_1^*, ..., x_n^*]||| = \max_k ||x_n^*||$. (Hint: given $\phi \in (\prod_k X_k)^*$, define $x_k^* x_k = \phi([0, ..., x_k, 0, ...0])$ for $x_k \in X_k$. Note that $\phi([x_1, ..., x_n]) = \sum_k x_k^* x_k$.)

Solution.

Given $\phi \in (\prod_k X_k)^*$, set $V\phi = [x_1^*, \cdots, x_n^*]$ with x_k^* as in the hint. Clearly x_k^* is linear and

$$|x_k^* x| \leq ||\phi||\, |||[0, \cdots, 0, x_k, 0, \cdots, 0]||| = ||\phi||\, ||x_k||.$$

Hence $||x_k^*|| \leq ||\phi|| < \infty$, and so $x_k^* \in X_k^*$ for all k, i.e., $[x_1^*, \cdots, x_n^*] \in \prod_k X_k^*$, and

$$||V\phi|| = |||[x_1^*, \cdots, x_n^*]||| := \max_k ||x_k^*|| \leq ||\phi||. \tag{1}$$

By linearity of ϕ, for any $[x_1, \cdots, x_n]$,

$$\phi([x_1, \cdots, x_n]) = \phi\Big([x_1, 0, \cdots, 0] + \cdots + [0, \cdots, 0, x_n]\Big) = \sum_k x_k^* x_k. \tag{2}$$

On the other hand, given $[x_1^*, \cdots, x_n^*] \in \prod_k X_k^*$ and *defining* ϕ by means of (2), we get an element $\phi \in (\prod X_k)^*$, and $V\phi = [x_1^*, \cdots, x_n^*]$. Thus V maps $\Big(\prod_k X_k\Big)^*$ *onto* $\prod_k X_k^*$. Furthermore

$$|\phi([x_1, \cdots, x_n])| = |\sum_k x_k^* x_k| \leq \sum_k |x_k^* x_k|$$

$$\leq \sum_k ||x_k^*||\,||x_k|| \leq \max_k ||x_k^*|| \sum_k ||x_k||$$

$$= ||[x_1^*, \cdots, x_n^*]||\,||[x_1, \cdots, x_n]||,$$

that is, $||\phi|| \leq ||[x_1^*, \cdots, x_n^*]|| = ||V\phi||$. Together with (1), this proves equality of the norms. Consequently, the (clearly linear) map $V : \phi \to [x_1^*, \cdots, x_n^*]$ is an isometric isomorphism of $\left(\prod_k X_k\right)^*$ onto $\prod_k X_k^*$.

4. Let X be a locally compact Hausdorff space. Let Y be a normed space, and $T \in B(C_c(X), Y)$. Prove that there exists a unique $P : \mathcal{B}(X) \to Y^{**} := (Y^*)^*$ such that $P(.)y^* \in M_r(X)$ for each $y^* \in Y^*$ and

$$y^*Tf = \int_X f d(P(\cdot)y^*)$$

for all $f \in C_c(X)$ and $y^* \in Y^*$. Moreover $||P(\cdot)y^*|| = ||y^* \circ T||$ for the appropriate norms (for all $y^* \in Y^*$) and $||P(\delta)|| \leq ||T||$ for all $\delta \in \mathcal{B}(X)$.

Solution.

For each $y^* \in Y^*$, we have $y^* \circ T \in C_c(X)^*$. By the Riesz Representation theorem (Theorem 4.9), there exists a unique measure $\mu = \mu(\cdot; y^*) \in M_r(X)$ such that

$$y^*Tf = \int_X f\, d\mu(\cdot; y^*) \qquad (f \in C_c(X)), \tag{3}$$

and furthermore

$$||\mu(\cdot; y^*)|| = ||y^* \circ T||. \tag{4}$$

The left hand side of (3) is linear in y^*. Therefore, by the uniqueness of the representation (3), the map

