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CHAPTER 3

MEASURE AND TOPOLOGY

Translations in L^p

1. Let L^p be the Lebesgue space on $\mathbb{R}^k$ with respect to Lebesgue measure. For each $t \in \mathbb{R}^k$, let

$$[T(t)f](x) = f(x+t) \qquad (f \in L^p; x \in \mathbb{R}^k).$$

This so-called "translation operator" is a linear isometry of L^p onto itself. Prove that $T(t)f \to f$ in L^p-norm as $t \to 0$, for each $f \in L^p$ $(1 \le p < \infty)$. (Hint: use Corollary 3.21 and an "$\epsilon/3$ argument".)

Solution.

Fix $p \in [1, \infty)$, $f \in L^p$, and $\epsilon > 0$. By Corollary 3.21, there exists $g \in C_c(\mathbb{R}^k)$ such that

$$||f - g||_p < \epsilon/3. \tag{1}$$

Let $K = \operatorname{supp} g$ and

$$K_1 = \{x + t; x \in K,\ |t| \le 1\}.$$

Since K_1 is compact, its Lebesgue measure $|K_1|$ is finite (and clearly positive). By uniform continuity of g on the compact set K, there exists $\delta \in (0,1)$ such that

$$|g(x+t) - g(x)| < \frac{\epsilon}{3|K_1|^{1/p}}$$

for all $x \in K$ and $|t| < \delta$. Since $g(x+t) - g(x) = 0$ for $x \in K_1^c$ if $|t| < \delta$, we have

$$||T(t)g - g||_p^p = \int_{K_1} |g(x+t) - g(x)|^p \, dx < (\epsilon/3)^p \tag{2}$$

for $|t| < \delta$.

We now use the fact that $T(t)$ is linear and isometric on L^p; by (1) and (2),

$$||T(t)f - f||_p \le ||T(t)(f-g)||_p + ||T(t)g - g||_p + ||g - f||_p$$

$$= 2||f - g||_p + ||T(t)g - g||_p < \epsilon$$

for $|t| < \delta$. Thus $T(t)f \to f$ in L^p-norm as $t \to 0$.

Automatic regularity

2. Let X be a locally compact Hausdorff space in which every open set is σ-compact (e.g., an Euclidean space). Then every positive Borel measure λ which is finite on compact sets is regular. (Hint: consider the positive linear functional $\phi(f) := \int_X f d\lambda$. If $(X, \mathcal{M}, \mu)$ is the associated measure space as in Theorem 3.18, show that $\lambda = \mu$ on open sets and use Theorem 3.19.)

Solution.

Let ϕ be as in the hint, let $f \in C_c(X)$ and $K = \operatorname{supp} f$. Since K is compact, $\lambda(K) < \infty$ by hypothesis, and consequently

$$|\phi(f)| \leq \int_K |f|\, d\lambda \leq ||f||_u \lambda(K) < \infty.$$

Therefore ϕ is a well defined (positive, linear) functional on $C_c(X)$. Let $(X, \mathcal{M}, \mu)$ be the measure space associated with ϕ as in the Riesz-Markov representation theorem (Theorem 3.18).

Claim 1. $\lambda = \mu$ on *open* sets.
Indeed, let $V \subset X$ be open. By the σ-compactness hypothesis, there exist compact sets $Q_i \subset V$ such that $Q_i \subset Q_{i+1}$ and $V = \bigcup Q_i$. Let $f_1 \in \Omega(V)$, with support K_1, equal 1 on $Q'_1 := Q_1$. Let $f_2 \in \Omega(V)$, with support K_2, equal 1 on $Q'_2 := Q'_1 \cup K_1$, etc. (i.e., $f_{n+1} \in \Omega(V)$, with support K_{n+1}, equals 1 on $Q'_{n+1} := Q'_n \cup K_n$). This inductively constructed sequence $\{f_n\}$ of non-negative continuous (hence Borel) functions is increasing ($f_{n+1} = 1 \geq f_n$ on Q'_{n+1}, while on the complement of that set, which is a subset of K_n^c, $f_{n+1} \geq 0 = f_n$), and $\lim f_n = I_V$ (since $\bigcup Q'_n = V$ and all the supports K_n are contained in V). By Theorem 3.18 and the Monotone Convergence theorem (1.13),

$$\lambda(V) = \int_X I_V d\lambda = \lim_n \int_X f_n d\lambda = \lim_n \phi(f_n)$$

$$= \lim_n \int_X f_n d\mu = \int_X I_V d\mu = \mu(V).$$

Claim 2. λ satisfies Property (4) in Theorem 3.18 for *closed* sets.
Let $F \subset X$ be closed. The σ-compactness hypothesis (applied to the open set X) implies the existence of compact sets Q_n such that $Q_n \subset Q_{n+1}$ and $X = \bigcup Q_n$. Then

$K_n := F \cap Q_n \subset F$ are compact, $K_n \subset K_{n+1}$, and $\bigcup K_n = F$. Hence, by Lemma 1.10, $\lambda(F) = \lim \lambda(K_n)$, and consequently

$$\lambda(F) = \sup_{K \in \mathcal{K}; K \subset F} \lambda(K).$$

We now complete the solution of Exercise 2. Let E be any Borel subset of X and $\epsilon > 0$. By Property (1) of μ (cf. Theorem 3.18), $E \in \mathcal{M}$. Since X is σ-compact, it follows from Theorem 3.19(1) that there exist F closed and V open such that

$$F \subset E \subset V; \qquad \mu(V - F) < \epsilon.$$

Since $V - F$ is open, this means that $\lambda(V - F) < \epsilon$ (by Claim 1), hence $\lambda(E - F) < \epsilon$. Therefore, by Claim 2,

$$\sup_{K \in \mathcal{K}; K \subset E} \lambda(K) \leq \lambda(E) = \lambda(F) + \lambda(E - F)$$

$$= \sup_{K \in \mathcal{K}; K \subset F} \lambda(K) + \lambda(E - F) \leq \sup_{K \in \mathcal{K}; K \subset E} \lambda(K) + \epsilon,$$

and consequently

$$\lambda(E) = \sup_{K \in \mathcal{K}; K \subset E} \lambda(K) \tag{3}$$

by the arbitrariness of ϵ.
Similarly, with V, F as above, we have

$$\lambda(V) = \lambda(E) + \lambda(V - E) \leq \lambda(E) + \lambda(V - F) \leq \lambda(E) + \epsilon.$$

Hence

$$\lambda(E) = \inf_{E \subset U \in \tau} \lambda(U) \tag{4}$$

By (3) and (4), λ is regular.

