.

CHAPTER 1

MEASURES

1. Let $(X, \mathcal{A}, \mu)$ be a positive measure space, and let f be a non-negative measurable function on X. Let $E := [f < 1]$. Prove

(a) $\mu(E) = \lim_n \int_E \exp(-f^n) d\mu$.

Solution.

On the set E, the non-negative sequence $\{f^n\}$ is decreasing and has limit 0. Therefore $\{\exp(-f^n)\}$ restricted to E is a monotonically increasing sequence of positive measurable functions, with limit equal to 1. By the Monotone Convergence theorem (1.13),

$$\lim_n \int_E \exp(-f^n)\, d\mu = \int_E 1\, d\mu = \mu(E).$$

(b) $\sum_{n=1}^{\infty} \int_E f^n d\mu = \int_E [f/(1-f)] d\mu$.

Solution.

The series $\sum_{n=1}^{\infty} f^n$ of non-negative measurable functions converges on E to the non-negative measurable function $f/(1-f)$. By the Beppo Levi theorem (1.16),

$$\int_E f/(1-f)\, d\mu = \int_E \sum_n f^n d\mu = \sum_n \int_E f^n\, d\mu.$$

2. Let $(X, \mathcal{A}, \mu)$ be a positive measure space, and let p, q be conjugate exponents. Prove that the map $[f, g] \in L^p(\mu) \times L^q(\mu) \to fg \in L^1(\mu)$ is continuous.

Solution.

Write L^p for $L^p(\mu)$. If $[f, g] \in L^p \times L^q$, then $fg \in L^1$ (by Theorem 1.33), so that the map $[f, g] \to fg$ indeed maps $L^p \times L^q$ into L^1. If $[f, g]$, $[h, k] \in L^p \times L^q$, then by Holder's inequality (Theorem 1.33),

$$||fg - hk||_1 = ||(f-h)g + h(g-k)||_1 \le ||(f-h)g||_1 + ||h(g-k)||_1$$

$$\leq ||f-h||_p||g||_q + ||h||_p||g-k||_q. \tag{1}$$

When $[f,g] \to [h,k]$ in $L^p \times L^q$, we have $||f-h||_p \to 0$ and $||g-k||_q \to 0$. In particular $||g||_q \to ||k||_q$ (by continuity of the norm). Therefore the expression in (1) has the limit $0.||k||_q + ||h||_p.0 = 0$.

3. Let $(X, \mathcal{A}, \mu)$ be a positive measure space, $f_n : X \to \mathbb{C}$ measurable functions converging pointwise to f, and $h : \mathbb{C} \to \mathbb{C}$ continuous and bounded. Prove that $\lim_n \int_E h(f_n) d\mu = \int_E h(f) d\mu$ for each $E \in \mathcal{A}$ with finite measure.

Solution.

The functions $h(f_n)$ are measurable complex functions (cf. Lemma 1.6), and have pointwise limit $h(f)$, by continuity of h. If M is a bound for $|h|$ and $E \in \mathcal{A}$ has finite measure, then

$$|h(f_n)I_E| = |h(f_n)|I_E \leq M\, I_E \in L^1(\mu).$$

By the Dominated Convergence theorem (1.20),

$$\lim_n \int_E h(f_n)\, d\mu = \lim_n \int_X h(f_n) I_E\, d\mu = \int_X h(f) I_E\, d\mu = \int_E h(f)\, d\mu.$$

4. Let $(X, \mathcal{A}, \mu)$ be a positive measure space, and $\mathcal{B} \subset \mathcal{A}$ be a σ-finite σ-algebra. If $f \in L^1(\mathcal{A}) := L^1(X, \mathcal{A}, \mu)$, consider the complex measure on $\mathcal{B}$ defined by

$$\lambda_f(E) := \int_E f\, d\mu \qquad (E \in \mathcal{B}).$$

Prove:
(a) There exists a unique element $Pf \in L^1(\mathcal{B}) := L^1(X, \mathcal{B}, \mu)$ such that

$$\lambda_f(E) = \int_E (Pf)\, d\mu \qquad (E \in \mathcal{B}).$$

Solution.

Note first that since each $f \in L^1(\mathcal{A})$ can be written as $\sum_{k=0}^{3} i^k f_k$ with $f_k \in L^1(\mathcal{A})$ non-negative, it follows that

$$\lambda_f = \sum_{k=0}^{3} i^k \lambda_{f_k}, \tag{2}$$

and λ_{f_k} is a finite positive measure (cf. Theorem 1.17). Hence λ_f is a complex measure, and $\lambda_f << \mu$. If $(X, \mathcal{B}, \mu)$ (with $\mathcal{B} \subset \mathcal{A}$ a σ-algebra) is σ-finite, the Radon-Nikodym theorem (cf. 1.41, 1.45) implies the existence of a unique $h \in L^1(\mathcal{B})$ such that

$$\lambda_f(E) = \int_E h\, d\mu \qquad (E \in \mathcal{B}). \tag{3}$$

Define $Pf = h$. Note that $Pf \geq 0$ (a.e.) if $f \geq 0$ (apply 1.38 on the σ-finite measure space $(X, \mathcal{B}, \mu)$).

(b) The map $P : f \to Pf$ is a linear continuous map of $L^1(\mathcal{A})$ onto the *subspace* $L^1(\mathcal{B})$, such that $P^2 = P$ (P^2 denotes the composition of P with itself). In particular, $L^1(\mathcal{B})$ is a *closed* subspace of $L^1(\mathcal{A})$.

Solution.

Linearity of P. If $f, g \in L^1(\mathcal{A})$ and $a, b \in \mathbb{C}$, then for all $E \in \mathcal{B}$

$$\lambda_{af+bg}(E) := \int_E (af + bg) d\mu = a \int_E f\, d\mu + b \int_E g\, d\mu$$

$$= a\lambda_f(E) + b\lambda_g(E) = a \int_E Pf\, d\mu + b \int_E Pg\, d\mu$$

$$= \int_E (aPf + bPg) d\mu.$$

The uniqueness of the representation (3) and the definition of P imply that $P(af + bg) = aPf + bPg$.

Writing $f = \sum_{k=0}^{3} i^k f_k$ as above, it follows from the linearity of P and the relations $Pf_k \geq 0$ (a.e.) and $\int_X f_k d\mu \leq ||f||_{L^1(\mathcal{A})}$, that

$$||Pf||_{L^1(\mathcal{B})} = ||\sum_{k=0}^{3} i^k Pf_k||_{L^1(\mathcal{B})} \leq \sum_k ||Pf_k||_{L^1(\mathcal{B})}$$

$$= \sum_k \int_X Pf_k d\mu = \sum_k \lambda_{f_k}(X) = \sum_k \int_X f_k d\mu \leq 4||f||_{L^1(\mathcal{A})}.$$

This inequality implies the *continuity* of P ($||Pf - Pg||_{L^1(\mathcal{B})} = ||P(f - g)||_{L^1(\mathcal{B})} \leq 4||f - g||_{L^1(\mathcal{A})}$ for all $f, g \in L^1(\mathcal{A})$).