$$\pi : y^* \to \mu(\cdot; y^*)$$

is a linear map of Y^* into $M_r(X)$. By (4), π is bounded with norm $\leq ||T||$ (because $||\pi y^*|| = ||\mu(\cdot; y^*)|| \leq ||T||\,||y^*||$). Thus $\pi \in B(Y^*, M_r(X))$.
For any $\delta \in \mathcal{B}(X)$, the map $y^* \to \mu(\delta; y^*)(= (\pi y^*)(\delta))$ is a linear functional on Y^*, and by (4)

$$|\mu(\delta; y^*)| \leq ||\mu(\cdot; y^*)|| \leq ||T||\,||y^*||,$$

that is $\mu(\delta; \cdot) \in (Y^*)^* := Y^{**}$ and the Y^{**}-norm $||\mu(\delta; \cdot)||$ is $\leq ||T||$. Denote

$$P(\delta)y^* = (\pi y^*)(\delta) \qquad (\delta \in \mathcal{B}(X);\ y^* \in Y^*).$$

We just observed that $P(\delta) \in Y^{**}$ and $||P(\delta)|| \leq ||T||$ for all $\delta \in \mathcal{B}(X)$. By (3), (4), and the definitions, $P(\cdot)y^* \in M_r(X)$, $||P(\cdot)y^*|| = ||\mu(\cdot; y^*)|| = ||y^* \circ T||$, and

$y^*Tf = \int_X f\, d(P(\cdot)y^*)$ for all $f \in C_c(X)$. The uniqueness of P follows from the uniqueness of the Riesz representation.

Convolution on L^p

5. Let L^p denote the Lebesgue spaces on $\mathbb{R}^k$ with respect to Lebesgue measure. Prove that if $f \in L^1$ and $g \in L^p$, then $f * g \in L^p$ and $||f * g||_p \leq ||f||_1 ||g||_p$. (Hint: use Theorems 4.6, 2.18, 1.33, and the translation invariance of Lebesgue measure; cf. Exercise 7, Chapter 2, in its $\mathbb{R}^k$ version.)

Solution.

The case $p = 1$ is proved in Exercise 7 (d), (e), (l), Chapter 2. We then assume $p > 1$. The conjugate exponent q is then finite.
In order to avoid repetitions, "measurable" means "Lebesgue measurable" on the relevant Euclidean space.
Let $B_n = \{[x,y] \in \mathbb{R}^k \times \mathbb{R}^k; |x| < n, |y| < n\}$, $(n \in \mathbb{N})$. Since $I_{B_n} \in L^q$, we have $g_n := gI_{B_n} \in L^1$ (cf. 1.33); by Exercise 7(d) in Chapter 2, it follows that $f(x-y)g_n(y) \in L^1(\mathbb{R}^k \times \mathbb{R}^k)$. Their pointwise limit $F(x,y) := f(x-y)g(y)$ (as $n \to \infty$) is then *measurable* on $\mathbb{R}^k \times \mathbb{R}^k$. If $V \subset \mathbb{C}$ is open, then $[F \in V] \cap B_n$ is measurable with finite measure. By Lemma 2.16, its x-sections are measurable for almost all $x \in \mathbb{R}^k$, that is, $[F \in V]_x \cap \{y; |y| < n\}$ is measurable for almost all x and for all n. Consequently $[F \in V]_x$ is measurable for almost all x. However

$$[F(x,\cdot) \in V] := \{y \in \mathbb{R}^k; F(x,y) \in V\} = \{y; [x,y] \in F^{-1}(V)\} = [F \in V]_x.$$

Therefore *the function $F(x,\cdot) := f(x-\cdot)g$ is measurable for almost all x.*

Case $p = \infty$.
Let x be such that $f(x-\cdot)g$ is measurable. By translation invariance of the Lebesgue measure,

$$\int_{\mathbb{R}^k} |f(x-y)g(y)|dy \leq ||g||_\infty \int |f(x-y)|dy = ||g||_\infty ||f||_1. \tag{5}$$

Thus $f(x-\cdot)g \in L^1$ for almost all x, and consequently $f * g$ is well-defined almost everywhere.
With notations as above, $f(x-\cdot)g$ is the poinwise limit of $f(x-\cdot)g_n$ as $n \to \infty$, and $|f(x-\cdot)g_n| \leq |f(x-\cdot)g| \in L^1$. By the Dominated Convergence theorem (1.20), it follows that $(f*g)(x) = \lim_n (f*g_n)(x)$ (for almost all x). Since $f*g_n$ are measurable (cf. Exercise 7(e)), $f*g$ is measurable, and by (5), $|f*g| \leq ||f||_1 ||g||_\infty$ almost everywhere. Thus $f*g \in L^\infty$ and $||f*g||_\infty \leq ||f||_1 ||g||_\infty$.