Hardy inequality

3. Let $1 < p < \infty$, and let $L^p(\mathbb{R}^+)$ denote the Lebesgue space for $\mathbb{R}^+ := (0, \infty)$ with respect to the Lebesgue measure. For $f \in L^p(\mathbb{R}^+)$, define

$$(Tf)(x) = (1/x) \int_0^x f(t)\, dt \qquad (x \in \mathbb{R}^+).$$

Prove:

(a) Tf is well defined, and $|(Tf)(x)| \leq x^{-1/p} ||f||_p$.

Solution.

Let $x > 0$. If $f \in L^p(\mathbb{R}^+)$, then $f \in L^p(0,x)$; since $1 \in L^q(0,x)$ (where $1/p + 1/q = 1$), it follows that $f = f.1 \in L^1(0,x)$ (cf. Theorem 1.26), hence $(Tf)(x)$ is well defined, and

$$|(Tf)(x)| \leq (1/x)\int_0^x |f(t)|dt \leq (1/x)(\int_0^x 1^q dt)^{1/q}(\int_0^x |f|^p dt)^{1/p}$$
$$\leq x^{(1/q)-1}||f||_p = x^{-1/p}||f||_p.$$

(b) Denote by D, M, and I the differentiation, multiplication by x, and identity operators, respectively (on appropriate domains). Verify the identities

$$MDT = I - T \quad \text{on } C_c^+(\mathbb{R}^+), \tag{5}$$

where multiplication of operators is their composition.

$$||Tf||_p^p = q\int_0^\infty f\,(Tf)^{p-1}\,dx \tag{6}$$

for all $f \in C_c^+(\mathbb{R}^+)$, where q is the conjugate exponent of p. (Hint: integrate by parts.)

Solution.

If $f \in C_c(\mathbb{R}^+)$, Tf is well defined, and

$$(MDTf)(x) = MD[x^{-1}\int_0^x f(t)\,dt] = M\Big[-x^{-2}\int_0^x f(t)\,dt + x^{-1}f(x)\Big] = (I-T)f(x).$$

This proves (5).
Let $f \in C_c^+(\mathbb{R}^+)$. Then $Tf \geq 0$, and (integrating by parts)

$$||Tf||_p^p = \int_0^\infty (Tf)^p dx$$

$$= \Big[x(Tf)(x)^p\Big]_0^\infty - p\int_0^\infty x(Tf)(x)^{p-1}(DTf)(x)\,dx. \tag{7}$$

Let $[0,N]$ be an interval containing the support of f. Then

$$x(Tf)(x)^p = x^{1-p}(\int_0^x f(t)\,dt)^p \leq x^{1-p}(\int_0^N f(t)\,dt)^p \to 0$$

as $x \to \infty$, since $1 - p < 0$. Therefore the first term in (7) vanishes. By (5) and (7),

$$||Tf||_p^p = -p\int_0^\infty (Tf)^{p-1} MDTf\,dx = p\int_0^\infty (Tf)^{p-1}(Tf - f)\,dx$$

$$= p\Big[||Tf||_p^p - \int_0^\infty f(Tf)^{p-1}dx\Big].$$

Dividing by p and rearranging, we get the identity (6).

(c) $||Tf||_p \leq q\,||f||_p \qquad f \in C_c^+(\mathbb{R}^+)$.

Solution.

By (6) and Holder's inequality (Theorem 1.24), we obtain (since $(p-1)q = p$)

$$||Tf||_p^p \leq q\,||f||_p||(Tf)^{p-1}||_q = q\,||f||_p\Big(\int_0^\infty (Tf)^{(p-1)q}dx\Big)^{1/q}$$

$$= q\,||f||_p||Tf||_p^{p/q}.$$

If $||Tf||_p \neq 0$, we divide this inequality by $||Tf||_p^{p/q}$ and obtain (c) (since $p-(p/q) = 1$). If $||Tf||_p = 0$, (c) is trivial.

(d) Extend the (Hardy) inequality (c) to all $f \in L^p(\mathbb{R}^+)$. (Hint: use Corollary 3.21.)

Solution.

Let $f \in C_c(\mathbb{R}^+)$. Applying (c) to $|f|$, we obtain (since $|Tf| \leq T|f|$)

$$||Tf||_p \leq ||\,T|f|\,||_p \leq q||f||_p \qquad (f \in C_c(\mathbb{R}^+)). \tag{8}$$

Next, let $f \in L^p(\mathbb{R}^+)$. By Corollary 3.21, there exists a sequence $\{f_n\} \subset C_c(\mathbb{R}^+)$ such that $f_n \to f$ in L^p-norm. For each $x > 0$, we have by Part (a)

$$|(Tf_n)(x) - (Tf)(x)| = |[T(f_n - f)](x)| \leq x^{-1/p}||f_n - f||_p \to 0.$$

Thus $Tf_n \to Tf$ pointwise. By (8) applied to $f_n - f_m$,

$$||Tf_n - Tf_m||_p = ||T(f_n - f_m)||_p \leq q\,||f_n - f_m||_p \to 0,$$

i.e., $\{Tf_n\}$ is Cauchy in $L^p(\mathbb{R}^+)$. Let $g = \lim_n Tf_n$ (in L^p-metric; cf. Theorem 1.29). By Lemma 1.30, there exists a subsequence Tf_{n_k} converging pointwise almost everywhere to g. However this subsequence converges pointwise to Tf, as we observed earlier. It follows that $g = Tf$ (as elements of L^p), i.e., $Tf_n \to Tf$ in L^p-norm. Letting $n \to \infty$ in the inequalities $||Tf_n||_p \leq q\,||f_n||_p$ (cf. (8)), we obtain $||Tf||_p \leq q\,||f||_p$, as wanted.

(e) Show that $\sup_{0\neq f\in L^p} \frac{||Tf||_p}{||f||_p} = q$. (Hint: consider the functions $f_n(x) = x^{-1/p}I_{[1,n]}$.)

Solution.