If $h \in L^1(\mathcal{B})$, surely $h \in L^1(\mathcal{A})$ ($\mathcal{B}$-measurability implies $\mathcal{A}$-measurability, because $\mathcal{B} \subset \mathcal{A}$, etc.), and since $\lambda_h(E) = \int_E h\, d\mu$ (for all $E \in \mathcal{B}$) by definition, with $h \in L^1(\mathcal{B})$, it follows from the uniqueness of the representation (3) and the definition of P that $Ph = h$. Thus P is *onto* $L^1(\mathcal{B})$. For any $f \in L^1(\mathcal{A})$, since $h := Pf \in L^1(\mathcal{B})$, we have $P^2(f) = Ph = h = Pf$, that is, $P^2 = P$.

Suppose $h_n \in L^1(\mathcal{B})$, and $h_n \to h$ in $L^1(\mathcal{A})$. By continuity of P (and since $Ph_n = h_n$, as observed above), $h_n = Ph_n \to Ph \in L^1(\mathcal{B})$ (convergence in $L^1(\mathcal{B})$, hence in

$L^1(\mathcal{A})$). By uniqueness of the limit, it follows that $h = Ph$, that is, $h \in L^1(\mathcal{B})$. This shows that $L^1(\mathcal{B})$ is a *closed* subspace of $L^1(\mathcal{A})$.

5. Let $(X, \mathcal{A}, \mu)$ be a finite positive measure space and $f_n \in L^p(\mu)$ for all $n \in \mathbb{N}$ (for some $p \in [1, \infty)$). Suppose there exists a measurable function $f : X \to \mathbb{C}$ such that $\sup_n \sup_X |f_n - f| < \infty$ and $f_n \to f$ in measure. Prove that $f \in L^p(\mu)$ and $f_n \to f$ in L^p-norm.

Solution.

By hypothesis, there exists a finite positive constant M such that $|f_n - f| \leq M$ on X for all n. Hence $||f_n - f||_p \leq M\mu(X)^{1/p} < \infty$, and therefore $f = (f - f_n) + f_n \in L^p(\mu)$.

Given $\epsilon > 0$, we have

$$||f_n - f||_p^p = \Big(\int_{[|f_n - f| < \epsilon]} + \int_{[|f_n - f| \geq \epsilon]}\Big)|f_n - f|^p d\mu$$

$$\leq \epsilon^p \mu(X) + M^p \mu([|f_n - f| \geq \epsilon]).$$

Since $f_n \to f$ in measure, we obtain

$$\limsup_n ||f_n - f||_p^p \leq \epsilon^p \mu(X).$$

The arbitrariness of ϵ implies that $\limsup_n ||f_n - f||_p^p = 0$, and therefore $||f_n - f||_p \to 0$ as $n \to \infty$.

6. Let λ and μ be positive σ-finite measures on the measurable space $(X, \mathcal{A})$. State and prove a version of the Lebesgue-Radon-Nikodym theorem for this situation.

Solution.

Since λ is a σ-finite positive measure on the measurable space $(X, \mathcal{A})$, there exist mutually disjoint sets $X_k \in \mathcal{A}$ $(k = 1, 2, ...)$, such that $\lambda(X_k) < \infty$ and $X = \bigcup_k X_k$. Fix k, and denote

$$\mathcal{A} \cap X_k := \{E \cap X_k;\ E \in \mathcal{A}\}.$$

The positive measure space $(X_k, \mathcal{A} \cap X_k, \mu)$ is σ-finite, and λ is a *finite* positive measure on $(X_k, \mathcal{A} \cap X_k)$. By the Lebesgue-Radon-Nikodym theorem (1.40), we have the unique decomposition

$$\lambda\Big|_{\mathcal{A} \cap X_k} = \lambda_{a,k} + \lambda_{s,k} \tag{4}$$

with $\lambda_{a,k}$ and $\lambda_{s,k}$ finite positive measures on $(X_k, \mathcal{A} \cap X_k)$ that are absolutely continuous and singular (respectively) with respect to the restriction of μ to $\mathcal{A} \cap X_k$. Also there exists a unique $h_k \in L^1(X_k, \mathcal{A} \cap X_k, \mu)$ such that

$$\lambda_{a,k}(G) = \int_G h_k d\mu \qquad (G \in \mathcal{A} \cap X_k). \tag{5}$$

We now define for any $E \in \mathcal{A}$

$$\lambda_a(E) = \sum_k \lambda_{a,k}(E \cap X_k), \tag{6}$$

$$\lambda_s(E) = \sum_k \lambda_{s,k}(E \cap X_k), \tag{7}$$

and we let h be the function on X which is equal to h_k on X_k.

A direct calculation with double series of non-negative terms shows that λ_a and λ_s are positive measures on $(X, \mathcal{A})$, and by (4), for all $E \in \mathcal{A}$,

$$\lambda_a(E) + \lambda_s(E) = \sum_k [\lambda_{a,k}(E \cap X_k) + \lambda_{s,k}(E \cap X_k)]$$

$$= \sum_k \lambda(E \cap X_k) = \lambda(E),$$

that is,

$$\lambda = \lambda_a + \lambda_s. \tag{8}$$

If $\mu(E) = 0$ for some $E \in \mathcal{A}$, then $\mu(E \cap X_k) = 0$ for all k, hence $\lambda_{a,k}(E \cap X_k) = 0$ for all k, and therefore $\lambda_a(E) = 0$ by (6). Thus $\lambda_a << \mu$.

Since $\lambda_{s,k}$ is singular with respect to the restriction of μ to $\mathcal{A} \cap X_k$, there exists a μ-null set $B_k \in \mathcal{A} \cap X_k$ which carries $\lambda_{s,k}$. Let B be the μ-null set $B = \bigcup_k B_k$. For all $E \in \mathcal{A}$,

$$\lambda_s(E) := \sum_k \lambda_{s,k}(E \cap X_k) = \sum_k \lambda_{s,k}(E \cap B_k)$$

$$= \sum_k \lambda_{s,k}(E \cap B \cap X_k) = \lambda_s(E \cap B),$$

that is, λ_s is carried by the μ-null set B. Thus $\lambda_s \perp \mu$.

(Uniqueness of the decomposition (8): the proof given in the text does not depend on the finiteness of the measure λ.)