Case $1 < p < \infty$.
For any $h \in L^q$, the function h is measurable on $\mathbb{R}^k$; hence $\tilde{h}(x,y) := h(x)$ is measurable on $\mathbb{R}^k \times \mathbb{R}^k$ (cf. Exercise 7(b), Chapter 2), and therefore the pointwise product

$(\tilde{h}F)(x,y) = h(x)f(x-y)g(y)$ is measurable on $\mathbb{R}^k \times \mathbb{R}^k$. By Tonelli's theorem (2.18), the translation invariance of the Lebesgue measure, and Holder's inequality

$$\int_{\mathbb{R}^k \times \mathbb{R}^k} \Big| h(x)f(x-y)g(y) \Big| \, dx dy = \int_{\mathbb{R}^k} |h(x)| \Big(\int_{\mathbb{R}^k} |f(x-y)g(y)| \, dy \Big) \, dx$$

$$= \int |h(x)| \Big(\int |f(t)g(x-t)| \, dt \Big) dx = \int |f(t)| \Big(\int |g(x-t)h(x)| \, dx \Big) \, dt$$

$$\leq \int |f(t)| \, ||g_t||_p ||h||_q dt = ||f||_1 ||g||_p ||h||_q \tag{6}$$

(where $g_t(x) := g(x-t)$). Thus $h(x)f(x-y)g(y) \in L^1(\mathbb{R}^k \times \mathbb{R}^k)$. By Fubini's theorem (2.17 (i)), $h(x)f(x-\cdot)g \in L^1$ for almost all x, for any $h \in L^q$. Since for any x we may choose $h \in L^q$ such that $h(x) \neq 0$, it follows that $f(x-\cdot)g \in L^1$ for almost all x, and therefore $(f*g)(x)$ is well-defined for almost all x. By Fubini's theorem (2.17 (ii) and (iii)) and (6),

$$[h(f*g)](x) = \int_{\mathbb{R}^k} h(x)f(x-y)g(y) dy \in L^1, \tag{7}$$

and

$$\Big| \int_{\mathbb{R}^k} h\,(f*g)\,dx \Big| = \Big| \int_{\mathbb{R}^k \times \mathbb{R}^k} h(x)f(x-y)g(y)\,dxdy \Big|$$

$$\leq \int_{\mathbb{R}^k \times \mathbb{R}^k} |h(x)f(x-y)g(y)| \, dxdy \leq ||f||_1 ||g||_p ||h||_q. \tag{8}$$

Thus the linear functional $\phi(h) := \int_{\mathbb{R}^k} h(f*g)\,dx$ is well-defined on L^q (by (7)) and has norm $||\phi|| \leq ||f||_1 ||g||_p$ (by (8)). Since $q < \infty$, it follows from Theorem 4.6 that there exists a unique $u \in L^p$ such that $\phi(h) = \int hu\,dx$ for all $h \in L^q$, and $||u||_p = ||\phi||$. (*Claim.* Let $(X, \mathcal{A}, \mu)$ be a complete positive measure space. Let k be an (almost everywhere defined) measurable function such that $hk \in L^1$ and $\int hk\,d\mu = 0$ for all $h \in L^q$, for some $q < \infty$. Then $k = 0$ a.e.
Proof. For each $E \in \mathcal{A}$ and $h \in L^q$, $I_E h$ is measurable and $|I_E h| \leq |h| \in L^q$. Hence $I_E h \in L^q$. Replacing h by $I_E h$ in the hypothesis, we get $\int_E hk\,d\mu = 0$ for all $E \in \mathcal{A}$. Since $hk \in L^1$, it follows from Proposition 1.22 that $hk = 0$ almost everywhere for all $h \in L^q$. Since for each x we can choose h such that $h(x) \neq 0$, it follows that $k = 0$ a.e.)
Take $k = f*g-u$. It clearly satisfies the hypothesis of the claim. Hence $f*g = u$ a.e., and therefore $f*g \in L^p$ and $||f*g||_p \leq ||f||_1 ||g||_p$ (by the corresponding properties of u).