Let f_n be as in the hint. We have

$$||f_n||_p = \Big(\int_1^n dx/x\Big)^{1/p} = (\log n)^{1/p}. \tag{9}$$

For $1 \le x \le n$,

$$(Tf_n)(x) = (1/x)\int_1^x t^{-1/p}dt = q(x^{-1/p} - x^{-1}).$$

Therefore, by the triangle inequality for norms,

$$||Tf_n||_p \ge ||Tf_n||_{L^p(1,n)} = q||x^{-1/p} - x^{-1}||_{L^p(1,n)}$$

$$\ge q\Big[||x^{-1/p}||_{L^p(1,n)} - ||x^{-1}||_{L^p(1,n)}\Big] = q\Big[(\int_1^n dx/x)^{1/p} - (\int_1^n x^{-p}dx)^{1/p}\Big]$$

$$= q\Big[(\log n)^{1/p} - (\frac{1-n^{1-p}}{p-1})^{1/p}\Big] \ge q\Big[(\log n)^{1/p} - [1/(p-1)]^{1/p}\Big]. \tag{10}$$

If s denotes the supremum in (e), then by (9) and (10),

$$s \ge \frac{||Tf_n||_p}{||f_n||_p} \ge q\Big[1 - \frac{1}{[(p-1)\log n]^{1/p}}\Big].$$

Letting $n \to \infty$, we get $s \ge q$. Since $s \le q$ by Part (d), we conclude that $s = q$.

Absolutely continuous and singular functions

4. Recall that a function $f : \mathbb{R} \to \mathbb{C}$ has *bounded variation* if its *total variation function* v_f is *bounded*, where

$$v_f(x) := \sup_P \sum_k |f(x_k) - f(x_{k-1})| < \infty,$$

and $P = \{x_k; k = 0, ..., n\}$, $x_{k-1} < x_k$, $x_n = x$ (the supremum is taken over all such "partitions" P of $(-\infty, x]$).

The *total variation* of f is $V(f) := \sup_{\mathbb{R}} v_f$.

It follows from a theorem of Jordan that such a function has a "canonical" (Jordan) decomposition $f = \sum_{k=0}^3 i^k f_k$ where f_k are non-decreasing real function. Therefore f has one sided limits at every point. We say that f is *normalized* if it is left-continuous and $f(-\infty) = 0$.

(a) Let μ be a complex Borel measure on $\mathbb{R}$. Show that $f(x) := \mu((-\infty, x))$ is a normalized function of bounded variation (briefly, f is NBV).

Solution.

Let $x \in \mathbb{R}$ and let $P : -\infty < x_0, \cdots, x_n = x$ be a "partition" of $(-\infty, x)$. Then the intervals $(-\infty, x_0), [x_0, x_1), \cdots, [x_{n-1}, x_n)$ form a partition of $(-\infty, x)$ in the sense of Definition 1.42. Therefore

$$\sum_{k=1}^{n} |f(x_k) - f(x_{k-1}| = \sum_k |\mu([x_{k-1}, x_k))| \leq |\mu|((-\infty, x))$$

$$\leq |\mu|(\mathbb{R}) := ||\mu|| < \infty. \tag{11}$$

Hence

$$v_f(x) \leq |\mu|((-\infty, x)) \leq ||\mu||, \tag{12}$$

and

$$V(f) \leq ||\mu||. \tag{13}$$

Thus, f is BV, and has therefore one-sided limits at every point. To show left-continuity of f at any x, it suffices to prove that $\lim_n f(x - h_n) = f(x)$ for any decreasing positive sequence $\{h_n\}$ with $\lim h_n = 0$. For such h_n, we have

$$[x - h_n, x) \subset [x - h_{n-1}, x); \qquad \bigcap_n [x - h_n, x) = \emptyset.$$

Therefore, by Lemma 1.11,

$$\lim_n |\mu|([x - h_n, x)) = |\mu|(\emptyset) = 0.$$

Since $|f(x) - f(x - h_n)| = |\mu([x - h_n, x))| \leq |\mu|([x - h_n, x))$, we conclude that

$$f(x - h_n) \to f(x).$$

Finally,

$$f(-\infty) = \lim_{n\to\infty} f(-n) = \lim_n \mu((-\infty, -n)).$$

But $(-\infty, -n) \subset (-\infty, -(n-1))$ and $\bigcap_n (-\infty, -n) = \emptyset$. Therefore, by Lemma 1.11, $\lim_n |\mu|((-\infty, -n)) = 0$; hence $\lim_n \mu((-\infty, -n)) = 0$, i.e., $f(-\infty) = 0$. Summing up, f is NBV.

(b) Conversely, if f is NBV and μ is the corresponding Lebesgue-Stieltjes measure (constructed through the Jordan decomposition of f as in Chapter 2, with left continuity replacing right continuity), then μ (restricted to $\mathcal{B} := \mathcal{B}(\mathbb{R})$) is a complex Borel measure such that $f(x)=\mu((-\infty,x))$ for all $x\in\mathbb{R}$. (Also $v_f(x)=|\mu|((-\infty,x))$ and $V(f)=||\mu||$.)

Solution.

If $f=\sum_{k=0}^{3} i^k f_k$ is the Jordan decomposition of f with f_k real non-decreasing left-continuous functions such that $f_k(-\infty)=0$, and μ_k are the corresponding positive Borel measures, then $\mu:=\sum_k i^k\mu_k$.
The measures μ_k satisfy (cf. page 65, with the appropriate change)

$$\mu_k([a,b))=f_k(b)-f_k(a);\qquad -\infty<a<b<\infty.$$

Therefore $\mu([a,b))=f(b)-f(a)$ for all a,b as above. Write $(-\infty,x)=\bigcup_{\{n;-n<x\}}[-n,x)$. Since $[-n,x)\subset[-(n+1),x)$, Lemma 1.10 implies that

$$\mu_k((-\infty,x))=\lim_n \mu_k([-n,x))=\lim_n[f_k(x)-f_k(-n)]=f_k(x),$$

and therefore $\mu((-\infty,x))=f(x)$.
By Part (a) (cf. (12)), this last relation implies that $v_f(x)\le|\mu|((-\infty,x))$ for all $x\in\mathbb{R}$. In order to prove the reverse inequality, we shall use the Riesz representation theorem (4.9) as a shortcut.
Observe that $|\mu|$ is regular (by Exercise 2); therefore μ is regular by definition. Consider the continuous linear functional ϕ on $C_c((-\infty,x))$ defined by

$$\phi(h)=\int_{(-\infty,x)} h\,d\mu;\qquad h\in C_c((-\infty,x)).$$

By uniqueness of the representating regular measure for ϕ, we know from Theorem 4,9 that $|\mu|((-\infty,x))=||\phi||$, where the right hand side in the norm of the functional, that is,

$$||\phi||:=\sup_{h\in C_c((-\infty,x));\,||h||_u\le 1}\left|\int_{(-\infty,x)} h\,d\mu\right|.$$

