CHAPTER 2

CONSTRUCTION OF MEASURES

1. Calculate (with appropriate justification).
(*Observation.* All integrands in this exercise are continuous, hence Borel (hence Lebesgue) measurable, and all improper Riemann integrals below converge absolutely. Therefore (cf. page 68) the integrals below can be considered interchangeably as Riemann or Lebesgue integrals.)

(a) $\lim_{n\to\infty}\int_{\mathbb{R}}(e^{-x^2/n})/(1+x^2)dx$.

Solution.

The integrands form a monotonely increasing sequence of positive Borel measurable functions converging pointwise to $1/(1+x^2)$. By the Monotone Convergence theorem (1.13), the wanted limit is $\int_{\mathbb{R}} dx/(1+x^2)=\pi$.

(b) $\lim_{t\to 0+}\int_0^{\pi/2}\sin[(\pi/2)e^{-tx^2}]\cos x\ dx$.

Solution.

The (Borel measurable) integrands f_t converge pointwise to $\cos x$ as $t\to 0+$, and satisfy $|f_t(x)|\le\cos x\in L^1(0,\pi/2)$. By the Dominated Convergence theorem (1.20), for any sequence $t_n\to 0+$, $\int_0^{\pi/2}f_{t_n}(x)dx\to\int_0^{\pi/2}\cos x\,dx=1$. Consequently the desired limit exists and is equal to 1.

(c) $\int_0^1\int_0^\infty[y\arctan(xy)]/[(1+x^2y^2)(1+y^2)]\,dy\,dx$.

Solution.

The integrand is Lebesgue measurable, and non-negative on $[0,1]\times[0,\infty)$. By the Tonelli theorem (2.18) applied to the Lebesgue measure spaces on $[0,1]$ and $[0,\infty]$, we may interchange the order of integration and get the iterated integral

$$\int_0^\infty\Big(\int_0^1\frac{\arctan(xy)}{1+(xy)^2}y\ dx\Big)\frac{dy}{1+y^2}.$$

The substitution $u = \arctan(xy)$ in the inner integral gives

$$\int_0^{\arctan y} u\,du = (1/2)\arctan^2 y.$$

We then substitute $v = \arctan y$ in the "outer" integral, and obtain the value $(1/2)\int_0^{\pi/2} v^2\,dv = \pi^3/48$.

2. Let $L^1(\mathbb{R})$ be the Lebesgue space with respect to the Lebesgue measure on $\mathbb{R}$. If $f \in L^1(\mathbb{R})$, define

$$F_u(t) = \int_{\mathbb{R}} \frac{\sin(t-s)u}{(t-s)u} f(s)\,ds \qquad (u > 0, t \in \mathbb{R}).$$

Prove:

(a) For each $u > 0$, the function $F_u : \mathbb{R} \to \mathbb{C}$ is well-defined, continuous, and bounded by $||f||_1$.

Solution.

Fix $u > 0$. Let

$$h_u(t,s) = \frac{\sin(t-s)u}{(t-s)u} \qquad (s \neq t)$$

and $h_u(t,t) = 1$. For each fixed t, $h_u(t,\cdot)$ is continuous (hence Lebesgue measurable) on $\mathbb{R}$, and $|h_u(t,\cdot)| \le 1$. Therefore, for each $f \in L^1(\mathbb{R})$, the function $h_u(t,\cdot)f$ is Lebesgue measurable and is dominated by $|f| \in L^1(\mathbb{R})$. It follows that $h_u(t,\cdot)f \in L^1(\mathbb{R})$, with $L^1(\mathbb{R})$-norm $\le ||f||_1$. This shows that F_u is well-defined, and is bounded by $||f||_1$. We prove next its continuity. Let $t_n \to t$ in $\mathbb{R}$. By the continuity of $h_u(\cdot,s)$ for each fixed s, $h_u(t_n,s)f(s) \to h_u(t,s)f(s)$, and $|h_u(t_n,\cdot)f| \le |f| \in L^1(\mathbb{R})$. By the Dominated Convergence theorem (1.20),

$$F_u(t_n) = \int_{\mathbb{R}} h_u(t_n,s)f(s)\,ds \to \int_{\mathbb{R}} h_u(t,s)f(s)\,ds = F_u(t).$$

Thus F_u is continuous on $\mathbb{R}$.

(b) $\lim_{u\to\infty} F_u = 0$ and $\lim_{u\to 0+} F_u = \int_{\mathbb{R}} f(s)\,ds$ pointwise.

Solution.

Fix $t \in \mathbb{R}$, and let $u_n \to \infty$. For all real $s \neq t$, $|h_{u_n}(t,s)| \le 1/[(t-s)u_n]$. Hence $h_{u_n}(t,\cdot)f \to 0$ a.e., and $|h_{u_n}(t,\cdot)f| \le |f| \in L^1(\mathbb{R})$. By the Dominated Convergence theorem (1.20)

$$F_{u_n}(t) = \int_{\mathbb{R}} h_{u_n}(t,s)f(s)\,ds \to 0$$

as $n \to \infty$. Since this is true for each sequence $\{u_n\}$ diverging to ∞ and for all $t \in \mathbb{R}$, we conclude that

$$\lim_{u \to \infty} F_u = 0.$$

If $u_n \to 0+$, $h_{u_n}(t,\cdot)f \to f$ pointwise on $\mathbb{R}$ and $|h_{u_n}(t,\cdot)f| \le |f| \in L^1(\mathbb{R})$; hence again by dominated convergence $F_{u_n}(t) \to \int_{\mathbb{R}} f(s)\,ds$, and therefore

$$\lim_{u \to 0+} F_u = \int_{\mathbb{R}} f(s)\,ds.$$

3. Let $h : [0,\infty) \to [0,\infty)$ have a non-negative continuous derivative, $h(0) = 0$ and $h(\infty) = \infty$. Prove that

$$\int_0^\infty \int_{[h' \ge s]} \exp[-h(t)^2]\,dt\,ds = \sqrt{\pi}/2.$$

Solution.