Approximate identities

6. Let m denote the normalized Lebesgue measure on $[-\pi, \pi]$. Let $K_n : [-\pi, \pi] \to [0, \infty)$ be Lebesgue measurable functions such that $\int_{-\pi}^{\pi} K_n dm = 1$ and

$$\sup_{\delta \le |x| \le \pi} K_n(x) \to 0 \tag{9}$$

as $n \to \infty$, for all $\delta > 0$. (Any sequence $\{K_n\}$ with these properties is called an *approximate identity.*)

Consider the convolutions

$$(K_n * f)(x) := \int_{-\pi}^{\pi} K_n(x-t) f(t) dm(t) = \int_{-\pi}^{\pi} f(x-t) K_n(t) dm(t)$$

with 2π-periodic functions f on $\mathbb{R}$ and K_n extended as a 2π-periodic function on $\mathbb{R}$. Prove:

(a) If f is continuous, $K_n * f \to f$ uniformly on $[-\pi, \pi]$. (Hint: $\int_{-\pi}^{\pi} = \int_{|t|<\delta} + \int_{\delta \le t \le \pi}$.)

Solution.

(Observe that if $h : \mathbb{R} \to \mathbb{C}$ is 2π-periodic and integrable on $(-\pi, \pi)$, then it is integrable on $(c - \pi, c + \pi)$, for any $c \in \mathbb{R}$, and its integrals over these intervals coincide.)

All functions f considered in this exercise are 2π-periodic and integrable on $(-\pi, \pi)$. Since K_n is 2π-periodic, the same is true of $f(x - \cdot)K_n$ for any given x. These functions are also integrable (in all parts of the exercise) on $(-\pi, \pi)$. By the above observation and translation invariance of the Lebesgue measure,

$$\int_{-\pi}^{\pi} K_n(x-t) f(t)\, dm(t) = \int_{x-\pi}^{x+\pi} f(x-t) K_n(t)\, dm(t) = \int_{-\pi}^{\pi} f(x-t) K_n(t) dm(t). \tag{10}$$

Let f be continuous (with $||f||_u \neq 0$) and $\epsilon > 0$. By *uniform continuity* of f on the closed interval $[-\pi, \pi]$, there exists $\delta > 0$ such that $|f(x-t) - f(x)| < \epsilon/2$ for all $x \in [-\pi, \pi]$ when $|t| < \delta$. For this δ, it follows from (9) that there exists $n_0 \in \mathbb{N}$ such that

$$\sup_{\delta \le |t| \le \pi} K_n(t) < \epsilon/(4||f||_u) \qquad (n > n_0). \tag{11}$$

By (10) and the hypothesis on K_n,

$$|(K_n * f)(x) - f(x)| = \left| \int_{-\pi}^{\pi} K_n(t)[f(x-t) - f(x)]\, dm(t) \right|$$

$$\le \int_{-\pi}^{\pi} K_n(t) |f(x-t) - f(x)|\, dm(t) = \int_{|t|<\delta} + \int_{\delta \le |t| \le \pi}. \tag{12}$$

In the first integral on the right hand side of (12), the integrand is $\leq (\epsilon/2)K_n(t)$, and therefore the integral is $\leq \epsilon/2$. In the second integral on the right hand side of (12), the integrand is $< \epsilon/2$ for $n > n_0$ (by (11)). Since m is normalized, the integral is $\leq \epsilon/2$ for $n > n_0$, and we conclude from (12) that

$$||K_n * f - f||_u \leq \epsilon \qquad (n > n_0),$$

as desired.