By Exercise 7(h) in Chapter 2, the last supremum remains unchanged when we let h vary over the complex step functions on $(-\infty,x)$ with $||h||_u\le1$, that is, over all h of the form

$$h=\sum_k c_k I_{[x_{k-1},x_k)};\qquad c_k\in\mathbb{C},\quad |c_k|\le 1,$$

where $\{x_k\}$ is a "partition" of $(-\infty,x)$. For such h, we have

$$\left|\int_{(-\infty,x)} h\,d\mu\right|=\left|\sum_k c_k[f(x_k)-f(x_{k-1})]\right|\le\sum_k|f(x_k)-f(x_{k-1})|\le v_f(x).$$

Hence

$$|\mu|((-\infty, x)) = ||\phi|| \leq v_f(x),$$

and the equality $|\mu|((-\infty, x)) = v_f(x)$ (for all $x \in \mathbb{R}$) follows. Consequently (cf. Lemma 1.10)

$$V(f) := \sup_x v_f(x) = \sup_x |\mu|((-\infty, x)) = |\mu|(\mathbb{R}) := ||\mu||.$$

(c) $f : \mathbb{R} \to \mathbb{C}$ is *absolutely continuous* if for each $\epsilon > 0$ there exists $\delta > 0$ such that whenever $\{(a_k, b_k);\ k = 1, ..., n\}$ is a finite family of disjoint intervals of total length $< \delta$, we have $\sum_k |f(b_k) - f(a_k)| < \epsilon$. If f is NBV and μ is the Borel measure associated to f as in Part (b), then $\mu << m$ iff f is absolutely continuous. Cf. Theorem 3.28 and Exercise 8(f) in Chapter 1.(m denotes the Lebesgue measure on $\mathbb{R}$, and also its restriction to the Borel sets on $\mathbb{R}$.)

Solution.

Taking $n = 1$ in the definition of absolute continuity, we see that if f is AC, then it is (uniformly) continuous on $\mathbb{R}$. If f is also NBV and μ is the associated measure, then for any $a, b \in \mathbb{R}$ with $a < b$,

$$\mu((a, b)) = f(b) - f(a). \tag{14}$$

Indeed, write (a, b) as the union of the intervals $I_n = [a + 1/n, b)$. Since $I_n \subset I_{n+1}$, we have $\mu_k((a, b)) = \lim_n \mu_k(I_n)$ for $k = 0, \cdots, 3$ (cf. Lemma 1.10), hence

$$\mu((a, b)) = \lim_n \mu(I_n) = \lim_n [f(b) - f(a + 1/n)] = f(b) - f(a)$$

by the continuity of f at a.
Let f be NBV and AC. Suppose E is a Borel set on $\mathbb{R}$ such that $m(E) = 0$, and let $\epsilon > 0$. Let $\delta > 0$ be associated with $\epsilon/2$ as in the definition of absolute continuity. Since m is regular, there exists an open set V such that $E \subset V$ and $m(V) < \delta$. Let W be *any* open set such that $E \subset W \subset V$. Write W as the countable union of the disjoint intervals (a_k, b_k) (any open set on $\mathbb{R}$ can be written in this form!). For any n,

$$\sum_{k=1}^{n} (b_k - a_k) < \sum_{k=1}^{\infty} (b_k - a_k) = m(W) \leq m(V) < \delta.$$

Hence

$$\sum_{k=1}^{n} |f(b_k) - f(a_k)| < \epsilon/2$$

for all n, and therefore

$$\sum_{k=1}^{\infty} |f(b_k) - f(a_k)| \leq \epsilon/2.$$

Consequently (by (14))

$$|\mu(W)| = |\sum_{k=1}^{\infty} \mu((a_k, b_k))| \leq \sum_k |\mu((a_k, b_k))|$$

$$= \sum_k |f(b_k) - f(a_k)| \leq \epsilon/2. \tag{15}$$

Since $|\mu|$ is regular (cf. Exercise 2), we can choose W such that

$$|\mu|(W) < |\mu|(E) + \epsilon/2.$$

Then

$$|\mu(W - E)| \leq |\mu|(W - E) = |\mu|(W) - |\mu|(E) < \epsilon/2,$$

and therefore (by (15))

$$|\mu(E)| = |\mu(W) - \mu(W - E)| \leq |\mu(W)| + |\mu(W - E)| < \epsilon.$$

Hence $\mu(E) = 0$ by the arbitrariness of ϵ. This proves that if f is AC, then $\mu << m$.

Conversely, suppose $\mu << m$. Then $|\mu| << m$ (cf. Exercise 8(d), Chapter 1). By Exercise 8(f) in Chapter 1, given $\epsilon > 0$, there exists $\delta > 0$ such that whenever E is a Borel set with $m(E) < \delta$, one has $|\mu|(E) < \epsilon$. If (a_k, b_k), $k = 1, \cdots, n$, are disjoint intervals with total length $< \delta$, take E to be their union. Then

$$m(E) = \sum_k m((a_k, b_k)) = \sum_k (b_k - a_k) < \delta,$$

and therefore (by (14))

$$\sum_k |f(b_k) - f(a_k)| = \sum_k |\mu((a_k, b_k)| \leq \sum_k |\mu|((a_k, b_k)) = |\mu|(E) < \epsilon.$$

This proves that f is AC.

(d) Let $h \in L^1 := L^1(\mathbb{R})$, $f(x) = \int_{-\infty}^{x} h(t)dt$, and $\mu(E) = \int_E h(t)dt \quad (E \in \mathcal{B})$. Conclude from Parts (a) and (c) that f is absolutely continuous and $D\mu = h$ m-a.e. (Cf. Theorem 3.28.)

Solution.