Define

$$f(s,t) := I_{[h' \ge s]}(t) \exp[-h(t)^2] \qquad (s,t \ge 0)$$

and

$$E := \{(s,t) \in [0,\infty) \times [0.\infty); h'(t) \ge s\}.$$

It follows from the continuity of h' that E is a closed set (if $(s_n,t_n) \in E$ converge to (s,t) in the plane, than $h'(t_n) \to h'(t)$ and $s_n \to s$, and since $h'(t_n) \ge s_n$, it follows that $h'(t) \ge s$, that is, $(s,t) \in E$). Similarly, for each $c > 0$, the set

$$F_c := \{(s,t) \in [0,\infty) \times [0,\infty); \exp[-h(t)^2] \ge c\}$$

is closed. Therefore the sets $[f \ge c] = E \cap F_c$ are closed, hence Borel (hence Lebesgue) measurable. This shows that f is Lebesgue measurable on $[0,\infty) \times [0,\infty)$ (and non-negative) (cf. appropriate version of Lemma 1.4). By the Tonelli theorem (2.18), we may intechange the order of integration. The given integral is thus equal to

$$\int_{[0,\infty)} \Big(\int_{[0,\infty)} f(s,t)\,ds \Big) dt = \int_{[0,\infty)} \Big(\int_{[0,h'(t)]} ds \Big) \exp[-h(t)^2] dt$$

$$= \int_0^\infty e^{-h(t)^2} h'(t) dt = \int_0^\infty e^{-u^2}\,du = \sqrt{\pi}/2.$$

4. Let $(X, \mathcal{A}, \mu)$ and $(Y, \mathcal{B}, \nu)$ be complete σ-finite positive measure spaces, and $p \in [1,\infty)$. Consider the map

$$[f,g] \in L^p(\mu) \times L^p(\nu) \to F(x,y) := f(x)g(y).$$

Prove:

(a) $F \in L^p(\mu \times \nu)$ and

$$||F||_{L^p(\mu\times\nu)} = ||f||_{L^p(\mu)}||g||_{L^p(\nu)}.$$

Solution.

If $f : X \to \mathbb{C}$ is $\mathcal{A}$-measurable, and $f_1(x, y) := f(x)$ on $X \times Y$, then for any open $V \subset \mathbb{C}$,

$$[f_1 \in V] = f^{-1}(V) \times Y \in \mathcal{A} \times \mathcal{B}.$$

This shows that f_1 is $\mathcal{A} \times \mathcal{B}$-measurable. The same is true for the function $g_2(x, y) := g(y)$, for any $\mathcal{B}$-measurable function g on Y. Therefore

$$F(x, y) := f(x)g(y) = f_1(x, y)g_2(x, y)$$

is $\mathcal{A}\times\mathcal{B}$-measurable on $X\times Y$ whenever f and g are functions on X and Y, measurable with respect to $\mathcal{A}$ and $\mathcal{B}$ respectively.
Suppose now that $f \in L^p(\mu)$ and $g \in L^p(\nu)$ for some $p \in [1, \infty)$, and let a, b be their respective norms. Then in particular f is $\mathcal{A}$-measurable and g is $\mathcal{B}$-measurable; hence F (and consequently $|F|^p$) is $\mathcal{A} \times \mathcal{B}$-measurable. By the Tonelli theorem (2.18; note that the hypothesis of completeness and σ-finiteness of the given measure spaces is satisfied!), we have

$$\int_{X\times Y} |F|^p d(\mu \times \nu) = \int_X \Big(\int_Y |f(x)|^p|g(y)|^p d\nu\Big) d\mu$$

$$= \int_X |f(x)|^p\Big(\int_Y |g(y)|^p d\nu\Big) d\mu = \int_X |f(x)|^p b^p d\mu = a^p b^p.$$

Thus $F \in L^p(\mu \times \nu)$ with norm ab, as desired.

(b) The map $[f, g] \to F$ is continuous from $L^p(\mu) \times L^p(\nu)$ to $L^p(\mu \times \nu)$.

Solution.

Let $[f, g]$, $[f', g'] \in L^p(\mu) \times L^p(\nu)$, and let $F(x, y) := f(x)g(y)$ and $F'(x, y) := f'(x)g'(y)$ (F, $F' \in L^p(\mu \times \nu)$ by Part (a)). Then by Part (a)

$$||F - F'||_{L^p(\mu\times\nu)} \le ||f(x)[g(y) - g'(y)]||_{L^p(\mu\times\nu)} + ||[f(x) - f'(x)]g'(y)||_{L^p(\mu\times\nu)}$$

$$= ||f||_{L^p(\mu)}||g - g'||_{L^p(\nu)} + ||f - f'||_{L^p(\mu)}||g'||_{L^p(\nu)} \to 0$$

when $[f', g'] \to [f, g]$ in $L^p(\mu) \times L^p(\nu)$ (note that $||g'||_{L^p(\nu)} \to ||g||_{L^p(\nu)}$ by continuity of the norm).

5. Let $f : \mathbb{R}^2 \to \mathbb{C}$ be Lebesgue measurable, such that $|f(x,y)| \leq M\, e^{-x^2} I_{[-|x|,|x|]}(y)$ on $\mathbb{R}^2$, for some constant $M > 0$. Prove:

(a) $f \in L^p(\mathbb{R}^2)$ for all $p \in [1, \infty)$, and $||f||_{L^p(\mathbb{R}^2)} \leq M(2/p)^{1/p}$.