By the averages lemma (1.38), $h_k \geq 0$ a.e. with respect to the restriction of μ to $\mathcal{A} \cap X_k$, and since h_k is only determined a.e., we may assume that $h_k \geq 0$. Then h is a non-negative $\mathcal{A}$-measurable function on X (so the integrals $\int_E h\, d\mu$ make sense for all $E \in \mathcal{A}$), and by (5) and (6)

$$\int_E h\, d\mu = \sum_k \int_{E\cap X_k} h\, d\mu = \sum_k \int_{E\cap X_k} h_k\, d\mu$$

$$= \sum_k \lambda_{a,k}(E \cap X_k) = \lambda_a(E).$$

This identity determines h uniquely (a.e.) on each X_k, hence (a.e.) on X. (Note that h is *σ-integrable*, in the sense that X is the disjoint union of countably many $X_k \in \mathcal{A}$, such that $h|_{X_k} \in L^1(X_k, \mathcal{A}\cap X_k, \mu)$, $k = 1, 2, \ldots.$) Collecting from the above discussion, we obtain the following version of the Lebesgue-Radon-Nikodym theorem for σ-finite positive measures λ, μ:

Let λ, μ be σ-finite positive measures on the measurable space $(X, \mathcal{A})$. Then

(a) λ has the unique decomposition

$$\lambda = \lambda_a + \lambda_s,$$

with λ_a and λ_s σ-finite positive measures on $(X, \mathcal{A})$, respectively absolutely continuous and singular with respect to μ;

(b) there exists a unique (up to μ-equivalence) non-negative σ-integrable function h on X such that

$$\lambda_a(E) = \int_E h\, d\mu \qquad (E \in \mathcal{A}).$$

7. Let $\{\lambda_n\}$ be a sequence of complex measures on the measurable space $(X, \mathcal{A})$ such that $\sum_n ||\lambda_n|| < \infty$. Prove

(a) For each $E \in \mathcal{A}$, the series $\sum_n \lambda_n(E)$ converges absolutely in $\mathbb{C}$ and defines a complex measure λ; the series $\sum_n |\lambda_n|(E)$ converges in $\mathbb{R}^+$, and defines a finite positive measure σ, and $\lambda << \sigma$.

Solution.

For all $E \in \mathcal{A}$, we have (cf. 1.43 and 1.44)

$$|\lambda_n(E)| \leq |\lambda_n|(E) \leq |\lambda_n|(X) := ||\lambda_n||. \qquad (9)$$

Since $\sum ||\lambda_n|| < \infty$, it follows that $\sum \lambda_n(E)$ converges absolutely. Set $\lambda(E) := \sum \lambda_n(E), \quad E \in \mathcal{A}$.
Let $E_k \in \mathcal{A}$ be mutually disjoint $(k = 1, 2, \ldots)$, and let E be their union. We have

$$\sum_n \sum_k |\lambda_n(E_k)| \le \sum_n \sum_k |\lambda_n|(E_k) = \sum_n |\lambda_n|(E) \le \sum_n ||\lambda_n|| < \infty.$$

Thus $\sum_n \sum_k \lambda_n(E_k)$ converges absolutely, and therefore its summation order may be interchanged. Hence

$$\lambda(E) = \sum_n \lambda_n(E) = \sum_n \sum_k \lambda_n(E_k) = \sum_k \sum_n \lambda_n(E_k) = \sum_k \lambda(E_k).$$

This proves that λ is a (complex) measure.
By (9) and the hypothesis $\sum ||\lambda_n|| < \infty$, the series $\sum |\lambda_n|(E)$ converges for each $E \in \mathcal{A}$. Denoting its sum by $\sigma(E)$, we obtain a finite positive measure (the σ-additivity is proved as above, using the fact that $|\lambda_n|$ are measures; the double series involved has non-negative terms, so that the change of summation order is trivial).
If $E \in \mathcal{A}$ is σ-null, necessarily $|\lambda_n|(E) = 0$ for all n (that is, $|\lambda_n| << \sigma$), hence $\lambda_n(E) = 0$ (thus $\lambda_n << \sigma$) for all n, and $\lambda(E) = 0$. Thus $\lambda << \sigma$.

(b)

$$\frac{d\lambda}{d\sigma} = \sum_n \frac{d\lambda_n}{d\sigma}.$$

Solution.

Since $\lambda << \sigma$, we may apply the Radon-Nikodym theorem on the *finite* positive measure space $(X, \mathcal{A}, \sigma)$ to the complex measure λ (cf. 1.45). Let then $h := \frac{d\lambda}{d\sigma}$ be the unique element of $L^1(\sigma)$ such that

$$\lambda(E) = \int_E h \, d\sigma \qquad (E \in \mathcal{A}). \tag{10}$$

We observed above that $\lambda_n << \sigma$ for all n. So we have uniquely determined elements $h_n := \frac{d\lambda_n}{d\sigma}$ in $L^1(\sigma)$ such that

$$\lambda_n(E) = \int_E h_n \, d\sigma \qquad (E \in \mathcal{A}). \tag{11}$$

By (11) and Theorem 1.47,

$$|\lambda_n|(E) = \int_E |h_n| d\sigma \qquad (E \in \mathcal{A}). \tag{12}$$

Therefore, for all $E \in \mathcal{A}$,

$$\sum_n \int_E |h_n| d\sigma = \sum_n |\lambda_n|(E) = \sigma(E). \tag{13}$$

If $\sigma(E) > 0$, we get by the Beppo Levi theorem (1.16) and (13)

$$\frac{1}{\sigma(E)} \int_E \sum_n |h_n| d\sigma = \frac{1}{\sigma(E)} \sum_n \int_E |h_n| d\sigma = 1.$$

Therefore, by the "averages lemma" (1.38), $\sum |h_n| = 1$ σ-a.e. In particular, the series $\sum h_n$ converges absolutely σ-a.e. Denote its N-th partial sum by g_N. Then $g_N \to \sum h_n$ σ-a.e. and $|g_N| \le \sum |h_n| = 1$ σ-a.e. By the Lebesgue Dominated Convergence theorem (1.21) on the *finite* positive measure space $(X, \mathcal{A}, \sigma)$, we have for all $E \in \mathcal{A}$,

$$\int_E \sum_n h_n \, d\sigma = \lim_N \int_E g_N d\sigma = \lim_N \sum_{n=1}^{N} \int_E h_n d\sigma$$

$$= \lim_N \sum_{n=1}^{N} \lambda_n(E) = \sum_{n=1}^{\infty} \lambda_n(E) := \lambda(E).$$

By (10) and uniqueness of the integral representation, we conclude that $h = \sum_n h_n$ σ-a.e. as desired.

8. Let $(X, \mathcal{A}, \mu)$ be a positive measure space, and let $M := M(\mathcal{A})$ denote the vector space (over $\mathbb{C}$) of all complex measures on $\mathcal{A}$. Set

$$M_a := \{\lambda \in M; \lambda << \mu\};$$

$$M_s := \{\lambda \in M; \lambda \perp \mu\}.$$

Prove:

(a) If $\lambda \in M$ is supported by E, then so is $|\lambda|$.

Solution.

Let $\{A_k\}$ be any partition of E^c. Since A_k is a measurable subset of E^c and E carries λ, we have $\lambda(A_k) = 0$ for all k, hence $\sum_k |\lambda(A_k)| = 0$. Therefore $|\lambda|(E^c) = 0$, i.e., $|\lambda|$ is supported by E.
Note that the converse is trivially true: if E carries $|\lambda|$, then for any measurable subset A of E^c, $|\lambda(A)| \le |\lambda|(A) = 0$, hence $\lambda(A) = 0$, and therefore E carries λ. Thus

(a') *$E \in \mathcal{A}$ carries λ iff it carries $|\lambda|$.*

(b) M_a and M_s are subspaces of M and $M_a \perp M_s$ (in particular, $M_a \cap M_s = \{0\}$).

