5.17.a. The model is described by the following equations.

$$\begin{aligned} y_i &= x_i'\beta + \varepsilon_i, & \text{(S 5.58)} \\ x_{ei} &= \Gamma z_i + v_i, \ v_i \sim \text{NID}\left(0, \Omega\right), & \text{(S 5.59)} \\ \varepsilon_i &= v_i'\alpha + w_i, \ w_i \sim \text{NID}\left(0, \sigma^2\right). & \text{(S 5.60)} \end{aligned}$$

Here $v_i'\alpha = E\left[\varepsilon_i | v_i\right]$, and because all involved random variables are normally distributed this implies that w_i is independent of v_i. The likelihood function can be expressed in terms of the independent variables w_i and v_i as

$$L\left(\theta\right) = p\left(y_1, ..., y_n, x_1, ..., x_n\right) = p\left(w_1, ..., w_n, v_1, ..., v_n\right) = \prod_{i=1}^{n} p\left(w_i\right) p\left(v_i\right).$$

As $w_i \sim N\left(0, \sigma^2\right)$ and $v_i \sim N\left(0, \Omega\right)$, the log-likelihood is

$$\begin{aligned} l\left(\theta\right) &= \sum_{i=1}^{n} \log p\left(w_i\right) + \sum_{i=1}^{n} \log p\left(v_i\right) \\ &= -n\log\left(2\pi\right) - \frac{n}{2}\log\left(\sigma^2\right) - \frac{1}{2\sigma^2}\sum_{i=1}^{n} w_i^2 + \frac{n}{2}\log\left(\det\left(\Omega^{-1}\right)\right) - \frac{1}{2}\sum_{i=1}^{n} v_i'\Omega^{-1}v_i. \end{aligned}$$

This depends on the parameters θ through the relationships (S 5.58), (S 5.59) and (S 5.60).

b. Writing the log-likelihood out explicitly, we get

$$\begin{aligned} l\left(\theta\right) &= -n\log\left(2\pi\right) - \frac{n}{2}\log\left(\sigma^2\right) - \frac{1}{2\sigma^2}\sum_{i=1}^{n}\left(y_i - x_i'\beta - \left(x_{ei} - \Gamma z_i\right)'\alpha\right)^2 \\ &\quad + \frac{n}{2}\log\left(\det\left(\Omega^{-1}\right)\right) - \frac{1}{2}\sum_{i=1}^{n}\left(x_{ei} - \Gamma z_i\right)'\Omega^{-1}\left(x_{ei} - \Gamma z_i\right). \qquad \text{(S 5.61)} \end{aligned}$$

Under the condition of the null hypothesis that $\alpha = 0$, this becomes

$$\begin{aligned} l_0 &= -n\log\left(2\pi\right) - \frac{n}{2}\log\left(\sigma^2\right) - \frac{1}{2\sigma^2}\sum_{i=1}^{n}\left(y_i - x_i'\beta\right)^2 \\ &\quad + \frac{n}{2}\log\left(\det\left(\Omega^{-1}\right)\right) - \frac{1}{2}\sum_{i=1}^{n}\left(x_{ei} - \Gamma z_i\right)'\Omega^{-1}\left(x_{ei} - \Gamma z_i\right) \\ &= l_{01}\left(\beta, \sigma^2\right) + l_{02}\left(\Gamma, \Omega\right). \end{aligned}$$

Observe that l_0 is the sum of two components that depend on different parameters, the function l_{01} that depends on β and σ^2 and the function l_{02} that depends on Γ and Ω^{-1}. These two components can be maximized separately. Maximizing l_{01} yields the familiar ML estimates $\widehat{\beta} = b$ and $\widehat{\sigma}^2 = e'e/n$ of the model $y_i = x_i'\beta + \varepsilon_i$ (with NID disturbances). The maximization of l_{02} requires the computation of its first order derivatives. For this we use the following results on matrix differentiation.