(b) If $f \in L^p := L^p(-\pi, \pi)$ for some $p \in [1, \infty)$, then $K_n * f \to f$ in L^p. (Hint: use the density of $C([-\pi, \pi])$ in L^p, cf. Corollary 3.21, Part (a), and Exercise 5.)

Solution.

Let $f \in L^p$ and $\epsilon > 0$. By Corollary 3.21, there exists a 2π- periodic continuous function g on $\mathbb{R}$ such that $||f - g||_p < \epsilon/3$ (where the norm is the L^p-norm). By Part (a), there exists n_0 such that $||K_n * g - g||_u \leq \epsilon/3$ for all $n > n_0$. Since m is normalized, it follows that $||K_n * g - g||_p \leq \epsilon/3$ for all $n > n_0$. Note that $K_n * f \in L^p$ by Exercise 5 (in its appropriate version for $[-\pi, \pi]$), and

$$||K_n * (f - g)||_p \leq ||K_n||_1 ||f - g||_p = ||f - g||_p < \epsilon/3.$$

Therefore

$$||K_n * f - f||_p \leq ||K_n * (f - g)||_p + ||K_n * g - g||_p + ||g - f||_p < \epsilon$$

for all $n > n_0$.

(c) If $f \in L^\infty$, then $K_n * f \to f$ in the *weak**-topology on L^∞ (cf. Theorem 4.6); this means that $\int (K_n * f) g \, dm \to \int fg \, dm$ for all $g \in L^1$.

Solution.

Let $f \in L^\infty$ and $g \in L^1$. Define $\tilde{K}_n(x) = K_n(-x)$. Then $\tilde{K}_n$ satisfies the conditions on K_n, and therefore, by Exercise 5, and Part (b), $\tilde{K}_n * g \to g$ in L^1. We now use Fubini's theorem (2.17) (justification analogous to the argument in Exercise 5). With all integrals extending over $[-\pi, \pi]$),

$$\Big| \int (K_n * f)(x) g(x) \, dm(x) - \int fg \, dm \Big|$$

$$= \Big| \int f(t) \int K_n(x - t) g(x) \, dm(x) \, dm(t) - \int fg \, dm \Big|$$

$$= \left| \int f(t)[(\tilde{K}_n * g)(t) - g(t)]\, dm(t) \right| \leq ||f||_\infty ||\tilde{K}_n * g - g||_1 \to 0$$

as $n \to \infty$.

7. Consider the measure space $(\mathbb{N}, \mathbb{P}(\mathbb{N}), \mu)$, where μ is the *counting measure* ($\mu(E)$ is the number of points in E if E is a finite subset of $\mathbb{N}$ and $= \infty$ otherwise). The space $l^p := L^p(\mathbb{N}, \mathbb{P}(\mathbb{N}), \mu)$ is the space of all complex sequences $x := \{x(n)\}$ such that $||x||_p := (\sum |x(n)|^p)^{1/p} < \infty$ (in case $p < \infty$) or $||x||_\infty := \sup |x(n)| < \infty$ (in case $p = \infty$). As a special case of Theorem 4.6, if $p \in [1, \infty)$ and q is its conjugate exponent, then $(l^p)^*$ is isometrically isomorphic to l^q through the map $x^* \in (l^p)^* \to y \in l^q$, where $y := \{y(n)\}$ is the unique element of l^q such that $x^*x = \sum x(n)y(n)$ for all $x \in l^p$. *Prove this directly!* (Hint: consider the unit vectors $e_m \in l^p$ with $e_m(n) = \delta_{n,m}$, the Kronecker delta.)

Solution.

We have $x = \sum_m x(m)e_m$ for every $x \in l^p$, with the series converging in l^p, because as $N \to \infty$,

$$|| \sum_{m>N} x(m)e_m||_p^p = \sum_{m>N} |x(m)|^p \to 0$$

(since $\sum_m |x(m)|^p < \infty$). Let $x^* \in (l^p)^*$. By linearity and continuity of x^*, we have then

$$x^*x = \sum_m x(m)(x^*e_m) \tag{13}$$

(meaning that the series on the right hand side *converges* to the left hand side). Define the sequence $y = \{y(m)\}$ by $y(m) = x^*e_m$. By (13),

$$x^*x = \sum_m x(m)y(m) \qquad (x \in l^p). \tag{14}$$

Since $||e_m||_p = 1$ (for any p), we have

$$|y(m)| = |x^*e_m| \leq ||x^*|| \, ||e_m||_p = ||x^*||. \tag{15}$$

Claim.

$$||y||_q \leq ||x^*||. \tag{16}$$

Case $p = 1$.
By (15), $||y||_\infty \leq ||x^*||$, and (16) is verified.