Solution.

Let $a, b \in \mathbb{C}$. Suppose λ_1, $\lambda_2 \in M_a$. If E is a μ-null set, then $\lambda_k(E) = 0$ for $k = 1, 2$, hence $a\lambda_1(E) + b\lambda_2(E) = 0$. Thus $a\lambda_1 + b\lambda_2 \in M_a$.
Suppose next that λ_1, $\lambda_2 \in M_s$, and let then E_k be μ-null sets that carry λ_k ($k = 1, 2$), respectively. Then $E = E_1 \cup E_2$ is μ-null, and for all measurable subsets $A \subset E^c$, we have $A \subset E_k^c$ ($k = 1, 2$), hence $\lambda_k(A) = 0$ for $k = 1, 2$, and therefore $a\lambda_1(A) + b\lambda_2(A) = 0$. Thus $a\lambda_1 + b\lambda_2$ is carried by the μ-null set E, hence belongs to M_s.
Next, let $\lambda_1 \in M_a$ and $\lambda_2 \in M_s$. Let then E be a μ-null set carrying λ_2. If A is a measurable subset of E, we have $\mu(A) = 0$, hence $\lambda_1(A) = 0$ since $\lambda_1 \in M_a$. This shows that λ_1 is carried by E^c, that is, the measures λ_1, λ_2 are carried by the disjoint sets E^c, E, respectively. Thus $\lambda_1 \perp \lambda_2$, and $M_a \perp M_s$ follows.
If $\lambda \in M_a \cap M_s$, then in particular $\lambda \perp \lambda$ (because $M_a \perp M_s$). This by itself (that is, without the particular definitions of M_a and M_s!) implies that $\lambda = 0$. Indeed, there exist disjoint sets $E, F \in \mathcal{A}$, each carrying λ. Hence for all $A \in \mathcal{A}$, write

$$\lambda(A) = \lambda(A \cap E) + \lambda(A \cap E^c).$$

The first term on the right vanishes because $A \cap E$ is a subset of F^c (and F carries λ); the second term vanishes because $A \cap E^c$ is a subset of E^c (and E carries λ). Thus $\lambda = 0$.

(c) If $(X, \mathcal{A}, \mu)$ is σ-finite, then $M = M_a \oplus M_s$.

Solution.

By the Lebesgue Decomposition theorem (1.45 (1)), each $\lambda \in M$ has the *unique* representation $\lambda = \lambda_a + \lambda_s$ with $\lambda_a \in M_a$ and $\lambda_s \in M_s$, i.e., $M = M_a \oplus M_s$.

(d) $\lambda \in M_a$ iff $|\lambda| \in M_a$ (and similarly for M_s).

Solution.

If $\lambda \in M_a$ and E is μ-null, then $\lambda(A) = 0$ for any measurable subset A of E, i.e., E^c carries λ; hence E^c carries $|\lambda|$ (by Part (a)), and therefore $|\lambda|(E) = 0$. This shows that $|\lambda| \in M_a$. On the other hand, if $|\lambda| \in M_a$ and E is μ-null, then $|\lambda(E)| \le |\lambda|(E) = 0$, hence $\lambda \in M_a$.
By Part (a'), a μ-null set E carries λ (that is, $\lambda \in M_s$) iff it carries $|\lambda|$ (that is, $|\lambda| \in M_s$). Thus $\lambda \in M_s$ iff $|\lambda| \in M_s$.

(e) If $\lambda_k \in M$ $(k = 1, 2)$, then $\lambda_1 \perp \lambda_2$ iff $|\lambda_1| \perp |\lambda_2|$.

Solution.

$\lambda_1 \perp \lambda_2$ iff there exist *disjoint* sets $E_k \in \mathcal{A}$ such that E_k carries λ_k, $k = 1, 2$. By Part (a'), the latter happens iff E_k carries $|\lambda_k|$, $k = 1, 2$, that is, iff $|\lambda_1| \perp |\lambda_2|$.

(f) $\lambda << \mu$ iff for each $\epsilon > 0$, there exists $\delta > 0$ such that $|\lambda(E)| < \epsilon$ for all $E \in \mathcal{A}$ with $\mu(E) < \delta$.

Solution.

Let $\lambda << \mu$, and suppose the ϵ, δ condition in Part (f) fails, that is: there exists $\epsilon > 0$ such that for all $\delta > 0$, there exists $F_\delta \in \mathcal{A}$ with $\mu(F_\delta) < \delta$ and $|\lambda(F_\delta)| \geq \epsilon$. Take $\delta_n = 1/2^n$ and define $E_n = F_{\delta_n}$ (thus $\mu(E_n) < 1/2^n$ and $|\lambda(E_n)| \geq \epsilon$, $n = 1, 2, \cdots$). Set

$$E = \limsup E_n := \bigcap_n G_n; \qquad G_n := \bigcup_{k \geq n} E_k.$$

Since $G_{n+1} \subset G_n$ and

$$\mu(G_1) \leq \sum_k \mu(E_k) < \sum_k 1/2^k = 1,$$

it follows from Lemma 1.11 that $\mu(E) = \lim_n \mu(G_n)$. But as $n \to \infty$,

$$\mu(G_n) \leq \sum_{k=n}^{\infty} \mu(E_k) < \sum_{k=n}^{\infty} 1/2^k = 1/2^{n-1} \to 0.$$

Hence $\mu(E) = 0$.
However $|\lambda|(E_n) \geq |\lambda(E_n)| \geq \epsilon$ for all n. We apply (4) on page 9 to the finite positive measure $|\lambda|$ (cf. Theorem 1.43). Thus

$$|\lambda|(E) \geq \limsup_n |\lambda|(E_n) \geq \epsilon.$$

Therefore $|\lambda| \notin M_a$, contradicting Part (d).
(The converse is trivial.)

9. Let $(X, \mathcal{A}, \mu)$ be a *probability space* (i.e. a positive measure space such that $\mu(X) = 1$). Let f, g be (complex) measurable functions. Prove that $||f||_1 ||g||_1 \geq \inf_X |fg|$.

Solution.