Clearly f and μ are associated as in Part (a), and $\mu << m$. By Part (c), f is AC (absolutely continuous). By definition, $h = d\mu/dm = d\mu_a/dm = D\mu$ m-a.e. (cf. Theorem 3.28).

(e) Let μ and f be as in Part (a), and let $x \in \mathbb{R}$ be fixed. Show that $(D\mu)(x)$ exists iff $f'(x)$ exists, and in that case $f'(x) = (D\mu)(x)$. In particular, if $\mu \perp m$, then $f'(x) = 0$ m-a.e. (such a function is called a *singular function*).(Cf. Theorem 3.28.)

Solution.

It suffices to consider *real* μ (and f).
Let $a, b \in \mathbb{R}$, $a < b$. The intervals $I_n = (a - 1/n, b)$ $(n \in \mathbb{N})$ satisfy $I_{n+1} \subset I_n$ and $\bigcap I_n = [a, b)$ Therefore, by Lemma 1.11 applied to μ^+ and μ^-,

$$\mu([a,b)) = \lim_n \mu((a - 1/n, b)). \tag{16}$$

Similarly, the intervals $J_n = [a + 1/n, b)$ (with $n \in \mathbb{N}$ large enough) satisfy $J_n \subset J_{n+1}$ and $\bigcup J_n = (a, b)$. Therefore, by Lemma 1.10 (applied to μ^+ and μ^-),

$$\mu((a,b)) = \lim_n \mu([a + 1/n, b)). \tag{17}$$

Suppose $(D\mu)(x)$ exists for some $x \in \mathbb{R}$. Let $0 < h < r$. For $n \in \mathbb{N}$ large enough, $h + 1/n < r$, i.e, the open interval $(x - 1/n, x + h)$ around x has diameter $< r$, hence

$$\inf_{x \in (a,b); \delta((a,b)) < r} \frac{\mu((a,b))}{m((a,b))} \leq \frac{\mu((x - 1/n, x + h))}{m((x - 1/n, x + h))} \leq \sup(\cdots),$$

where the sup applies to the same set of numbers as the inf above. Letting $n \to \infty$, we get from (16) that

$$\inf(\cdots) \leq h^{-1}\mu([x, x + h)) \leq \sup(\cdots),$$

that is,

$$\inf(\cdots) \leq h^{-1}[f(x + h) - f(x)] \leq \sup(\cdots). \tag{18}$$

When $r \to 0$, the end sides of (18) converge to $(D\mu)(x)$ (by hypothesis). Hence the right derivative of f at x, $f'_+(x)$, exists and equals $(D\mu)(x)$. An analogous argument shows that the left derivative $f'_-(x)$ exists and equals $(D\mu)(x)$. Hence $f'(x)$ exists and equals $(D\mu)(x)$.

Conversely, suppose $f'(x)$ exists. Given $\epsilon > 0$, there exists $\eta > 0$ such that whenever $a < x < b$ and $b - a < \eta$,

$$(b - x)[f'(x) - \epsilon] < f(b) - f(x) < (b - x)[f'(x) + \epsilon]$$

and

$$(x-a)[f'(x)-\epsilon] < f(x)-f(a) < (x-a)[f'(x)+\epsilon].$$

Adding, we get

$$(b-a)[f'(x)-\epsilon] < f(b)-f(a) < (b-a)[f'(x)+\epsilon],$$

that is

$$f'(x)-\epsilon < \frac{\mu([a,b))}{b-a} < f'(x)+\epsilon$$

whenever $b-a<\eta$. For such $a<b$, also $a+1/n<b$ (for n large enough) and $b-(a+1/n)<\eta$, so that

$$f'(x)-\epsilon < \frac{\mu([a+1/n,b))}{b-(a+1/n)} < f'(x)+\epsilon.$$

Letting $n\to\infty$, it follows from (17) that

$$f'(x)-\epsilon \le \frac{\mu((a,b))}{m((a,b))} \le f'(x)+\epsilon$$

whenever $a<x<b$ and $b-a<\eta$. For any r such that $0<r<\eta$, we then get

$$f'(x)-\epsilon \le \inf_{b-a<r} \frac{\mu((a,b))}{m((a,b)} \le \sup(\cdots) \le f'(x)+\epsilon.$$

Letting $r\to 0$, we conclude that

$$f'(x)-\epsilon \le (\underline{D}f)(x) \le (\overline{D}f)(x) \le f'(x)+\epsilon.$$

Since ϵ is arbitrary, it follows that $(Df)(x)$ exists and is equal to $f'(x)$.

(f) With h and f as in Part (d), conclude from Parts (d) and (e) (and Theorem 3.28) that $f'=h$ m-a.e.

Solution.

If h, f, and μ are as in Part (d), then f and μ are related as in Part (a) and $\mu << m$. By Part (d), $D\mu = h$ m-a.e.. By Part (e), this means that $f'=h$ m-a.e.

(g) If f is NBV, show that f' exists m-a.e. and is in L^1, and $f(x)=f_s(x)+\int_{-\infty}^{x} f'(t)dt$ where f_s is a singular NBV function. (Apply Parts (b), (e), and (f), and the Lebesgue decomposition.)

Solution.

Let f be NBV, let μ be associated to f as in Part (b). By the Lebesgue-Radon-Nikodym theorem (1.45), we have the unique (Lebesgue) decomposition

$$\mu = \mu_s + \mu_a; \qquad \mu_s \perp m, \ \mu_a << m, \tag{19}$$

and

$$\mu_a(E) = \int_E h \, dm \qquad (E \in \mathcal{B}) \tag{20}$$

for a unique $h \in L^1(m)$ (also denoted $d\mu_a/dm$, the "Radon-Nikodym derivative" of μ_a with respect to m).
By Part (b), f and μ are related as in Part (a). We define $f_s(x) = \mu_s((-\infty, x))$ and $f_a(x) = \mu_a((-\infty, x))$. Then f_s and f_a are NBV and by (19)

$$f = f_s + f_a \tag{21}$$

and

$$f_a(x) = \int_{-\infty}^{x} h(t) \, dt \qquad (x \in \mathbb{R}). \tag{22}$$

By Part (e), f_s is a singular function. By Part (f), (21), and (22), $h = f_a' = f'$ m-a.e. (since $f_s = 0$ m-a.e.), and the wanted conclusion follows from (21) and (22).