$$\begin{aligned} \frac{\partial \log\left(\det\left(\Omega^{-1}\right)\right)}{\partial \Omega^{-1}} &= \Omega, \quad \frac{\partial\left(a'\Omega^{-1}a\right)}{\partial\Omega^{-1}} = aa', \quad \frac{\partial\left(a'\Gamma b\right)}{\partial\Gamma} = ab', \quad \frac{\partial\left(a'\Gamma' b\right)}{\partial\Gamma} = ba', \\ \frac{\partial\left(a'\Gamma'\Omega^{-1}\Gamma b\right)}{\partial\Gamma} &= \Omega^{-1}\Gamma\left(ab' + ba'\right), \end{aligned}$$

where Ω^{-1} is a symmetric matrix, Γ is a matrix, and a and b are column vectors. This gives

$$\begin{aligned} \frac{\partial l_{02}}{\partial\Gamma} &= \Omega^{-1}\left(\sum_{i=1}^{n} x_{ei}z_i' - \sum_{i=1}^{n} z_i z_i'\right) = \Omega^{-1}\left(X_e'Z - \Gamma Z'Z\right) = 0, \\ \frac{\partial l_{02}}{\partial\Omega^{-1}} &= \frac{1}{2}\left[n\Omega - \sum_{i=1}^{n}\left(x_{ei} - \Gamma z_i\right)\left(x_{ei} - \Gamma z_i\right)'\right] = \frac{1}{2}\left(n\Omega - V'V\right) = 0, \end{aligned}$$

where X_e is the $n \times k_0$ matrix with rows x'_{ei}, Z is the $n \times m$ matrix with rows z'_i and V is the $n \times k_0$ matrix with rows v'_i. Solving these equations we obtain $\widehat{\Gamma} = X'_e Z \left(Z'Z\right)^{-1}$ (so that $\widehat{\Gamma}' = \left(Z'Z\right)^{-1} Z'X_e$) and $\widehat{\Omega} = \frac{1}{n}\widehat{V}'\widehat{V} = \frac{1}{n}\sum \widehat{v}_i\widehat{v}'_i$ where $\widehat{V} = (\widehat{v}_1, ..., \widehat{v}_n)'$ with $\widehat{v}_i = x_{ei} - \widehat{\Gamma} z_i$.

c. The first order derivatives of the unrestricted log-likelihood $l(\theta)$ in (S 5.61) are:

$$\begin{aligned}
\frac{\partial l}{\partial \alpha} &= \frac{1}{\sigma^2}\sum_{i=1}^{n}\left(y_i - x'_i\beta - (x_{ei} - \Gamma z_i)'\alpha\right)(x_{ei} - \Gamma z_i) = \frac{1}{\sigma^2}\sum_{i=1}^{n} w_i v_i, \\
\frac{\partial l}{\partial \beta} &= \frac{1}{\sigma^2}\sum_{i=1}^{n}\left(y_i - x'_i\beta - (x_{ei} - \Gamma z_i)'\alpha\right)x_i = \frac{1}{\sigma^2}\sum_{i=1}^{n} w_i x_i, \\
\frac{\partial l}{\partial \Gamma} &= -\frac{1}{\sigma^2}\sum_{i=1}^{n} w_i \alpha z'_i + \Omega^{-1}\left(X'_e Z - \Gamma Z'Z\right) = -\frac{1}{\sigma^2}\sum_{i=1}^{n} w_i \alpha z'_i + \Omega^{-1}\sum_{i=1}^{n} v_i z'_i, \\
\frac{\partial l}{\partial \sigma^2} &= -\frac{n}{2\sigma^2} + \frac{1}{2\sigma^4}\sum_{i=1}^{n} w_i^2, \\
\frac{\partial l}{\partial \Omega^{-1}} &= \frac{1}{2}\left(n\Omega - V'V\right) = \frac{1}{2}\left(n\Omega - \sum_{i=1}^{n} v_i v'_i\right).
\end{aligned}$$