Note first that since only the absolute values $|f|$, $|g|$ are involved in the inequality, and the latter is trivial if either f or g vanishes at some point, we may assume that $0 < f,\ g < \infty$.
Consider first the case when f, g are *simple* (measurable) functions. Write (all indices below range over a finite segment of $\mathbb{N}$)

$$f = \sum a_k I_{E_k}; \qquad g = \sum b_j I_{F_j},$$

where $\{a_k\}$ is the (finite, positive) range of f, $E_k = [f = a_k]$, and similarly for g. Since X is the disjoint union of E_k (and of F_j), we have $\sum_k \mu(E_k) = \mu(X) = 1$, and similarly $\sum_j \mu(F_j) = 1$. Therefore

$$||f||_1||g||_1 = \sum_k a_k\mu(E_k)\sum_j b_j\mu(F_j) = \sum_k\sum_j a_k b_j \mu(E_k)\mu(F_j)$$

$$\geq \min_{k,j}(a_k b_j)\sum_k\sum_j \mu(E_k)\mu(E_j) = \min_{k,j}(a_k b_j) = \inf_X fg,$$

as desired.
Consider next the case of *bounded* and *bounded away from zero* measurable functions f, g, that is, there exists constants $0 < \delta < M < \infty$ such that $\delta \leq f, g \leq M$ on X. Set $c := \inf_X fg$.
By the Approximation theorem (1.8 (1)), there exist simple measurable functions ϕ_n, ψ_n such that

$$0 \leq \phi_1 \leq \phi_2 \leq \cdots \leq \phi_n \leq \cdots \leq f,$$
$$0 \leq \psi_1 \leq \psi_2 \leq \cdots \leq \psi_n \leq \cdots \leq g,$$

and

$$\phi_n \to f, \qquad \psi_n \to g$$

uniformly on X. Let $0 < \epsilon < \delta$. By uniform convergence, there exists $n_0 \in \mathbb{N}$ such that for all $n > n_0$,

$$\phi_n > f - \epsilon \quad (> f - \delta \geq 0)$$

and

$$\psi_n > g - \epsilon \quad (> g - \delta \geq 0).$$

Hence, for all $n > n_0$,

$$\phi_n\psi_n > (f-\epsilon)(g-\epsilon) > fg - \epsilon(f+g) \geq c - \epsilon(f+g) \geq c - 2\epsilon M,$$

i.e., $\inf_X \phi_n\psi_n \geq c - 2\epsilon M$. By the special case of the inequality for *simple* functions, fixing $n > n_0$, we get

$$||f||_1||g||_1 \geq ||\phi_n||_1||\psi_n||_1 \geq \inf_X \phi_n\psi_n \geq c - 2\epsilon M,$$

and the desired inequality $||f||_1||g||_1 \geq c$ follows from the arbitrariness of ϵ.

We consider next the case where f, g are bounded away from zero but not necessarily bounded. Let N be an arbitrary positive integer. Set

$$E_N = [f \leq N]; \qquad F_N = [g \leq N].$$

Then

$$E_1 \subset E_2 \subset \cdots; \qquad \bigcup_N E_N = X,$$

and similarly for F_N. By Lemma 1.10 for the positive measures μ, $\nu(E) := \int_E f\, d\mu$, and $\sigma(E) := \int_E g\, d\mu$, we have

$$\lim_N \mu(E_N) = \mu(X) = 1, \qquad \lim_N \mu(F_N) = 1, \tag{14}$$

$$\lim_N \int_{E_N} f\, d\mu = \nu(X) = ||f||_1, \tag{15}$$

and similarly

$$\lim_N \int_{F_N} g\, d\mu = ||g||_1. \tag{16}$$

Define

$$f_N = fI_{E_N} + aI_{E_N^c}, \qquad g_N = gI_{F_N} + aI_{F_N^c},$$

where $a = \max(c/\delta, \delta)$. Then

$$\delta \leq f_N,\ g_N \leq \max(N, a).$$

By the bounded case,

$$||f_N||_1||g_N||_1 \geq \inf_X f_N g_N. \tag{17}$$

Write

$$X = (E_N \cap F_N) \cup (E_N \cap F_N^c) \cup (E_N^c \cap F_N) \cup (E_N^c \cap F_N^c).$$

On the first set, $f_N g_N = fg \geq c$. On the second set, $f_N g_N = fa \geq \delta(c/\delta) = c$. On the third set $f_N g_N = ag \geq (c/\delta)\delta = c$. On the fourth set, $f_N g_N = a.a \geq (c/\delta)\delta = c$. Hence $f_N g_N \geq c$ on X, and we conclude from (17) that

$$||f_N||_1||g_N||_1 \geq c. \tag{18}$$

However by (14) and (15)

$$||f_N||_1 = \int_{E_N} f\, d\mu + a(1 - \mu(E_N)) \to ||f||_1$$

and similarly $||g_N||_1 \to ||g||_1$ when $N \to \infty$, by (14) and (16). Letting $N \to \infty$, we then obtain the wanted inequality $||f||_1||g||_1 \geq c$.

Finally, in case f, g are not necessarily bounded away from zero, we consider the sets

$$G_N = [f \geq 1/N]; \qquad H_N = [g \geq 1/N]$$

and the bounded away from zero functions

$$f'_N = fI_{G_N} + bI_{G_N^c}; \qquad g'_N = gI_{H_N} + bI_{H_N^c},$$

where $b = \max(Nc, 1/N)$. Calculations analogous to those of the preceding case, together with the result for that case, yield the result in the present (general) situation.

10. Let $(X, \mathcal{A}, \mu)$ be a positive measure space and f a complex measurable function on X.

(a) If $\mu(X) < \infty$, prove that

$$\lim_{p\to\infty} ||f||_p = ||f||_\infty. \tag{*}$$

Solution.

(We do not assume $\mu(X) < \infty$ until stated specifically.) Let f be a complex measurable function on the measure space $(X, \mathcal{A}, \mu)$. If $||f||_\infty = 0$, then $f = 0$ a.e., hence $||f||_p = 0$ for all p, and (*) is trivial. If $||f||_\infty = \infty$, any $N \in \mathbb{N}$ is *not* an a.e. upper bound for $|f|$, that is, for each $N \in \mathbb{N}$, there exists $E_N \in \mathcal{A}$ with $\mu(E_N) > 0$ such that $|f| > N$ on E_N. Then for all p

$$||f||_p \geq \Big(\int_{E_N} |f|^p d\mu\Big)^{1/p} \geq N\mu(E_N)^{1/p} \tag{19}$$

($= \infty$ if $\mu(E_N) = \infty$). Hence for all N

$$\liminf_{p\to\infty} ||f||_p \geq N. \tag{19'}$$

Therefore

$$\limsup_p ||f||_p \geq \liminf_p ||f||_p = \infty,$$

and consequently $\lim_p ||f||_p = \infty = ||f||_\infty$. This shows that *throughout the exercise*, we need to consider only the case

$$0 < ||f||_\infty < \infty.$$

If (*) has been verified in case $||f||_\infty = 1$, then for arbitrary f (with $0 < ||f||_\infty < \infty$!), the function $g := f/||f||_\infty$ satisfies $||g||_\infty = 1$, hence $\lim_{p\to\infty} ||g||_p = 1$, and therefore

$$\lim_p ||f||_p = \lim_p ||(||f||_\infty) g||_p = ||f||_\infty \lim_p ||g||_p = ||f||_\infty.$$

We now assume $\mu(X) < \infty$, and consider the case $||f||_\infty = 1$.
Let $0 < \epsilon < 1$. Since $||f||_\infty = 1$, $1-\epsilon$ is not an a.e. upper bound for $|f|$, that is, there exists $E \in \mathcal{A}$ with $\mu(E) > 0$ such that $|f| > 1-\epsilon$ on E. The argument leading to (19) (with $1-\epsilon$ and E replacing N and E_N, respectively) gives $||f||_p \geq (1-\epsilon)\mu(E)^{1/p}$. On the other hand, since $|f| \leq ||f||_\infty = 1$ a.e., we have $||f||_p \leq \mu(X)^{1/p}$. Thus

$$(1-\epsilon)\mu(E)^{1/p} \leq ||f||_p \leq \mu(X)^{1/p}. \tag{20}$$

Since $0 < \mu(E) \leq \mu(X) < \infty$, it follows from (20) that

$$1 - \epsilon \leq \liminf_{p\to\infty} ||f||_p \leq \limsup_{p\to\infty} ||f||_p \leq 1.$$

The arbitrariness of ϵ implies that the limit $\lim_p ||f||_p$ exists and is equal to 1.