Solution.

Let

$$E = \{(x,y) \in \mathbb{R}^2; |y| \leq |x|\}.$$

The set E is closed, hence Borel (hence Lebesgue measurable) in $\mathbb{R}^2$. Since $I_{[-|x|,|x|]}(y) = I_E(x,y)$, and e^{-x^2} is continuous on $\mathbb{R}^2$ (hence Lebesgue measurable on $\mathbb{R}^2$), it follows that $g(x,y) := e^{-x^2} I_{[-|x|,|x|]}(y)$ is a (non-negative) Lebesgue measurable function on $\mathbb{R}^2$, and by the Tonelli theorem (2.18)

$$\int_{\mathbb{R}^2} g^p\, dx dy = \int_{\mathbb{R}} e^{-px^2} \Big(\int_{\mathbb{R}} I_{[-|x|,|x|]}(y) dy \Big) dx = \int_{\mathbb{R}} 2|x| e^{-px^2} dx$$

$$= (2/p) \int_0^\infty 2px e^{-px^2} dx = (2/p) \int_0^\infty e^{-u} du = 2/p.$$

Consequently

$$||f||_{L^p(\mathbb{R}^2)} \leq M\, ||g||_{L^p(\mathbb{R}^2)} = M\, (2/p)^{1/p}.$$

(b) Suppose $h : \mathbb{R} \to \mathbb{C}$ is continuous and vanishes outside the interval $[-1, 1]$. Define $f : \mathbb{R} \to \mathbb{C}$ by $f(x,y) = e^{-x^2} h(y/x)$ for $x \neq 0$ and $f(0,y) = 0$. Then $\int_{\mathbb{R}^2} f\, dx\, dy = \int_{-1}^1 h(t)\, dt$.

Solution.

The function f is continuous on each of the Borel sets $\{(x,y) \in \mathbb{R}^2; x \neq 0\}$ and $\{(0,y); y \in \mathbb{R}\}$. It is therefore Borel (hence Lebesgue) measurable on $\mathbb{R}^2$. We apply the Tonelli theorem (2.18) to the non-negative Lebesgue measurable function $|f|$ on $\mathbb{R}^2$; since h vanishes outside $[-1, 1]$, we get

$$||f||_{L^1(\mathbb{R}^2)} = \int_{\mathbb{R}} e^{-x^2} \int_{|y| \leq |x|} |h(y/x)| dy\, dx.$$

The substitution $y/x = t$ in the "inner integral" shows that it is equal to $|x| \int_{-1}^{1} |h(t)|dt = |x|\, ||h||_1$. Therefore, the relevant integrand being even, we get

$$||f||_{L^1(\mathbb{R}^2)} = 2||h||_1 \int_0^\infty e^{-x^2} x\, dx = ||h||_1 < \infty.$$

Thus $f \in L^1(\mathbb{R}^2)$, and we may apply Fubini's theorem (2.17) to it:

$$\int_{\mathbb{R}^2} f\, dx\, dy = \int_{\mathbb{R}} e^{-x^2} \int_{|y| \le |x|} h(y/x) dy\, dx = \int_{-1}^{1} h(t) dt$$

(by elementary calculations as before).

6. Let $f : \mathbb{R}^2 \to \mathbb{C}$. Prove:

(a) If $f(x, \cdot)$ is Borel for all real x and $f(\cdot, y)$ is continuous for all real y, then f is Borel on $\mathbb{R}^2$.

Solution.

The solution is based on the fact that if f, g are Borel (Lebesgue) measurable on $\mathbb{R}$, then the function $F(x, y) = f(x)g(y)$ is Borel (Lebesgue, resp.) measurable on $\mathbb{R}^2$ (cf. solution of Exercise 4). The given function $f : \mathbb{R}^2 \to \mathbb{C}$ will be represented as a pointwise limit (a.e. in Part (b)) of a sequence of functions f_n, where each f_n is a finite sum of functions of the above type.

For each $n \in \mathbb{N}$, define

$$f_n(x, y) := (k - nx) f(\frac{k-1}{n}, y) + (nx + 1 - k) f(\frac{k}{n}, y) \tag{1}$$

for $(k-1)/n < x \le k/n$, $k \in \mathbb{Z}$, and $y \in \mathbb{R}$. The functions $k - nx$ and $nx + 1 - k$ are continuous, hence Borel measurable on $\mathbb{R}$; the functions $f((k-1)/n, \cdot)$ and $f(k/n, \cdot)$ are Borel measurable on $\mathbb{R}$ by hypothesis; hence f_n, restricted to each strip $((k-1)/n, k/n] \times \mathbb{R}$, are Borel measurable. Therefore f_n are Borel measurable on $\mathbb{R}^2$.
For $x \in ((k-1)/n, k/n]$ and all $y \in \mathbb{R}$,

$$|f_n(x,y) - f(x,y)| = \Big|(k-nx)\Big[f((k-1)/n, y) - f(x,y)\Big] + (nx+1-k)\Big[f(k/n, y) - f(x,y)\Big]\Big|$$

$$\le |f((k-1)/n, y) - f(x, y)| + |f(k/n, y) - f(x, y)|. \tag{2}$$

Given $(x, y) \in \mathbb{R}^2$, the unique k such that $x \in ((k-1)/n, k/n]$ satisfies $0 \le (k/n) - x < 1/n$. Since $f(\cdot, y)$ is continuous for each y, it follows from (2) that $f_n(x, y) \to f(x, y)$ as $n \to \infty$. We conclude from Lemma 1.5 that f is Borel measurable on $\mathbb{R}^2$.