Evaluated under the restriction that $\alpha = 0$, with corresponding ML estimates $\widehat{\theta}_0$ of part (b), this gives (with $\widehat{w}_i = e_i$ the OLS residuals in $y_i = x'_i\beta + \varepsilon_i$)

$$\begin{aligned}
\left.\frac{\partial l}{\partial \alpha}\right|_{\widehat{\theta}_0} &= \frac{1}{\widehat{\sigma}^2}\sum_{i=1}^{n} e_i \widehat{v}_i = \frac{1}{\widehat{\sigma}^2}\widehat{V}'e, \\
\left.\frac{\partial l}{\partial \beta}\right|_{\widehat{\theta}_0} &= \frac{1}{\widehat{\sigma}^2}\sum_{i=1}^{n} e_i x_i = 0, \\
\left.\frac{\partial l}{\partial \Gamma}\right|_{\widehat{\theta}_0} &= \Omega^{-1}\left(X_e Z - \widehat{\Gamma} Z'Z\right) = 0, \\
\left.\frac{\partial l}{\partial \sigma^2}\right|_{\widehat{\theta}_0} &= -\frac{n}{\widehat{\sigma}^2} + \frac{1}{2\widehat{\sigma}^4}\sum_{i=1}^{n} e_i^2 = 0, \\
\left.\frac{\partial l}{\partial \Omega^{-1}}\right|_{\widehat{\theta}_0} &= \frac{1}{2}\left(n\widehat{\Omega} - \sum_{i=1}^{n} \widehat{v}_i \widehat{v}'_i\right) = 0.
\end{aligned}$$

d. We prove the expressions for B_{11}, B_{12} and B_{22}. For instance, B_{11} is obtained by differentiating $-\frac{\partial l}{\partial \alpha}$ with respect to α'. Using the expressions for $\frac{\partial l}{\partial \alpha}$ and $\frac{\partial l}{\partial \beta}$ obtained in part (c) it follows that

$$\begin{aligned}
B_{11} &= -\frac{\partial^2 l}{\partial \alpha \partial \alpha'} = -\frac{\partial}{\partial \alpha'}\left(\frac{1}{\sigma^2}\sum w_i v_i\right) = -\frac{1}{\sigma^2}\sum v_i \frac{\partial\left(y_i - x'_i\beta - v'_i\alpha\right)}{\partial \alpha'} = \frac{1}{\sigma^2}\sum v_i v'_i \\
&= \frac{1}{\sigma^2}V'V, \\
B_{22} &= -\frac{\partial^2 l}{\partial \beta \partial \beta'} = -\frac{\partial}{\partial \beta'}\left(\frac{1}{\sigma^2}\sum w_i x_i\right) = -\frac{1}{\sigma^2}\sum x_i \frac{\partial\left(y_i - x'_i\beta - v'_i\alpha\right)}{\partial \beta'} = \frac{1}{\sigma^2}\sum x_i x'_i \\
&= \frac{1}{\sigma^2}X'X, \\
B_{12} &= -\frac{\partial^2 l}{\partial \alpha \partial \beta'} = -\frac{\partial}{\partial \beta'}\left(\frac{\partial l}{\partial \alpha}\right) = -\frac{\partial}{\partial \beta'}\left(\frac{1}{\sigma^2}\sum w_i v_i\right) = -\frac{1}{\sigma^2}\sum v_i \frac{\partial\left(y_i - x'_i\beta - v'_i\alpha\right)}{\partial \beta'} \\
&= \frac{1}{\sigma^2}\sum v_i x'_i = \frac{1}{\sigma^2}V'X.
\end{aligned}$$

Evaluating these expressions at the ML estimates under the null hypothesis we obtain that $B_{11} = \frac{1}{\widehat{\sigma}^2}\widehat{V}'\widehat{V}$, $B_{22} = \frac{1}{\widehat{\sigma}^2}X'X$, $B_{12} = \frac{1}{\widehat{\sigma}^2}\widehat{V}'X$.