Cantor functions

5. Let $\{r_n\}_{n=0}^{\infty}$ be a positive decreasing sequence with $r_0 = 1$. Denote $r = \lim_n r_n$. Let $C_0 = [0,1]$, and for $n \in \mathbb{N}$, let C_n be the union of the 2^n disjoint closed intervals of length $r_n/2^n$ obtained by removing open intervals at the center of the 2^{n-1} intervals comprising C_{n-1} (note that the removed intervals have length $(r_{n-1} - r_n)/2^{n-1} > 0$ and $m(C_n) = r_n$). Let $C = \bigcap_n C_n$.

(a) C is a compact set of Lebesgue measure r.

Solution.

The set C_n is the disjoint union of 2^n closed intervals J_k. Hence C_n is closed. Therefore $C := \bigcap C_n$ is closed and bounded ($\subset [0,1]$), hence compact.
Assume that the 2^{n-1} intervals of C_{n-1} have length $r_{n-1}/(2^{n-1})$. When we remove from each of these intervals a central open interval of length $(r_{n-1} - r_n)/(2^{n-1})$, we obtain the $2 \times 2^{n-1} = 2^n$ intervals of C_n, each of length

$$(1/2)\Big[r_{n-1}/(2^{n-1}) - (r_{n-1} - r_n)/(2^{n-1})\Big] = r_n/(2^n).$$

Since the assumption is clearly valid for $n = 1$, it is valid for all n (by induction). Therefore $m(C_n) = 2^n \times r_n/(2^n) = r_n$. We have $C_n \subset C_{n-1} \subset [0,1]$. Therefore, by Lemma 1.11, $m(C) = \lim_n m(C_n) = \lim r_n = r$.

(b) Let $g_n = r_n^{-1} I_{C_n}$ and $f_n(x) = \int_0^x g_n(t)dt$. Then f_n is continuous, non-decreasing, constant on each open interval comprising C_n^c, $f_n(0) = 0$, $f_n(1) = 1$, and f_n converge *uniformly* in $[0,1]$ to some function f. The function f is continuous, non-decreasing, has range equal to $[0,1]$, and $f' = 0$ on C^c. (In particular, if $r = 0$, $f' = 0$ m-a.e., but f is *not* constant. Such so-called *Cantor functions* are examples of continuous non-decreasing non-constant singular functions.)

Solution.

Since g_n is an integrable non-negative Borel function, f_n is absolutely continuous (cf. Exercise 4(d)), hence continuous, and non-decreasing; $f_n(0) = 0$, and $f_n(1) = r_n^{-1} m(C_n) = 1$. The set $C_n^c := C_0 - C_n$ is the finite disjoint union of open intervals O_j, on which g_n vanishes; therefore f_n is constant on each O_j.
Each interval J of C_n contributes two such intervals J' of C_{n+1}, hence

$$\int_J g_{n+1} dt = 2r_{n+1}^{-1} m(J') = 2r_{n+1}^{-1} r_{n+1}/2^{n+1} = 1/2^n$$

$$= r_n^{-1} m(J) = \int_J g_n dt. \tag{23}$$

If $x \in C_n^c$, there exists an index set $H(x)$ such that $J_k \subset (0,x)$ iff $k \in H(x)$ and $J_k \subset (x,1]$ for the remaining indices. Therefore, by (23),

$$f_n(x) = \sum_{k\in H(x)} \int_{J_k} g_n(t)dt = \sum_{k\in H(x)} \int_{J_k} g_{n+1}(t)dt = f_{n+1}(x) \qquad (x \in C_n^c). \tag{24}$$

If $x \in C_n$, we have a unique interval J^* of C_n which contains x. For all other intervals J of C_n contained in $[0,x]$, $\int_J g_{n+1} dt = \int_J g_n dt$ by (23). Therefore (since $g_n \geq 0$, and using (23))

$$|f_{n+1}(x) - f_n(x)| = |\int_{J^*\cap[0,x]} (g_{n+1} - g_n)dt|$$

$$\leq \int_{J^*} g_{n+1} dt + \int_{J^*} g_n dt = 1/2^{n-1} \qquad (x \in C_n). \tag{25}$$

By (24) and (25),

$$|f_{n+1}(x) - f_n(x)| \leq 1/2^{n-1} \qquad (x \in [0,1]).$$

Hence the series

$$f_0 + \sum_{n=0}^{\infty}(f_{n+1} - f_n)$$

converges uniformly on $[0,1]$; equivalently, f_n converge uniformly on $[0,1]$ to some function f. By the relevant properties of f_n, the function f is *continuous*, non-decreasing, $f(0) = 0$, and $f(1) = 1$ (therefore f has the range $[0,1]$, by the intermediate value theorem for continuous functions).
If $x \in C^c = \bigcup_n C_n^c$, and $n_0 \in \mathbb{N}$ is such that $x \in C_{n_0}^c$, then $x \in C_n^c$ for all $n \geq n_0$, and it follows from (24) that $f(x) = f_{n_0}(x)$. If O_j be the unique open interval in $C_{n_0}^c$ such that $x \in O_j$, then $f = f_{n_0}$ on O_j, and we saw before that f_{n_0} is constant on O_j; hence f is constant on an open neighborhood of x. This shows that $f' = 0$ on C^c. If $r = 0$, we have $m(C) = 0$, and therefore $f' = 0$ m-a.e.

Semi-continuity

6. Let X be a topological space. A function $f : X \to \overline{\mathbb{R}}$ is *lower semi-continuous* (l.s.c.) if $[f > c]$ is open for all real c; f is *upper semi-continuous* (u.s.c.) if $[f < c]$ is open for all real c. Prove:

(a) f is continuous iff it is both l.s.c. and u.s.c.

Solution.

Suppose f is continuous. Since $(c,\infty]$ and $[-\infty,c)$ are open sets in $\overline{\mathbb{R}}$ for all real c, it follows that $[f > c] = f^{-1}((c,\infty])$ and $[f < c] = f^{-1}([-\infty,c))$ are open for all c, so that f is both l.s.c. and u.s.c. Conversely, if f is both l.s.c. and u.s.c., then for all real $a < b$,

$$f^{-1}((a,b)) = f^{-1}([-\infty,b) \cap (a,\infty]) = [f^{-1}([-\infty,b))] \cap [f^{-1}((a,\infty])]$$

is open. Since every open set V in $\overline{\mathbb{R}}$ is the (countable) union of intervals of the form $(a,b), (c,\infty]$, and $[-\infty,c)$, it follows that $f^{-1}(V)$ is open for every open $V \subset \overline{\mathbb{R}}$, so that f is continuous.