(b) If $f(x,\cdot)$ is Lebesgue measurable for all x in some dense set $E \subset \mathbb{R}$ and $f(\cdot,y)$ is continuous for almost all $y \in \mathbb{R}$, then f is Lebesgue measurable on $\mathbb{R}^2$.

Solution.

Let

$$G := \{y \in \mathbb{R}; f(\cdot,y) \text{ is continuous}\}.$$

By hypothesis, its complement in $\mathbb{R}$, G^c, is a Lebesgue null set. Therefore the complement in $\mathbb{R}^2$ of $\mathbb{R} \times G$, which is $\mathbb{R} \times G^c$, is a Lebesgue null set in $\mathbb{R}^2$ ($0 \cdot \infty = 0$ in the arithmetic of ∞!).
Fix $n \in \mathbb{N}$. For each $k \in \mathbb{Z}$, pick a point $x_{n,k} \in E \cap ((k-1)/n, k/n]$ (this is possible, since E is dense in $\mathbb{R}$). Define f_n by

$$f_n(x,y) = f(x_{n,k}, y) \quad (x,y) \in ((k-1)/n, k/n] \times G. \tag{3}$$

These are Lebesgue measurable a.e. defined functions on $\mathbb{R}^2$ (cf. preceding remarks). For each given $(x,y) \in \mathbb{R} \times G$, $|x_{n,k} - x| < 1/n$, hence $f_n(x,y) \to f(x,y)$ as $n \to \infty$, by continuity of $f(\cdot,y)$ for $y \in G$ fixed. Thus $f_n \to f$ a.e. on $\mathbb{R}^2$, and we conclude that f is Lebesgue measurable (by the a.e. version of Lemma 1.5). (Note that we could simplify the definition of f_n in Part (a), without the "polygonal interpolation" used in (2). However we wished to have f_n "as regular as" f, that is: $f_n(\cdot,y)$ *continuous* for each given y, and $f_n(x,\cdot)$ Borel for each given x.)

Convolution and Fourier transform

7. If $E \subset \mathbb{R}$, denote

$$\tilde{E} := \{(x,y) \in \mathbb{R}^2; x - y \in E\}$$

and

$$\mathcal{S} := \{E \subset \mathbb{R}; \tilde{E} \in \mathcal{B}(\mathbb{R}^2)\},$$

where $\mathcal{B}(\mathbb{R}^2)$ is the Borel σ-algebra on $\mathbb{R}^2$. Prove:

(a) $\mathcal{S}$ is a σ-algebra on $\mathbb{R}$ which contains the open sets (hence $\mathcal{B}(\mathbb{R}) \subset \mathcal{S}$).

Solution.

Let $\eta : (x,y) \in \mathbb{R}^2 \to \mathbb{R}$ be defined by $\eta(x,y) = x - y$. Then η is continuous and $\tilde{E} = \eta^{-1}(E)$. Hence $\tilde{E}$ is open (hence Borel) in $\mathbb{R}^2$ for E open in $\mathbb{R}$, that is, $\mathcal{S}$ contains the open sets in $\mathbb{R}$. In particular $\mathcal{S}$ contains X.
If $E \in \mathcal{S}$, then $\eta^{-1}(E^c) = [\eta^{-1}(E)]^c \in \mathcal{B}(\mathbb{R}^2)$ because $\eta^{-1}(E) \in \mathcal{B}(\mathbb{R}^2)$. Thus $E^c \in \mathcal{S}$.
If $E_n \in \mathcal{S}$ for $n = 1, 2, \cdots$, then $\eta^{-1}(E_n) \in \mathcal{B}(\mathbb{R}^2)$, hence

$$\eta^{-1}(\bigcup_n E_n) = \bigcup_n \eta^{-1}(E_n) \in \mathcal{B}(\mathbb{R}^2),$$

and therefore $\bigcup_n E_n \in \mathcal{S}$.

(b) If f is a Borel function on $\mathbb{R}$, then $f(x-y)$ is a Borel function on $\mathbb{R}^2$.

Solution.

For any open set $V \subset \mathbb{C}$,

$$[f(x-y) \in V] = \{(x,y) \in \mathbb{R}^2; f(\eta(x,y)) \in V] = (f \circ \eta)^{-1}(V) = \eta^{-1}[f^{-1}(V)]. \quad (4)$$

Since f is Borel, $f^{-1}(V) \in \mathcal{B}(\mathbb{R}) \subset \mathcal{S}$ (by Part (a)). Therefore the set on the right end side of (4) belongs to $\mathcal{B}(\mathbb{R}^2)$, by the definition of $\mathcal{S}$.

(c) If f, g are integrable Borel functions on $\mathbb{R}$, then $f(x-y)g(y)$ is an integrable Borel function on $\mathbb{R}^2$ and its $L^1(\mathbb{R}^2)$-norm is equal to the product of the $L^1(\mathbb{R})$ norms of f and g.

Solution.