(b) For an arbitrary positive measure space, if $||f||_r < \infty$ for some $r \in [1, \infty)$, then (*) is valid.

Solution.

Consider the positive measure ν defined by

$$\nu(E) = \int_E |f|^r d\mu \qquad (E \in \mathcal{A}). \tag{21}$$

(cf. Theorem 1.17.) It is finite, since $\nu(X) = ||f||_r^r < \infty$ by hypothesis. Therefore, by Part (a),

$$\lim_{p\to\infty} ||f||_{L^p(\nu)} = ||f||_{L^\infty(\nu)}. \tag{22}$$

For all $p \geq r+1$, we have by Theorem 1.17,

$$||f||_{L^{p-r}(\nu)}^{1-r/p} = \left(\int_X |f|^{p-r} d\nu\right)^{1/p} = \left(\int_X |f|^{p-r}|f|^r d\mu\right)^{1/p} = ||f||_p. \tag{23}$$

By Part (a) with the *finite* measure ν, the left hand side of (23) has the limit $||f||_{L^\infty(\nu)}$. Thus

$$\lim_{p\to\infty} ||f||_p = ||f||_{L^\infty(\nu)}. \tag{24}$$

It remains to show that

$$||f||_{L^\infty(\nu)} = ||f||_\infty. \tag{25}$$

If the left hand side of (25) vanishes, then $f = 0$ ν-a.e., i.e., $\nu([|f| > 0]) = 0$, hence

$$\int_X |f|^r d\mu = \int_{[|f|>0]} |f|^r d\mu := \nu([|f| > 0]) = 0,$$

and therefore $f = 0$ μ-a.e., so that $||f||_\infty = 0$, and (25) is correct in this case.
We may then assume that $||f||_{L_\infty(\nu)} > 0$.
Since $||f||_\infty$ is a μ-a.e. upper bound for $|f|$ (cf. (22) on page 28) and $\nu << \mu$, it follows that $||f||_\infty$ is a ν-a.e. upper bound for $|f|$, and therefore

$$||f||_{L^\infty(\nu)} \leq ||f||_\infty. \tag{26}$$

If the inequality (26) is *strict*, we may choose $t > 1$ such that $c := t||f||_{L^\infty(\nu)} < ||f||_\infty$. Set $E = [|f| > c]$. Then $\mu(E) > 0$, and by (19') (with N and E_N replaced by c and E respectively)

$$\liminf_{p\to\infty} ||f||_p \geq c.$$

By (24), it follows that

$$||f||_{L^\infty(\nu)} \geq c := t||f||_{L^\infty(\nu)}.$$

Hence $1 \geq t$, contradiction. This shows that the inequality (26) is *not* strict, and (25) follows.

11. Let $(X, \mathcal{A}, \mu)$ be a positive measure space, $1 \leq p < \infty$, and $\epsilon > 0$.

(a) Suppose f_n, f are unit vectors in $L^p(\mu)$ such that $f_n \to f$ a.e. Consider the probability measure $d\nu = |f|^p d\mu$. Show that there exists $E \in \mathcal{A}$ such that $f \neq 0$ and $f_n/f \to 1$ uniformly on E, and $\nu(E^c) < \epsilon$.

Solution.

Let $G = [f \neq 0]$. Then $f_n/f \to 1$ a.e. on G, and $(G, \mathcal{A} \cap G, \nu)$ is a probability space, because

$$\nu(G) = \int_G |f|^p d\mu = \int_X |f|^p d\mu = 1.$$

Let $\epsilon > 0$ be given. By Egoroff's theorem (1.57), there exists $E \in \mathcal{A} \cap G$ such that $\nu(E^c) < \epsilon$ and $f_n/f \to 1$ *uniformly* on E (cf. 1.49 (1)).

(b) For E as in Part (a), show that $\limsup_n \int_{E^c} |f_n|^p d\mu < \epsilon$.

Solution.

Let $0 < \delta < 1$. By Part (a), there exists $n_0 \in \mathbb{N}$ such that for all $n > n_0$, $|(f_n/f) - 1| < \delta$ on E (and $|f| > 0$ on E). Equivalently,

$$|f_n - f| < \delta |f|, \tag{27}$$

and therefore $|f_n| > (1-\delta)|f|$ on E, for all $n > n_0$. Consequently

$$\int_E |f_n|^p d\mu \geq (1-\delta)^p \nu(E) \qquad (n > n_0).$$

Since $||f_n||_p^p = 1$ by hypothesis,

$$\int_{E^c} |f_n|^p d\mu = 1 - \int_E |f_n|^p d\mu \leq 1 - (1-\delta)^p \nu(E)$$

for all $n > n_0$, hence

$$\limsup_n \int_{E^c} |f_n|^p d\mu \leq 1 - (1-\delta)^p \nu(E).$$

Letting $\delta \to 0+$, we obtain

$$\limsup_n \int_{E^c} |f_n|^p d\mu \leq 1 - \nu(E) = \nu(E^c) < \epsilon.$$

(c) Deduce from Parts (a) and (b) that $f_n \to f$ in $L^p(\mu)$-norm.

Solution.

We have

$$||f_n - f||_p^p = \Big(\int_E + \int_{E^c}\Big)|f_n - f|^p d\mu. \tag{28}$$

By (27), the first integral in (28) is

$$\leq \delta^p \int_E |f|^p d\mu \leq \delta^p ||f||_p^p = \delta^p \tag{29}$$

for all $n > n_0$.
By Minkovsky's inequality, the second integral in (28) is

$$\leq \Big[\Big(\int_{E^c} |f_n|^p d\mu\Big)^{1/p} + \Big(\int_{E^c} |f|^p d\mu\Big)^{1/p}\Big]^p. \tag{30}$$

The second term in the square brackets in (30) is equal to $\nu(E^c)^{1/p} < \epsilon^{1/p}$. Hence for all $n > n_0$,

$$||f_n - f||_p^p \leq \delta^p + \Big[\Big(\int_{E^c} |f_n|^p d\mu\Big)^{1/p} + \epsilon^{1/p}\Big]^p.$$

Therefore by Part (b),

$$\limsup_n ||f_n - f||_p^p \leq \delta^p + 2^p \epsilon.$$

By the arbitrariness of ϵ and δ, we conclude that $\lim_n ||f_n - f||_p = 0$

(d) If $g_n, g \in L^p(\mu)$ are such that $g_n \to g$ a.e. and $||g_n||_p \to ||g||_p$, then $g_n \to g$ in $L^p(\mu)$-norm.