We do not prove that the other off-diagonal elements of the Hessian matrix can be approximated by 0. However, note that the unrestricted log-likelihood (S 5.61) can be written as $l(\theta) = l_1\left(\alpha,\beta,\Gamma,\sigma^2\right) + l_2\left(\Gamma,\Omega^{-1}\right)$. This implies, for instance, that $B_{15} = \frac{\partial^2 l}{\partial\alpha\partial\Omega^{-1}} = 0$, $B_{25} = \frac{\partial^2 l}{\partial\beta\partial\Omega^{-1}} = 0$ and $B_{45} = \frac{\partial^2 l}{\partial\sigma^2\partial\Omega^{-1}} = 0$. The other blocks require more tedious computations.

e. Denoting by $*$ non-zero parts of vectors and matrices, we can conclude the following from the results in part (c) and (d). Here all expressions are evaluated at the ML estimator $\widehat{\theta}_0$ under the null hypothesis, and the blocks are ordered according to the components of $\theta = \left(\alpha,\beta,\Gamma,\sigma^2,\Omega^{-1}\right)$.

$$
\begin{aligned}
LM &= \left(\frac{\partial l}{\partial\theta}\right)'\left(-\frac{\partial^2 l}{\partial\theta\partial\theta'}\right)^{-1}\left(\frac{\partial l}{\partial\theta}\right)\\
&= \begin{pmatrix} \frac{1}{\widehat{\sigma}^2}\widehat{V}'e \\ 0 \\ 0 \\ 0 \\ 0 \end{pmatrix}'
\begin{pmatrix} \frac{1}{\widehat{\sigma}^2}\widehat{V}'\widehat{V} & \frac{1}{\widehat{\sigma}^2}\widehat{V}'X & 0 & 0 & 0 \\ \frac{1}{\widehat{\sigma}^2}X'\widehat{V} & \frac{1}{\widehat{\sigma}^2}X'X & 0 & 0 & 0 \\ 0 & 0 & * & * & * \\ 0 & 0 & * & * & * \\ 0 & 0 & * & * & * \end{pmatrix}^{-1}
\begin{pmatrix} \frac{1}{\widehat{\sigma}^2}\widehat{V}'e \\ 0 \\ 0 \\ 0 \\ 0 \end{pmatrix}\\
&= \frac{1}{\widehat{\sigma}^2}\begin{pmatrix} e'\widehat{V} & 0 \end{pmatrix}\begin{pmatrix} \widehat{V}'\widehat{V} & \widehat{V}'X \\ X'\widehat{V} & X'X \end{pmatrix}^{-1}\begin{pmatrix} \widehat{V}'e \\ 0 \end{pmatrix} = \frac{1}{\widehat{\sigma}^2}e'U\left(U'U\right)^{-1}U'e,
\end{aligned}
$$

where we used that $e'U = e'\begin{pmatrix} \widehat{V} & X \end{pmatrix} = \begin{pmatrix} e'\widehat{V} & e'X \end{pmatrix} = \begin{pmatrix} e'\widehat{V} & 0 \end{pmatrix}$ because $X'e = 0$.

f. The auxiliary regression (5.83) can be written in matrix form as $e = X\delta + \widehat{V}\alpha + \eta = U\lambda + \eta$ where $\lambda = \begin{pmatrix} \delta \\ \alpha \end{pmatrix}$. OLS gives $\widehat{\lambda} = \left(U'U\right)^{-1}U'e$ with explained part $\widehat{e} = U\widehat{\lambda} = U\left(U'U\right)^{-1}U'e$ and explained sum of squares $\widehat{e}'\widehat{e} = e'U\left(U'U\right)^{-1}U'e$. The total sum of squares is $e'e$. Then the regression (5.83) gives

$$
nR^2 = n\frac{SSE}{SST} = n\frac{e'U\left(U'U\right)^{-1}U'e}{e'e} = \frac{1}{\widehat{\sigma}^2}e'U\left(U'U\right)^{-1}U'e
$$

because $\widehat{\sigma}^2 = \frac{1}{n}e'e$. This is the same expression as the one obtained in part (e), which proves that $LM = nR^2$.