For any open set $V \subset \mathbb{C}$,

$$\{(x,y) \in \mathbb{R}^2; g(y) \in V\} = \mathbb{R} \times g^{-1}(V) \in \mathcal{B}(\mathbb{R}^2)$$

because $g^{-1}(V) \in \mathcal{B}(\mathbb{R})$ by Borel measurability of g. Therefore the function $(x,y) \to g(y)$ is Borel on $\mathbb{R}^2$. Together with Part (b), this shows that $f(x-y)g(y)$ is Borel on $\mathbb{R}^2$ (as the product of two such functions).
We now apply Tonelli's theorem (2.18) to the non-negative *Lebesgue* measurable function $|f(x-y)g(y)|$ (the shift to Lebesgue measure spaces is done because of the completeness hypothesis in Tonelli's theorem!):

$$\int_{\mathbb{R}^2} |f(x-y)g(y)|dx\,dy = \int_{\mathbb{R}} \Big(\int_{\mathbb{R}} |f(x-y)|dx \Big) |g(y)|dy. \quad (5)$$

By translation invariance of Lebesgue measure, the inner integral in (5) is equal to the constant $||f||_1$ (cf. page 67). Therefore the right hand side of (5) is equal to $||f||_1||g||_1 < \infty$, and the desired conclusion follows.

(d) Let $L^1(\mathbb{R})$ and $L^1(\mathbb{R}^2)$ be the Lebesgue spaces for the Lebesgue measure spaces on $\mathbb{R}$ and $\mathbb{R}^2$ respectively. If $f, g \in L^1(\mathbb{R})$, then $f(x-y)g(y) \in L^1(\mathbb{R}^2)$,

$$||f(x-y)g(y)||_{L^1(\mathbb{R}^2)} = ||f||_1||g||_1, \quad (6)$$

and

$$\int_{\mathbb{R}} |f(x-y)g(y)|dy < \infty \quad (7)$$

for almost all x. Hint: use Corollary 3.22, Chapter 3.

Solution.

By Corollary 3.22, there exists a sequence $\{f_n\}$ of continuous functions with compact support such that $f_n \to f$ poinwise a.e. on $\mathbb{R}$. Let $f_0 := \lim_n f_n$ on the set E of all points where the limit exists, and set $f_0 = 0$ elsewhere. Since the functions f_n are Borel functions, it follows from Exercise 13, Chapter 1, that E is a Borel set in $\mathbb{R}$, with E^c null, f_0 is a Borel function (cf. Lemma 1.5 and page 20), and $f_0 = f$ on E. Similarly, associate a Borel set $F \subset \mathbb{R}$ and a Borel function g_0 such that F^c is null and $g_0 = g$ on F. By Part (c), the function $f_0(x-y)g_0(y)$ is Borel on $\mathbb{R}^2$, integrable on $\mathbb{R}^2$, with $L^1(\mathbb{R}^2)$-norm equal to $||f_0||_1||g_0||_1$.
By Part (a),

$$W := \tilde{E} \cap (\mathbb{R} \times F) \in \mathcal{B}(\mathbb{R}^2).$$

On W, $f_0(x-y)g_0(y) = f(x-y)g(y)$, and the complement of W in $\mathbb{R}^2$ is

$$W^c = (\tilde{E})^c \cup (\mathbb{R} \times F)^c \tag{8}$$

(complements in $\mathbb{R}^2$). The first set in this union is $(\eta^{-1}(E))^c = \eta^{-1}(E^c)$; it is a Borel, hence Lebesgue, measurable set, whose Lebesgue measure can be calculated by Tonelli's theorem applied to its indicator:

$$|\eta^{-1}(E^c)| = \int_{\mathbb{R}^2} I_{\eta^{-1}(E^c)} dx\, dy = \int_{\mathbb{R}} \Big(\int_{y+E^c} dx \Big) dy = \int_{\mathbb{R}} |y + E^c| dy = 0,$$

since $|y + E^c| = |E^c| = 0$, by the translation invariance of Lebesgue measure. (The Lebesgue measure of a set $A \subset \mathbb{R}$ or $A \subset \mathbb{R}^2$ is denoted by $|A|$.)
The second set in the union (8) is equal to $\mathbb{R} \times F^c$, which is a Lebesgue null set in $\mathbb{R}^2$. Thus $|W^c| = 0$, that is, $f_0(x-y)g_0(y) = f(x-y)g(y)$ almost everywhere on $\mathbb{R}^2$. We then conclude that $f(x-y)g(y)$ is Lebesgue measurable on $\mathbb{R}^2$, and (by Part (c))

$$\int_{\mathbb{R}^2} |f(x-y)g(y)| dx\, dy = \int_{\mathbb{R}^2} |f_0(x-y)g_0(y)| dx\, dy$$

$$= ||f_0||_1 ||g_0||_1 = ||f||_1 ||g||_1.$$

Thus $f(x-y)g(y) \in L^1(\mathbb{R}^2)$, and (6) is valid. By Tonelli's theorem, the function $x \to \int_{\mathbb{R}} |f(x-y)g(y)| dy$ is Lebesgue measurable, and its integral over $\mathbb{R}$ is equal to $||f||_1||g||_1 < \infty$; it is then finite for almost all x (cf. page 15).

(e) For x such that (7) holds, define

$$(f * g)(x) = \int_{\mathbb{R}} f(x-y)g(y)dy. \tag{9}$$

Show that the (almost everywhere defined and finite-valued) function $f * g$ (called the *convolution* of f and g) is in $L^1(\mathbb{R})$, and

$$||f * g||_1 \leq ||f||_1 ||g||_1. \tag{10}$$

Solution.

Since $f(x-y)g(y) \in L^1(\mathbb{R}^2)$ by Part (d), the first statement follows from Fubini's theorem (2.17(ii)), and by (6) (and Tonelli's theorem),

$$||f * g||_1 = \int_{\mathbb{R}} \Big| \int_{\mathbb{R}} f(x-y)g(y)\, dy \Big|\, dx$$

$$\leq \int_{\mathbb{R}} \Big(\int_{\mathbb{R}} |f(x-y)g(y)|\, dy \Big)\, dx = ||f||_1 ||g||_1.$$

(f) For $f \in L^1(\mathbb{R})$, define its *Fourier transform* Ff by

$$(Ff)(t) = \int_{\mathbb{R}} f(x) e^{-ixt} dx \qquad (t \in \mathbb{R}).$$

Show that $Ff : \mathbb{R} \to \mathbb{C}$ is continuous, bounded by $||f||_1$, and $F(f * g) = (Ff)(Fg)$ for all $f, g \in L^1(\mathbb{R})$.