Solution.

If $||g||_p = 0$, then g is the zero element of $L^p(\mu)$. Hence $||g_n - g||_p = ||g_n||_p \to 0$, and the claim is trivial.
We may then assume that $||g||_p > 0$. Since $||g_n||_p \to ||g||_p$, there exists n_0 such that $||g_n||_p > 0$ for all $n > n_0$. Define $f = g/||g||_p$ and $f_n = g_n/||g_n||_p$ for $n > n_0$. These are unit vectors in $L^p(\mu)$, and $f_n \to f$ a.e. By Part (c), $||f_n - f||_p \to 0$ as $n \to \infty$. Therefore, for $n > n_0$,

$$||g_n - g||_p = \Big|\Big| f_n ||g_n||_p - f\, ||g||_p \Big|\Big|_p$$

$$\leq ||f_n - f||_p ||g_n||_p + ||f||_p \Big(||g_n||_p - ||g||_p \Big) \to 0 \cdot ||g||_p + ||f||_p \cdot 0 = 0$$

as $n \to \infty$.

12. Let $(X, \mathcal{A}, \mu)$ be a measurable space and $f \in L^p(\mu)$ for some $p \in [1, \infty)$. Prove that the set $[f \neq 0]$ has σ-finite measure.

Solution.

Let $f \in L^p(\mu)$ for some $p \in [1, \infty)$. We write (cf. page 16)

$$[f \neq 0] = [|f| > 0] = \bigcup_{n=1}^{\infty} [|f| > 1/n]. \tag{31}$$

By (2) on page 50 (with $y = 1/n$)

$$\mu([|f| > 1/n]) \leq (n||f||_p)^p < \infty.$$

Together with (31), this shows that the set $[f \neq 0]$ has σ- finite measure.

13. Let $(X, \mathcal{A})$ be a measurable space, and let $f_n : X \to \mathbb{C}$, $n \in \mathbb{N}$, be measurable functions. Prove that the set of all points $x \in X$ for which the complex sequence $\{f_n(x)\}$ converges in $\mathbb{C}$ is measurable.

Solution.

By the Cauchy criterion for convergence of the (complex) sequence $\{f_n(x)\}$,

$$[f_n \text{ converge}] = \bigcap_{r=1}^{\infty} \bigcup_{s=1}^{\infty} \bigcap_{m,n=s}^{\infty} [|f_n - f_m| < 1/r]. \tag{32}$$

Since $|f_n - f_m|$ is a (positive) measurable function (for each n, m), the sets $[|f_n - f_m| < 1/r]$ are in $\mathcal{A}$, and it follows from (32) that $[f_n \text{ converge}] \in \mathcal{A}$, since $\mathcal{A}$ is a σ- algebra.

14. Let $(X, \mathcal{A})$ be a measurable space, and let E be a dense subset of $\mathbb{R}$. Suppose $f : X \to \mathbb{R}$ is such that $[f \geq c] \in \mathcal{A}$ for all $c \in E$. Prove that f is measurable.

Solution.

Let $a \in \mathbb{R}$. Since E is dense in $\mathbb{R}$, there exist $c_n \in E$ such that $a - (1/n) < c_n < a$ for $n = 1, 2, \cdots$. (In particular, $c_n \to a$.) Then

$$[f \geq a] = \bigcap_{n=1}^{\infty} [f \geq c_n].$$

Since $[f \geq c_n] \in \mathcal{A}$ for all n (by hypothesis), it follows that $[f \geq a] \in \mathcal{A}$ for all real a, and therefore f is measurable by (one of the versions of) Lemma 1.4 (cf. page 3).

15. Let $(X, \mathcal{A})$ be a measurable space, and let $f : X \to \mathbb{R}^+$ be measurable. Prove that there exist $c_k > 0$ and $E_k \in \mathcal{A}$ $(k \in \mathbb{N})$ such that $f = \sum_{k=1}^{\infty} c_k I_{E_k}$. Conclude that for any positive measure μ on $\mathcal{A}$, $\int f \, d\mu = \sum_{k=1}^{\infty} c_k \mu(E_k)$; in particular, if $f \in L^1(\mu)$, the series converges (in the strict sense) and $\mu(E_k) < \infty$ for all k.

Solution.

(The case $f = 0$ identically being trivial, we assume otherwise, so that the following discussion is not "empty".) By the approximation theorem, there exist simple measurable functions

$$0 \leq \phi_0 \leq \phi_1 \leq \phi_2 \leq \cdots \leq \phi_n \leq \cdots \leq f$$

such that $\lim \phi_n = f$.
Define $\psi_0 = \phi_0$ and $\psi_n = \phi_n - \phi_{n-1}$ for $n \geq 1$. Then ψ_n are non-negative simple

measurable functions such that $f = \sum_n \psi_n$. Each ψ_n is a finite linear combination of measurable indicators with non-negative coefficients, and therefore f is an infinite linear combination of indicators of (not necessarily disjoint) measurable sets E_k. Omitting the summands with zero coefficients, we can write

$$f = \sum_{k=1}^{\infty} c_k I_{E_k}, \qquad c_k > 0.$$

By the Beppo Levi theorem (1.16),

$$\int_X f\, d\mu = \sum_{k=1}^{\infty} c_k \mu(E_k). \tag{33}$$

If $f \in L^1(\mu)$, the series (of non-negative terms) in (33) converges in the strict sense (to $||f||_1$), and since $c_k > 0$, we have necessarily $\mu(E_k) \le ||f||_1/c_k < \infty$ for all $k \in \mathbb{N}$.

16. Let $(X, \mathcal{A}, \mu)$ be a positive measure space, and let $\{E_k\} \subset \mathcal{A}$ be such that $\sum_k \mu(E_k) < \infty$. Prove that almost all $x \in X$ lie in at most finitely many of the sets E_k.

Solution.

Let F denote the set of all x belonging to *infinitely* many sets E_n. Then (cf. page 9)

$$F = \limsup_n E_n := \bigcap_n G_n, \qquad G_n = \bigcup_{k \ge n} E_k.$$

By σ-subadditivity of μ (cf. page 8),

$$\mu(G_n) \le \sum_{k=n}^{\infty} \mu(E_k),$$

hence $\mu(G_n) \to 0$ as $n \to \infty$, because $\sum_k \mu(E_k) < \infty$. Since $F \subset G_n$ for all n, we have $\mu(F) \le \mu(G_n) \to 0$, and therefore $\mu(F) = 0$. This means that *almost all* x belong to F^c, which is precisely the set of all x belonging to at most *finitely many* of the sets E_k.