(b) If f is l.s.c. (u.s.c.) and α is a positive constant, then αf is l.s.c. (u.s.c., respectively). Also $-f$ is u.s.c. (l.s.c., respectively).

(*Solution*: trivial.)

(c) If f, g are l.s.c. (u.s.c.), then $f + g$ is l.s.c. (u.s.c., respectively).

Solution.

(We need of course to assume that $f+g$ is well defined, that is f and g do not assume opposite infinite value at any point.)
Let $c \in \mathbb{R}$. For any $a \in \mathbb{R}$, if $f(x) > a$ and $g(x) > c-a$, then $(f+g)(x) > c$. Hence

$$\bigcup_{a\in\mathbb{R}} \Big([f>a] \cap [g > c-a]\Big) \subset [f+g>c]. \tag{26}$$

On the other hand, if $f(x)+g(x) > c$ (i.e., $f(x) > c-g(x)$), choose any a such that $f(x) > a > c-g(x)$. Then $x \in [f>a]\cap[g>c-a]$ (for our choice of a). This shows that *equality* holds in (26).
Suppose now that f and g are l.s.c. Then by the *equality* (26) the sets $[f+g>c]$ are open for all c, i.e., $f+g$ is l.s.c.
If f and g are u.s.c., then $-f$ and $-g$ are l.s.c. (by Part (b)). Hence $(-f)+(-g)$ is l.s.c., and therefore $f+g$ is u.s.c. (by Part (b)).

(d) The supremum (infimum) of any family of l.s.c. (u.s.c.) functions is l.s.c. (u.s.c., respectively).

Solution.

Let $\{f_\alpha; \alpha \in J\}$ be a family of l.s.c. functions, and let $f = \sup f_\alpha$. For any real c,

$$[f>c] = \bigcup_{\alpha\in J} [f_\alpha > c]$$

is open, hence f is l.s.c.
If all f_α are u.s.c, then $-f_\alpha$ are l.s.c. by Part (b). Hence $\sup(-f_\alpha)$ is l.s.c., and therefore $\inf f_\alpha = -\sup(-f_\alpha)$ is u.s.c.

(e) If $\{f_n\}$ is a sequence of non-negative l.s.c. functions, then $f := \sum_n f_n$ is l.s.c.

Solution.

By Part (c) and induction,the partial sums s_n of the series are l.s.c. Since the summands are non-negative, the sequence $\{s_n\}$ is non-decreasing, and therefore $f = \sup_n s_n$, and consequently f is l.s.c. by Part (d).

(f) The indicator I_A is l.s.c. (u.s.c.) iff $A \subset X$ is open (closed, respectively).

Solution.

Let $c \in \mathbb{R}$. Then $[I_A > c]$ is equal to X, A, or $\emptyset$ iff $c < 0$, $0 \le c < 1$, or $c \ge 1$, respectively. Therefore $[I_A > c]$ is open for all c iff A is open. Equivalently, I_A is l.s.c.

iff A is open.
Similarly, $[I_A < c]$ is equal to $\emptyset$, A^c, or X iff $c \leq 0$, $0 < c \leq 1$, or $c > 1$, respectively. Therefore $[I_A < c]$ is open for all c iff A^c is open. Equivalently, I_A is u.s.c. iff A is closed.

7. Let $(X, \mathcal{M}, \mu)$ be a positive measure space as in the Riesz-Markov theorem.

(a) Let $0 \leq f \in L^1(\mu)$ and $\epsilon > 0$. Represent $f = \sum_{j=1}^{\infty} c_j I_{E_j}$ as in Exercise 15, Chapter 1, and choose K_j compact and V_j open such that $K_j \subset E_j \subset V_j$ and $\mu(V_j - K_j) < \epsilon/(c_j 2^{j+1})$. Fix n such that $\sum_{j>n} c_j \mu(E_j) < \epsilon/2$ and define $u = \sum_{j=1}^{n} c_j I_{K_j}$ and $v = \sum_{j=1}^{\infty} c_j I_{V_j}$. Prove that u is u.s.c., v is l.s.c., $u \leq f \leq v$, and $\int_X (v-u) d\mu < \epsilon$.

Solution.

(Note that $v - u = v + (-u)$ is l.s.c. by Exercise 6. It is therefore a (non-negative) Borel function, since l.s.c. and u.s.c. functions are Borel, by definition and Lemma 1.4, although this fact is obvious for the explicitly given function $v - u$. In particular, the integral $\int_X (v-u)\, d\mu$ makes sense, cf. 3.18 (1).)

Since $f \in L^1(\mu)$, we have necessarily $\mu(E_j) < \infty$ (cf. Exercise 15, Chapter 1), and therefore, by the properties (3) and (4)(ii) of the measure space in Theorem 3.18, sets K_j and V_j as above do exist.
Since $\sum_j c_j \mu(E_j) = \int_X f\, d\mu < \infty$, there exists n as above.
The sets K_j are compact, hence closed (since X is Hausdorff). Therefore $c_j I_{K_j}$ are u.s.c. (by Exercise 6, Parts (f) and (b)). Hence u is u.s.c., by Exercise 6, Part (c) (and induction). By Exercise 6, Part (f), (b), and (e) (in this order), v is l.s.c. Since $K_j \subset E_j \subset V_j$, we have $I_{K_j} \leq I_{E_j} \leq I_{V_j}$ for all j, and therefore $u \leq f \leq v$. Also

$$\int_X (v-u) d\mu = \sum_{j=1}^{n} c_j \mu(V_j - K_j) + \sum_{j>n} c_j \mu(V_j). \tag{27}$$

Each summand in the first sum is $< \epsilon/(2^{j+1})$. Since

$$\mu(V_j) = \mu(E_j) + \mu(V_j - E_j) < \mu(E_j) + \epsilon/(c_j 2^{j+1}),$$

it follows from (27) that

$$\int_X (v-u) d\mu < \sum_{j>n} c_j \mu(E_j) + \sum_{j=1}^{\infty} \epsilon/(2^{j+1}) < \epsilon/2 + \epsilon/2 = \epsilon.$$

(b) Generalize the above conclusion to any *real* function $f \in L^1(\mu)$. (This is the *Vitali-Caratheodory theorem.*) (Hint: Exercise 6.)