Solution.

For each given $t \in \mathbb{R}$, the function $x \to f(x)e^{-ixt}$ is Lebesgue measurable on $\mathbb{R}$ (as the product of the Lebesgue measurable function f and the continuous (hence Borel, hence Lebesgue measurable) function $x \to e^{-ixt}$), and since its absolute value is $|f| \in L^1(\mathbb{R})$, it is integrable, so that $(Ff)(t)$ is well-defined, and $|(Ff)(t)| \leq ||f||_1$. If $\{t_n\} \subset \mathbb{R}$ converges to t, then $f(x)e^{-ixt_n} \to f(x)e^{-ixt}$ by continuity of the exponential function, at all points x where f is defined (that is, almost everywhere on $\mathbb{R}$). Also $|f(x)e^{-ixt_n}| = |f(x)| \in L^1(\mathbb{R})$. By the Dominated Convergence theorem, it follows that $(Ff)(t_n) \to (Ff)(t)$. We conclude that Ff is a continuous function bounded by $||f||_1$.

Let $f, g \in L^1(\mathbb{R})$, and fix $t \in \mathbb{R}$. The function $(x,y) \in \mathbb{R}^2 \to f(x-y)g(y)e^{-ixt}$ is Lebesgue measurable on $\mathbb{R}^2$ (cf. Part (d)), and has the absolute value $|f(x-y)g(y)| \in L^1(\mathbb{R}^2)$ (cf. Part (d)). It is therefore in $L^1(\mathbb{R}^2)$, and consequently, by Part (e), Fubini's theorem (2.17(iii)), and the translation invariance of Lebesgue measure,

$$[F(f * g)](t) = \int_{\mathbb{R}} \Big(\int_{\mathbb{R}} f(x-y)g(y)e^{-ixt} dy \Big)\, dx$$

$$= \int_{\mathbb{R}} \Big(\int_{\mathbb{R}} f(x-y) e^{-ixt} dx \Big)\, g(y)\, dy$$

$$= \int_{\mathbb{R}} \Big(\int_{\mathbb{R}} f(x) e^{-i(x+y)t} dx \Big)\, g(y)\, dy$$

$$= \int_{\mathbb{R}} (Ff)(t) e^{-iyt} g(y)\, dy = (Ff)(t)(Fg)(t).$$

(g) If $f = I_{(a,b]}$ for $-\infty < a < b < \infty$, then

$$\lim_{|t|\to\infty} (Ff)(t) = 0. \tag{11}$$

Solution.

Fix $t \in \mathbb{R}$. The function $x \to f(x)e^{-ixt}$ is continuous except at the two points a, b; therefore its (improper) Riemann integral over $\mathbb{R}$ exists and converges (trivially) absolutely; it coincides consequently with the Lebesgue integral defining $(Ff)(t)$ (cf. page 68). Hence

$$(Ff)(t) = \int_a^b e^{-ixt} dx = (e^{-iat} - e^{-ibt})/it \qquad (t \neq 0),$$

and $(Ff)(0) = b - a$. In particular, $|(Ff)(t)| \le 2/|t|$ for all $t \neq 0$, and (11) follows.

(h) Show that the *step functions* (i.e finite linear combinations of indicators of disjoint intervals $(a_k, b_k]$) are dense in $C_c(\mathbb{R})$ (the normed space of continuous complex functions on $\mathbb{R}$ with compact support, with pointwise operations and supremum norm), and hence also in $L^p(\mathbb{R})$ for any $1 \le p < \infty$. Hint: use Corollary 3.21, Chapter 3. ("Density" of the step functions in $C_c(\mathbb{R})$ means that each $f \in C_c(\mathbb{R})$ is the uniform limit on $\mathbb{R}$ of a sequence of step functions.)

Solution.

Let $f \in C_c(\mathbb{R})$, and fix an interval (a, b) containing the (compact) support of f. Let $\epsilon > 0$. Since f is uniformly continuous on $[a, b]$, we may fix $n \in \mathbb{N}$ such that $|f(x) - f(y)| < \epsilon$ if $|x - y| < (b - a)/n$. Let $a = x_0 < x_1 < x_2 < \cdots < x_n = b$ be an equipartition of $[a, b]$ (i.e., $x_k - x_{k-1} = (b - a)/n$, $k = 1, \cdots, n$). Define

$$f_n = \sum_{k=1}^{n} f(x_k) I_{(x_{k-1}, x_k]}.$$

Then f_n is a step function. For each $x \in [a, b]$, there is a unique k such that $x \in (x_{k-1}, x_k]$. For this k, $f_n(x) = f(x_k)$, so that $|f(x) - f_n(x)| = |f(x) - f(x_k)| < \epsilon$,

since $|x - x_k| < (b-a)/n$. Note that $f = f_n = 0$ on $[a,b]^c$. Thus $||f - f_n||_u := \sup_{\mathbb{R}} |f - f_n| \leq \epsilon$. This proves the density of the step functions in $C_c(\mathbb{R})$.
By Corollary 3.21, Chapter 3, $C_c(\mathbb{R})$ is dense in $L^p(\mathbb{R})$ for each $p \in [1, \infty)$. Fix such p, and let $f \in L^p(\mathbb{R})$. Given $\epsilon > 0$, there exists $h \in C_c(\mathbb{R})$ such that $||f - h||_p < \epsilon/2$. Let $[a,b]$ be an interval containing the support of h. By the preceding construction, there exists a step function g such that $g = 0$ on $[a,b]^c$ and $||h - g||_u < \epsilon/[2(b-a)^{1/p}]$. Hence $||h-g||_p \leq \epsilon/2$, and therefore $||f - g||_p \leq ||f-h||_p + ||h-g||_p < \epsilon$. This proves the density of the step functions in $L^p(\mathbb{R})$.