17. Let X be a (complex) normed space. Define

$$f(x, y) = \frac{||x+y||^2 + ||x-y||^2}{2||x||^2 + 2||y||^2} \qquad (x, y \in X).$$

(We agree that the fraction is 1 when $x = y = 0$.) Prove:

(a) $1/2 \leq f \leq 2$.

Solution.

Let $x, y \in X$. We may assume that x, y are not both zero (since $f(0,0) := 1 \in [1/2, 2]$). By the triangle inequality for norms,

$$||x+y||^2 + ||x-y||^2 \leq 2(||x|| + ||y||)^2 = 2(||x||^2 + ||y||^2 + 2||x||\,||y||).$$

However $2||x||\,||y|| \leq ||x||^2 + ||y||^2$. Therefore $||x+y||^2 + ||x-y||^2 \leq 4(||x||^2 + ||y||^2)$ and consequently $f(x,y) \leq 2$.
Consider the vectors $u = (x+y)/2$ and $v = (x-y)/2$ (they are not both zero, because otherwise $x = y = 0$). Since $u + v = x$ and $u - v = y$, we get from the preceding inequality with the vectors u, v replacing x, y,

$$2 \geq f(u,v) = \frac{||x||^2 + ||y||^2}{2||(x+y)/2||^2 + 2||(x-y)/2||^2} = \frac{2||x||^2 + 2||y||^2}{||x+y||^2 + ||x-y||^2} := \frac{1}{f(x,y)},$$

hence $1/2 \leq f(x,y)$.

(b) X is an inner product space (i.p.s.) iff $f = 1$ (identically).

Solution.

The identity $f = 1$ is equivalent to the parallelogram identity, which is indeed a necessary condition for X to be an i.p.s. (cf. (8), page 30). We must prove the *sufficiency* of the identity (for X to be an i.p.s.).
Define for all $x, y \in X$

$$(x,y) := (1/4)\Big[(||x+y||^2 - ||x-y||^2) + i(||x+iy||^2 - ||x-iy||^2)\Big] \tag{34}$$

(cf. the *polarization identity* (11) on page 31). We shall verify that $(\cdot,\cdot)$ is an inner product on X.
Note that

$$4\Re(x,y) = ||x+y||^2 - ||x-y||^2; \qquad \Im(x,y) = \Re(x,iy). \tag{35}$$

We have by (35)

$$4\Im(x,x) = ||x+ix||^2 - ||x-ix||^2 = (|1+i|^2 - |1-i|^2)||x||^2 = 0,$$

and $4\Re(x,x) = ||2x||^2 = 4||x||^2$, so that $(x,x) = ||x||^2$. It follows in particular that the form $(\cdot,\cdot)$ is positive definite (and the "induced norm" $(\cdot,\cdot)^{1/2}$ coincides with the

given norm).
By (35) with $x = u + v$ $(u, v \in X)$,

$$8\Re(u+v,y) = 2||u+v+y||^2 - 2||u+v-y||^2.$$

We add and subtract $2||y||^2$ and use the given parallelogram identity with the pairs of vectors $[u+v+y, y]$ and $[u+v-y, y]$; thus

$$8\Re(u+v,y) = ||u+v+2y||^2 + ||u+v||^2 - [||u+v||^2 + ||u+v-2y||^2]$$

$$= ||u+v+2y||^2 - ||u+v-2y||^2. \quad (36)$$

On the other hand, after regrouping and using the parallelogram identity, we obtain

$$8\Re[(u,y)+(v,y)] = 2||u+y||^2 - 2||u-y||^2 + 2||v+y||^2 - 2||v-y||^2$$

$$= 2[||u+y||^2 + ||v+y||^2] - 2[||u-y||^2 + ||v-y||^2] = ||u+v+2y||^2 - ||u+v-2y||^2. \quad (37)$$

By (36) and (37),

$$\Re(u+v,y) = \Re[(u,y)+(v,y)]. \quad (38)$$

By (35) and (38),

$$\Im(u+v,y) = \Re(u+v,iy) = \Re(u,iy) + \Re(v,iy) = \Im[(u,y)+(v,y)],$$

and therefore, by (38),

$$(u+v,y) = (u,y) + (v,y). \quad (39)$$

In particular $(2x, y) = (x+x, y) = (x, y)+(x, y) = 2(x, y)$, and proceeding by induction, we obtain

$$(2^k x, y) = 2^k (x, y) \quad (40)$$

for all $k \in \mathbb{N}$ and $x, y \in X$. Replacing $2^k x$ by x, we get that the identity (40) is valid for all $k \in \mathbb{Z}$. Next, for any *positive* real number α with *finite* diadic expansion $\alpha = \sum_{k\in J} 2^k$ (where J is a finite subset of $\mathbb{Z}$), we obtain from (39) and (40)

$$(\alpha x, y) = \sum_{k\in J} (2^k x, y) = \sum_k 2^k (x, y) = \alpha(x, y).$$

By (34) and the continuity of the norm on X, the function $(\cdot, y)$ is continuous (for each given y). Since the above α's are dense in $\mathbb{R}^+$, we conclude that $(\alpha x, y) = \alpha(x, y)$ for all positive real α.
We have

$$4\Re(-x,y) = ||-x+y||^2 - ||-x-y||^2 = -[||x+y||^2 - ||x-y||^2] = -4\Re(x,y),$$

hence also $\Im(-x,y) = \Re(-x,iy) = -\Re(x,iy) = -\Im(x,y)$, and therefore $(-x,y) = -(x,y)$. Together with the "positive" homogeneity obtained before, we get homogeneity over $\mathbb{R}$ of the functional $(\cdot,y)$ (the fact $(0,y)=0$ follows trivially from the definition (34)). By (39), $(\cdot,y)$ is linear over the real field of scalars.
We have by definition and rearrangement

$$4(ix,y) = ||ix+y||^2 - ||ix-y||^2 + i||i(x+y)||^2 - i||i(x-y)||^2$$

$$= i\Big[||x+y||^2 - ||x-y||^2 + i||i(x+iy)||^2 - i||i(x-iy)||^2\Big] = 4i(x,y).$$

Together with the linearity over $\mathbb{R}$ of the functional $(\cdot,y)$, this implies that the functional is linear over $\mathbb{C}$.
By (35)

$$4\Re(y,x) = ||y+x||^2 - ||y-x||^2 = 4\Re(x,y)$$

and

$$4\Im(y,x) = 4\Re(y,ix) = ||y+ix||^2 - ||y-ix||^2 = -||-i(iy+x)||^2 + ||i(-iy+x)||^2$$

$$= -||x+iy||^2 + ||x-iy||^2 = -4\Re(x,iy) = -4\Im(x,y).$$

Hence $(y,x) = \overline{(x,y)}$ for all $x,y \in X$. This concludes the verification of the claim that $(\cdot,\cdot)$ is an inner product on X (whose induced norm coincides with the given norm); thus X is an i.p.s.