Solution.

Let $f \in L^1(\mu)$ be real and $\epsilon > 0$. By Part (a) for the non-negative functions $f^+, f^- \in L^1(\mu)$, there exist u.s.c. functions u_j and l.s.c. function v_j $(j = 1, 2)$ such that

$$u_1 \leq f^+ \leq v_1; \qquad u_2 \leq f^- \leq v_2; \tag{28}$$

$$\int_X (v_j - u_j) d\mu < \epsilon/2 \qquad (j = 1, 2). \tag{29}$$

By (28), $-v_2 \leq -f^- \leq -u_2$; adding these inequalities with the first inequalities in (28), we get

$$u \leq f \leq v,$$

where $u := u_1 + (-v_2)$ is u.s.c. by Exercise 6, Parts (b) and (c); $v := v_1 + (-u_2)$ is l.s.c. (same reference), and by (29)

$$\int_X (v - u) d\mu = \int_X (v_1 - u_1) d\mu + \int_X (v_2 - u_2) d\mu < \epsilon.$$

Fundamental theorem of calculus

8. Let $f : [a, b] \to \mathbb{R}$ be differentiable at every point of $[a, b]$, and suppose $f' \in L^1 := L^1([a, b])$ (with respect to Lebesgue measure dt). Denote $\int_a^b f'(t)dt = c$ and fix $\epsilon > 0$. By Exercise 7 above, there exists v l.s.c. such that $f' \leq v$ and $\int_a^b v\, dt < c + \epsilon$. Fix a constant $r > 0$ such that $r(b - a) < c + \epsilon - \int_a^b v\, dt$, and let $g = v + r$. Observe that g is l.s.c., $g > f'$, and $\int_a^b g\, dt < c + \epsilon$. By the l.s.c. property of g and the differentiability of f, we may associate to each $x \in [a, b)$ a number $\delta(x)$ such that $g(t) > f'(x)$ *and* $f(t) - f(x) < (t - x)[f'(x) + \epsilon]$ for all $t \in (x, x + \delta(x))$.

Define

$$F(x) = \int_a^x g(t)dt - f(x) + f(a) + \epsilon(x - a).$$

(F is clearly continuous and $F(a) = 0$.)

(a) Show that $F(t) > F(x)$ for all $t \in (x, x + \delta(x))$.

Solution.

Let $x \in [a, b)$ and $s \in (x, x + \delta(x))$. Then $g(t) > f'(x)$ for $t \in [x, s]$ and $f(s) - f(x) < (s - x)f'(x) + \epsilon(s - x)$. Therefore

$$F(s) - F(x) = \int_x^s g(t)\, dt - [f(s) - f(x)] + \epsilon(s - x) > \int_x^s f'(x)\, dt - (s - x)f'(x) = 0.$$

(b) Conclude that $F(b) \geq 0$, and consequently $f(b) - f(a) < c + \epsilon(1 + b - a)$. Hence $f(b) - f(a) \leq c$

Solution.

Since F is continuous, the set $[F = 0]$ is closed (and $\subset [a,b]$), hence compact and non-empty (it contains a). Let $x_0 := \max[F = 0]$ (the maximum exists by compactness). Thus $F(x_0) = 0$. Suppose $x_0 < b$. By Part (a), $F(x) > F(x_0) = 0$ for all $x \in (x_0, x_0 + \delta(x_0))$. Fix x_1 in the latter interval. If $F(b) < 0$, the intermediate value theorem for the continuous function F implies the existence of x_2, $x_1 < x_2 < b$, such that $F(x_2) = 0$, contradicting the maximality of x_0 in $[F = 0]$. Thus $F(b) \geq 0$ (under the assumption $x_0 < b$, and trivially if $x_0 = b$, since then $F(b) = 0$).
Now

$$\begin{aligned} 0 \leq F(b) &= \int_a^b g\,dt - f(b) + f(a) + \epsilon(b - a) \\ &< c - f(b) + f(a) + \epsilon(1 + b - a), \end{aligned}$$

and the wanted conclusions follow.

(c) Conclude that $\int_a^b f'(t)dt = f(b) - f(a)$. (Hint: replace f by $-f$ in the conclusion of Part (b).)

Solution.

Replace f by $-f$ (so that c is replaced by $-c$). By Part (b), $-f(b) + f(a) \leq -c$, i.e., $f(b) - f(a) \geq c$. Together with Part (b) for f, this implies the wanted equality.

Approximation almost everywhere by continuous functions

9. Let $(X, \mathcal{M}, \mu)$ be a positive measure space as in the Riesz-Markov theorem. Let $f : X \to \mathbb{C}$ be a bounded measurable function vanishing outside some measurable set of finite measure. Prove that there exists a sequence $\{g_n\} \subset C_c(X)$ such that $||g_n||_u \leq ||f||_u$ and $g_n \to f$ almost everywhere. (Hint: Lusin and Exercise 16 of Chapter 1.)

Solution.

By Lusin's theorem (Theorem 3.20) with $\epsilon = 1/2^n$, $n = 1, 2, \cdots$, there exist functions $g_n \in C_c(X)$ such that $||g_n||_u \leq ||f||_u$ and $\mu([g_n \neq f]) < 1/2^n$. Let $E_n = [g_n \neq f]$. Then $\sum \mu(E_n) < \infty$. Therefore, by Exercise 16, Chapter 1 , *almost all* $x \in X$ lie in at most finitely many E_n. Fix any *such* x, and let $n_1, \cdots, n_p$ be the indices n for which $x \in E_n$. Let $n_0 = n_0(x) = \max_{1 \leq j \leq p} n_j$. If $n > n_0$, we have necessarily $x \in E_n^c$, that is, $g_n(x) = f(x)$. Hence $\lim_n g_n(x) = f(x)$. Since this is true for almost all x, we conclude that $g_n \to f$ a.e.

(Clearly, $f \in L^1(\mu)$; therefore, by Corollary 3.22, there exists a sequence $\{h_n\} \subset C_c(X)$ that converges to f almost everywhere. However the property $||h_n||_u \leq ||f||_u$ is not garanteed.)