(k) Prove (11) for any $f \in L^1(\mathbb{R})$. (This is the *Riemann-Lebesgue lemma.*)

Solution.

The validity of (11) extends from indicators of intervals to step functions by linearity. Next, let $f \in L^1(\mathbb{R})$ and $\epsilon > 0$. By Part (h), there exists a step function g such that $||f - g||_1 < \epsilon/2$. By (11) for step functions, there exists $K > 0$ such that $|(Fg)(t)| < \epsilon/2$ for $|t| > K$. Then by the linearity of the map $f \to (Ff)(t)$ on $L^1(\mathbb{R})$ and by Part (f),

$$|(Ff)(t)| \leq |(Fg)(t)| + |[F(f-g)](t)| \leq |(Fg)(t)| + ||f - g||_1 < \epsilon$$

for $|t| > K$, which proves (11) for any $f \in L^1(\mathbb{R})$.

(l) Generalize the above statements to functions on $\mathbb{R}^k$.

Solution.

The statements and proofs of Parts (a)-(f) are identical to the case $k=1$. (Replace $\mathbb{R}$ by $\mathbb{R}^k$ throughout, and let dx (or dy) and $dx\,dy$ denote the Lebesgue measure on $\mathbb{R}^k$ and $\mathbb{R}^k \times \mathbb{R}^k$ respectively.)
In the definition of Ff, the product xt is replaced by the inner product $x \cdot t := \sum_j x_j t_j$ of $x, t \in \mathbb{R}^k$.
In Part (g), the interval $(a,b]$ is replaced by the cell $T = \prod_j (a_j, b_j]$ in $\mathbb{R}^k$, and $|t| := \sqrt{(t \cdot t)}$ is the Euclidean norm in $\mathbb{R}^k$. Note that

$$I_T(t) = \prod_j I_{(a_j,b_j]}(t_j), \qquad t = (t_1, \cdots, t_k) \in \mathbb{R}^k.$$

Since $e^{-ix\cdot t} = \prod_j e^{-ix_j t_j}$, it follows from Fubini's theorem (2.17) that

$$(FI_T)(t) = \prod_{j=1}^{k} (FI_{(a_j,b_j]})(t_j) \qquad (t \in \mathbb{R}^k). \tag{12}$$

(We used the same notation F for the Fourier transform in $\mathbb{R}$ and $\mathbb{R}^k$.) Denote

$$H = \max_{1 \le l \le k} \prod_{j=1,\cdots,k; j \ne l} (b_j - a_j). \tag{13}$$

Let $\epsilon > 0$ be given. By Part (g) (on $\mathbb{R}$), there exists $M_0 > 0$ such that $|(FI_{(a_j,b_j]})(t_j)| < \epsilon/H$ for $|t_j| > M_0$, for all $j = 1, \cdots, k$. Let $M = M_0\sqrt{k}$. If $t \in \mathbb{R}^k$ is such that $|t| > M$, there exists l, $1 \le l \le k$, such that $|t_l| > M_0$ (otherwise $|t| \le M_0\sqrt{k} = M$). For this l,

$$|(FI_{(a_l,b_l]})(t_l)| < \epsilon/H. \tag{14}$$

By Part (f) (on $\mathbb{R}$) and (13),

$$\prod_{j=1,\cdots,k; j \ne l} |(FI_{(a_j,b_j]})(t_j)| \le \prod_{j \ne l} ||I_{(a_j,b_j]}||_1 = \prod_{j \ne l}(b_j - a_j) \le H.$$

Therefore, by (12) and (14), $|(FI_T)(t)| < (\epsilon/H)\, H = \epsilon$ for $|t| > M$. This proves the said version of (11) in $\mathbb{R}^k$:

$$\lim_{|t| \to \infty} (FI_T)(t) = 0. \tag{15}$$

In Part (h), intervals are replaced by cells as above in the definition of step functions; the statements and proofs of Parts (h) and (k) for $\mathbb{R}^k$ are then identical to the case $k = 1$. (Note that Corollary 3.21, which was used in the proof for $k = 1$, applies to very general measure spaces on locally compact Hausdorff spaces, and in particular to the Lebesgue measure space on $\mathbb{R}^k$.)

8. Let $p \in [1, \infty)$ and let q be its conjugate exponent. Let $K : \mathbb{R}^2 \to \mathbb{C}$ be Lebesgue measurable such that

$$\tilde{K}(y) := \int_{\mathbb{R}} |K(x,y)|\, dx \in L^q(\mathbb{R}).$$

Denote

$$(Tf)(x) = \int_{\mathbb{R}} K(x,y) f(y)\, dy.$$

Prove:

(a) $\int_{\mathbb{R}} \int_{\mathbb{R}} |K(x,y)f(y)|\, dy\, dx \le ||\tilde{K}||_q ||f||_p$ for all $f \in L^p(\mathbb{R})$. Conclude that $K(x,\cdot)f \in L^1(\mathbb{R})$ for almost all x, and therefore Tf is well defined a.e. (when $f \in L^p$).

Solution.