Case $1 < p < \infty$.
Consider the sequence x_0 with $x_0(m) = |y(m)|^{q-1}\theta(y(m))$ (cf. proof of Theorem 4.6 for notation and inspiration). For any sequence z and $N \in \mathbb{N}$, denote by z^N the

sequence with $z^N(m) = z(m)$ for $m \le N$ and $z^N(m) = 0$ for $m > N$.
Since $(q-1)p = q$, we have $|x_0(m)|^p = |y(m)|^q$, and therefore

$$||x_0^N||_p^p = ||y^N||_q^q \qquad (N \in \mathbb{N}). \tag{17}$$

We also have by (17)

$$||y^N||_q^q = \sum_{m \le N} |y(m)|^q = \sum_{m \le N} x_0(m)y(m) = \sum_{m \le N} x_0(m)x^*e_m$$

$$= x^*x_0^N \le ||x^*||\,||x_0^N||_p = ||x^*||\,||y^N||_q^{q/p}.$$

If $||y^N||_q \ne 0$, we divide the inequality by $||y^N||_q^{q/p}$; since $q - q/p = 1$, we conclude that $||y^N||_q \le ||x^*||$ for all N (the inequality is trivial if $||y^N||_q = 0$). Hence $||y||_q = \sup_N ||y^N||_q \le ||x^*||$, as claimed above.
In particular, $y \in l^q$.
On the other hand, for *any* $y \in l^q$, we may define $x^* \in l^p$ by (14); this x^* is well-defined, since by Holder's inequality for sequences (special case of Theorem 1.33),

$$\sum_m |x(m)y(m)| \le ||x||_p||y||_q,$$

so that the series in (14) converges (absolutely). This shows also that $|x^*x| \le ||x||_p||y||_q$ for all $x \in l^p$, that is $||x^*|| \le ||y||_q$. The vector y (with components x^*e_m) associated above to x^* clearly coincides with the given y. In particular, the inequality (16) implies that $||x^*|| = ||y||_q$.

The map $V : x^* \to \{x^*e_m\}$ of $(l^p)^*$ into l^q is trivially linear. We proved that V is norm-preserving; it is therefore isometric (and one-to-one); we also observed that V is onto.

(The case $p = \infty$ is false. See observation following the solution of Exercise 8.)

8. Consider $\mathbb{N}$ with the discrete topology, and let $c_0 := C_0(\mathbb{N})$ (this is the space of all complex sequences $x := \{x_n\} = \{x(n)\}$ with $\lim x_n = 0$). As a special case of Exercise 2, if $x^* \in c_0^*$, there exists a unique complex Borel measure μ on $\mathbb{N}$ such that $x^*x = \sum_n x(n)\mu(\{n\})$. Denote $y(n) = \mu(\{n\})$. Then $||y||_1 = \sum |\mu(\{n\})| \le |\mu|(\mathbb{N}) = ||\mu|| = ||x^*||$, that is, $y \in l^1$ and $||y||_1 \le ||x^*||$. The reversed inequality is trivial. This shows that c_0^* is isometrically isometric to l^1 through the map $x^* \to y$, where $x^*x = \sum_n x(n)y(n)$. *Prove this directly!*

Solution.