(All measurabilities below are *Lebesgue* measurabilities on the appropriate spaces.) By Tonelli's theorem (1.18), $|K(x,\cdot)|$ is measurable for almost all x, and $|K(\cdot,y)|$

is measurable for almost all y. In particular, $\tilde{K}$ is well defined a.e., and is itself measurable (cf. 1.18). For any $f \in L^p(\mathbb{R})$, the function $y \to |f(y)|$ is measurable on $\mathbb{R}$, and it follows that $(x,y) \to |f(y)|$ is measurable on $\mathbb{R}^2$, and so $(x,y) \to |K(x,y)f(y)|$ is measurable on $\mathbb{R}^2$. By Tonelli's theorem, $|K(x,\cdot)f|$ is measurable for almost all x, $\int_{\mathbb{R}} |K(x,y)f(y)|\,dy$ is a measurable (a.e. defined) function of x, and

$$\int_{\mathbb{R}^2} |K(x,y)f(y)|\,dxdy = \int_{\mathbb{R}} \Big(\int_{\mathbb{R}} |K(x,y)f(y)|\,dy \Big)\,dx$$

$$= \int_{\mathbb{R}} \Big(\int_{\mathbb{R}} |K(x,y)f(y)|\,dx \Big)\,dy = \int_{\mathbb{R}} \tilde{K}(y)|f(y)|\,dy \le ||\tilde{K}||_q ||f||_p, \tag{16}$$

where we used Holder's inequality (1.33). In particular, $K(x,y)f(y) \in L^1(\mathbb{R}^2)$. Therefore, by Fubini's theorem, $K(x,\cdot)f \in L^1(\mathbb{R})$ for almost all x (cf. 1.17(i)), and consequently Tf is well defined (almost everywhere).

(b) T is a continuous (linear) map of $L^p(\mathbb{R})$ into $L^1(\mathbb{R})$, and $||Tf||_1 \le ||\tilde{K}||_q ||f||_p$.

Solution.

As observed after (16), $K(x,y)f(y) \in L^1(\mathbb{R}^2)$. By Fubini's theorem (1.17(ii)), it follows that $Tf \in L^1(\mathbb{R})$, and by (16)

$$||Tf||_1 = \int_{\mathbb{R}} \Big| \int_{\mathbb{R}} K(x,y)f(y)dy \Big| dx \le \int_{\mathbb{R}} \int_{\mathbb{R}} |K(x,y)f(y)|dy\,dx \le ||\tilde{K}||_q ||f||_p. \tag{17}$$

The map T is thus a (trivially linear) map of $L^p(\mathbb{R})$ into $L^1(\mathbb{R})$. Replacing f by $f-g$ in the conclusion of (17) (for $f,g \in L^p(\mathbb{R})$), we obtain

$$||Tf - Tg||_1 = ||T(f-g)||_1 \le ||\tilde{K}||_q ||f-g||_p,$$

which proves the continuity of T from $L^p(\mathbb{R})$ to $L^1(\mathbb{R})$.

9. Apply Fubini's theorem to the function $e^{-xy}\sin x$ in order to prove the (Dirichlet) formula

$$\int_0^\infty \frac{\sin x}{x} dx = \pi/2.$$

Solution.

Observe first that the function $(\sin x)/x$, defined as 1 at 0, is continuous on $[0,\infty)$. Its Riemann integral over $[0,N]$ exists for all $N > 0$, and the Dirichlet integral is understood as the improper Riemann integral defined as

$$\lim_{N\to\infty} \int_0^N \frac{\sin x}{x} dx.$$

(Of course, the Riemann integrals $\int_0^N$ coincide with the Lebesgue integrals $\int_{[0,N]}$; cf. Proposition on page 68.)

The function $f(x,y) = e^{-xy}\sin x$ is continuous on $E_N := [0,N] \times [0,\infty)$, hence Borel, hence Lebesgue measurable. By Tonelli's theorem (2.18),

$$\int_{E_N} |f|\,dx\,dy = \int_{[0,N]} \Big(\int_{[0,\infty)} e^{-xy} dy\Big) |\sin x|\,dx = \int_{[0,N]} \frac{|\sin x|}{x} dx < \infty.$$

(The inner integral is equal to $1/x$ for all $x > 0$, hence a.e. on $[0,N]$.)
Thus $f \in L^1(E_N)$. Therefore, by Fubini's theorem (1.17)

$$\int_{[0,\infty)} \Big(\int_{[0,N]} f\,dx\Big)\,dy = \int_{[0,N]} \Big(\int_{[0,\infty)} e^{-xy} dy\Big)\,\sin x\,dx = \int_{[0,N]} \frac{\sin x}{x} dx. \tag{18}$$

Let $J_N(y) := \int_0^N f\,dx$. Two integrations by parts show that

$$J_N(y) = \frac{1 - e^{-Ny}[\cos N + y \sin N]}{1+y^2}. \tag{19}$$

Thus $J_N(y) \to 1/(1+y^2)$ as $N \to \infty$, for all $y > 0$. In addition, for all $N \geq 1$,

$$|J_N(y)| \leq \frac{1 + e^{-y}(1+y)}{1+y^2} \leq g(y),$$

where $g(y)$ is equal to $1/(1+y^2) + e^{-y}$ for $y \geq 1$ and to $3/(1+y^2)$ for $0 \leq y < 1$, and is evidently in $L^1([0,\infty))$. Therefore, by the Dominated Convergence theorem,

$$\int_{[0,\infty)} J_N(y)\,dy \to \int_0^\infty \frac{dy}{1+y^2} = \pi/2 \tag{20}$$

(cf. page 68). By (18), this proves that the improper Riemann integral $\int_0^\infty (\sin x)/x\,dx$ exists and is equal to $\pi/2$.