(Notation as in Exercise 7.) Observe that $e_m \in c_0$; hence, given $x^* \in c_0^*$, $y(m) := x^* e_m$ are well defined for all $m \in \mathbb{N}$. The series representation $x = \sum_m x(m) e_m$ converges in c_0 (for all $x \in c_0$); indeed, given $\epsilon > 0$, there exists $n_0 \in \mathbb{N}$ such that $|x(m)| < \epsilon$ for all $n > n_0$ (since $x(m) \to 0$); therefore, if $N \geq n_0$,

$$||x - \sum_{m=1}^{N} ||_u = ||\{0, \cdots, 0, x(N+1), x(N+2), \cdots\}||_u$$

$$= \sup_{m>N} |x(m)| \leq \epsilon.$$

By linearity and continuity of x^*, it follows that

$$x^* x = \sum_m x(m)\, x^* e_m = \sum_m x(m) y(m) \qquad (x \in c_0). \tag{18}$$

Fix $N \in \mathbb{N}$. Let $x(m) = \theta(y(m))$ for $m \leq N$, and $x(m) = 0$ for $m > N$. Then $x \in c_0$, $||x||_u \leq 1$, and by (18)

$$\sum_{m=1}^{N} |y(m)| = \sum_m x(m) y(m) = x^* x \leq ||x^*||\, ||x||_u \leq ||x^*||.$$

Letting $N \to \infty$, we get

$$||y||_1 \leq ||x^*||. \tag{19}$$

In particular $y \in l^1$.
On the other hand, for *any* $y \in l^1$, we may *define* x^* on c_0 by the right hand side of (18). This x^* is a well-defined element of c_0^*, because

$$\sum_m |x(m) y(m)| \leq ||x||_u \sum |y(m)| = ||x||_u ||y||_1 < \infty$$

for all $x \in c_0$, so that the series in (18) converges (absolutely), and the clearly linear functional x^* satisfies $||x^*|| \leq ||y||_1$. Note also that the special y associated above to *this* x^* is precisely y, because

$$\{x^* e_m\} = \{\sum_n e_m(n) y(n)\} = \{y(m)\} = y.$$

This shows that the map $V : x^* \in c_0^* \to y := \{x^* e_m\} \in l^1$ is *onto* and norm-preserving (in view of (19) and the last inequality). Since V is clearly linear, it is an isomorphic isomorphism of c_0^* and l^1, and we have the representation

$$x^* x = \sum_m x(m) y(m) \qquad (x \in c_0).$$

(Clearly c_0 is a proper closed subspace of l^∞. It follows that $(l^\infty)^*$ is a proper subspace of c_0^* (identified with l^1 by means of V). Indeed, by Corollary 5.4, there exists $0 \neq \phi \in (l^\infty)^*$ such that $\phi = 0$ on c_0 (because c_0 is a proper closed subspace of l^∞). If $(l^\infty)^*$ coincides with l^1 (through V), there exists $x \in l^1$ such that $\phi(y) = \sum x(m)y(m)$ for all $y \in l^\infty$. In particular $\phi(e_n) = \sum x(m)e_n(m) = x(n)$. Since $e_n \in c_0$, it follows that $x(n) = \phi(e_n) = 0$ for all n, and therefore $\phi = 0$, contradiction! In other words, the statement of Exercise 7 is false in case $p = \infty$.)

9. Let c denote the space of all *convergent* complex sequences $x = \{x(n)\}$ with pointwise operations and the supremum norm. Show that c is a Banach space and c^* is isometrically isomorphic to l^1. (Hint: given $x^* \in c^*$, $x^*|_{c_0} \in c_0^*$; apply Exercise 8, and note that for each $x \in c$, $x - (\lim x)e \in c_0$, where $e(\cdot) = 1$.)

Solution.

(The completeness of c is easily verified; details are omitted.)
Let $x^* \in c^*$. Since c_0 is a (closed) subspace of c, the restriction $x^*|_{c_0}$ of x^* to c_0 belongs to c_0^*. By Exercise 8, there exists a unique $y_0 \in l_1$ such that $x^*x = \sum x(m)y_0(m)$ for all $x \in c_0$.
For any $x \in c$, we have $x_0 := x - (\lim x)e \in c_0$ and $x = x_0 + (\lim x)e$. Therefore

$$x^*x = x^*x_0 + (\lim x)x^*e = \sum_{m=1}^{\infty} x_0(m)y_0(m) + (\lim x)x^*e$$

$$= \sum_{m=1}^{\infty} x(m)y_0(m) + \lambda(\lim x) \tag{20}$$

where $\lambda = x^*e - \sum_m y_0(m) \in \mathbb{C}$. Define $y \in l^1$ by

$$y(1) = \lambda; \qquad y(m) = y_0(m-1) \quad (m \geq 2). \tag{21}$$

By (20) and (21),

$$x^*x = y(1)\lim x + \sum_m x(m)y(m+1) \quad (x \in c). \tag{22}$$

Fix integers $1 < N < M$, and define the sequence x by

$$x(m) = \theta(y(m+1)) \text{ for } m < N; x(m) = 0 \text{ for } N \leq m < M;$$

$$x(m) = \theta(\lambda) \text{ for } m \geq M.$$

Then $x \in c$ with norm ≤ 1. By (21) and (22),

$$\sum_{m=1}^{N} |y(m)| = y(1) \lim x + \sum_{m=1}^{N-1} x(m)y(m+1) = x^*x - \theta(\lambda) \sum_{m=M}^{\infty} y_0(m)$$

$$\leq ||x^*|| + \sum_{m \geq M} |y_0(m)|.$$

Since $y_0 \in l^1$, letting $M \to \infty$, we obtain $\sum_{m=1}^{N} |y(m)| \leq ||x^*||$ for all N, hence $y \in l^1$ and $||y||_1 \leq ||x^*||$.

On the other hand, given *any* $y \in l^1$ and defining x^* by (22), we obtain a well-defined element of c^* with norm $\leq ||y||_1$, and a simple calculation shows that the special vector in l^1 associated with it as above coincides with y. It follows that the map $V : x^* \in c^* \to y \in l^1$ defined above is (clearly linear) norm-preserving and onto. Hence V is an isometric isomorphism of c^* onto l^1.
(Note that although c_0 is a proper (closed) subspace of c, the adjoints c_0^* and c^* are isomorphically isomorphic.)

10. Let $(X, \mathcal{A}, \mu)$ be a positive measure space, $q \in (1, \infty]$, and $p = q/(q-1)$. Prove that for all $h \in L^q(\mu)$

$$||h||_q = \sup |\sum_k \alpha_k \int_{E_k} h d\mu|,$$

where the supremum is taken over all finite sums with $\alpha_k \in \mathbb{C}$ and $E_k \in \mathcal{A}$ with $0 < \mu(E_k) < \infty$, such that $\sum |\alpha_k|^p \mu(E_k) \leq 1$. (In case $q = \infty$, assume that the measure space is σ-finite.)

Solution.

Note that $1 \leq p < \infty$. The simple functions (represented canonically as) $\phi = \sum \alpha_k I_{E_k}$ with $E_k \in \mathcal{A}$ such that $0 < \mu(E_k) < \infty$ are precisely the simple functions in $L^p(\mu)$. By Theorem 1.27, these functions are dense in $L^p(\mu)$. Clearly

$$||\phi||_p^p = \sum_k |\alpha_k|^p \mu(E_k).$$

Consider the functional $x^* \in L^p(\mu)^*$ defined by

$$x^* f = \int_X fh \, d\mu \qquad (f \in L^p(\mu)).$$

By Theorem 4.6 and the density mentioned above,

$$||h||_q = ||x^*|| := \sup_{f \in L^p; ||f||_p \leq 1} |x^* f| = \sup_{\phi \in L^p; \phi \text{ simple}; ||\phi||_p \leq 1} |x^* \phi|$$

$$= \sup_{\phi} \left| \int_X \phi h \, d\mu \right| = \sup \left| \int_X \sum_k \alpha_k h I_{E_k} d\mu \right|$$

$$= \sup \left| \sum_k \alpha_k \int_{E_k} h \, d\mu \right|,$$

where the last two suprema are taken over all $\alpha_k \in \mathbb{C}$ and disjoint collections of sets $E_1, \cdots, E_n \in \mathcal{A}$ such that $0 < \mu(E_k) < \infty$ and $\sum_k |\alpha_k|^p \mu(E_k) \leq 1$.